Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2022, 9(9):48-59**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Group Actions on Incidence Matrices of *X*-Labeled graphs

Wadhah S. Jassim

College of Bilad Alrafidain-Khanagin- Iraq

Abstract The main aim of this work is to define an action of a group on incidence matrix of *X*-labeled graph and, then constructing the incidence matrix of *X*-labeled graph of groups and their directed incidence matrix of X-labeled graph of groups.

Keywords Incidence Matrices, *X*-labeled Graphs,

1. Introduction

In [1] we gave the definition of Incidence of *X*- labeled graph and in [2] we gave an application of the incidence matrix of X- labeled graph which is the incidence matrix of directed graph of groups and their up-down pregroup. In this work we give new concepts which are called the action of group on the incidence matrix of *X*-labeled graph and the incidence matrix of *X*-labeled graph of groups which is called the incidence matrix of a directed graph of groups that in [1]. Moreover, we can write a computer program for this algorithm. Therefore, this paper is divided six sections, in section one we give an introduction, in section two we give the basic concepts that we use in the rest of this work, such as the graph, group actin on graphs, incidence matrix of *X*-labeled graph. in section three we give the definition of group action incidence matrix and other concepts. In section four we give the method of construction of the graph of groups by using the action of the group on *X*-labeled graph. In section five we give the conclusion and in section six we give an example to show the construction of the incidence matrix of *X*-labeled core graph.

2. Preliminaries

A **graph** Γ is a collection of two disjoint sets ($V(\Gamma)$ and $E(\Gamma)$ (such that $V(\Gamma)$ is a nonempty set) which are called the sets of **vertices** and **edges** respectively of the graph Γ, Together with two functions $i : E(\Gamma) \to V(\Gamma)$, $t : E(\Gamma) \to V(\Gamma)$ (the functions *i* and *t* join the vertices $i(e)$ and $t(e)$ to the edge *e* of *Γ*. The vertex $i(e)$ is called **the initial** vertex of *e* and $t(e)$ is called the **terminal** vertex of *e*. Moreover for each *e* in $E(\Gamma)$, there is an element $\overline{e}$ in *E*, is called the **inverse** of *e*, such that

$i(\overline{e}) = t(e)$, $t(\overline{e}) = i(e)$ and $\overline{\overline{e}} = e$.

A directed graph Γ is called a ***X*- labeled graph**, if each directed edge *e* of Γ is labeled by a letter *x* of the set *X*. Therefore $\Gamma(F, X)$ Cayley graph, $\Gamma(F, X)/H$ Cayley coset graph $\Gamma(H)$ and $\Gamma^*(H)$ Core graph of Cayley coset graph are *X*- labeled graphs. **The product of *X*- Labeled graphs Γ and Δ** is the graph $\Gamma \tilde{\times} \Delta$ with set of vertices $V(\Gamma) \times V(\Delta) = \{(u, v) : u \in V(\Gamma), v \in V(\Delta)\}$ and edges

$\{((u,v),y):(u,y)\in E(\Gamma),(v,y)\in E(\Delta),y\in X\}$.

An *X*- labeled graph Γ is called **folded graph**, if for each vertex *v* of Γ is not incident with two edges e_1, e_2 labeled *x*, x or x^{-1}, x^{-1} respectively, $x\in X$. Otherwise Γ is called **non - folded graph** (or unfolded graph). The operation of folded graph is called **folding** that by identifying the edges which are incident with the vertex *v* and both of them labeled *x* or x^{-1} into single edge labeled *x* or x^{-1} respectively.

Lemma 2.1: If Γ is any connected non- folded *X* – Labeled graph, then the folded *X*- Labeled graph Γ' is also connected.

Proof: See [3].■

2.2. Group action on graphs

Let *G* be a group and Γ be a graph, then we say that G **acts** on Γ **, if** it acts on the sets of vertices $V(\Gamma)$ and edges $E(\Gamma)$, such that for any vertices u,u' in $V(\Gamma)$, edges e,e' in $E(\Gamma)$ of the graph Γ and for any g in *G*, then $g(u)=u'$, $g(e)=e'$. Moreover if *G* acts on a graph Γ, then we say that *G* acts on a graph Γ without inversions if $ge\neq\bar{e}$ for any g in *G* and e in $E(\Gamma)$, and we say that *G* acts on a graph Γ with inversions if $ge=\bar{e}$, for some g in *G* and some e in $E(\Gamma)$.

Now for any vertex *v* in $V(\Gamma)$ and any g in *G*, then we say that g **stabilize** the vertex *v* if $g(v)=v$. Therefore **the set of the stabilizers of the vertex *v*** is denoted by G_v. i.e. $G_v=\{g\in G; g(v)=v\}$. Also for any e in $E(\Gamma)$ and any g in *G*, then we say that g **stabilize** the edge *e* if $g(e)=e$. Therefore **the set of the stabilizers** of the edge *e* is denoted by G_e. i.e. $G_e=\{g\in G; g(e)=e\}$.

Lemma 2.2.1: G_v and G_e are subgroups of *G*.■

Now for any vertex *v* in $V(\Gamma)$ and any g in *G*, then we say that $g(v)$ is **an orbit** of the vertex *v*. Therefore **the set of orbits of the vertex** *v* is denoted by $G(v)$. i.e. $G(v)=\{g(v)\in\Gamma; g\in G\}$. Also for any e in $E(\Gamma)$ and any g in *G*, then we say that $g(e)$ is **an orbit** of the edge *e*. Therefore **the set of orbits** of the edge *e* is denoted by $G(e)$. i.e. $G(e)=\{g(e)\in\Gamma; g\in G\}$. Hence $G(v)$ and $G(e)$ are subsets of Γ.

2.3. Incidence matrices of *X*-labeled graphs.

In this section we will assume that all *X*- labeled graphs are without loops.

Let Γ be any *X* – Labeled graph (where $X=\{a,b\}$), then the **incidence matrix** of *X* – Labeled graph Γ **[1]** is an $n\times m$ incidence matrix $[x_{ij}]$, where $1\le i\le n, 1\le j\le m$) with x_{ij} entries such that

$$x_{ij}=\begin{cases} x \text{ if } v_i = i(e_j) \text{ and } e_j \text{ lables } x\in X \\ 0 \text{ if } v_i \text{ is not incident with } e_j \\ x^{-1} \text{ if } v_i = \tau(e_j) \text{ and } e_j \text{ labeles } x\in X \end{cases}$$

N.B. Incidence matrices of *X* – Labeled graphs Γ will be denoted by $M_X(\Gamma)$.

Now let $M_X(\Gamma)$ be an $n\times m$ incidence matrix $[x_{ij}]$ of *X* – Labeled graph Γ and let r_i and c_j be a row and a column in $M_X(\Gamma)$ respectively. If x_{ij} is a non – zero entry in the row r_i, then r_i is called an

incidence row with the column c_j at the non – zero entry $x_{ij} \in X \cup X^{-1}$ and if $x_{ij} \in X$, then the row r_i is called the **starting row** (denoted by $s(c_j)$) of the column c_j and the row r_i is called the **ending row** (denoted by $e(c_j)$) of the column c_j if $x_{ij} \in X^{-1}$. If the rows r_i and r_k are incidence with column c_j at the non – zero entries x_{ij} and x_{kj} respectively, then we say that the rows r_i and r_k are **adjacent**. If c_j and c_h are two distinct columns in $M_X(\Gamma)$ such that the row r_i is incidence with the columns c_j and c_h at the non – zero entries x_{ij} and x_{ih} respectively (where $x_{ij}, x_h \in X \cup X^{-1}$), then we say that c_j and c_h are **adjacent columns**. For each column c there is an inverse column denoted by $\bar{c}$ such that $s(\bar{c}) = e(c), e(\bar{c}) = s(c)$ and $\bar{\bar{c}} = c$. The degree of a row r_i of $M_X(\Gamma)$ is the number of the columns incidence to r_i and is denoted by $\deg(r_i)$. If the row r_i is incident with at least three distinct columns c_j , c_h and c_k at the non – zero entries x_{ij}, x_{ih} and x_{ik} respectively,(where x_{ij}, x_{ih} , $x_{ik} \in X \cup X^{-1}$), then the row r_i is called a **branch row**. If the row r_i is incident with only one column c_j at the non- zero entry $x_{ij} \in X \cup X^{-1}$ and all other entries of r_i are zero, then the row r_i is called **isolated row**.

A **scale** in $M_X(\Gamma)$ is a finite sequence of form $S = r_1, c_1^{\in_1}, r_2, c_2^{\in_2}, \ldots, r_{k-1}, c_{k-1}^{\in_{k-1}}, r_k$, where $k \geq 1$, $\in = \mp$, $s(c_j^{\in_j}) = r_j$, and $e(c_j^{\in_j}) = r_{j+1} = s(c_{j+1}), 1 \leq j \leq k$. The starting row of a scale $S = r_1, c_1^{\in_1}, r_2, c_2^{\in_2}, \ldots, r_{k-1}, c_{k-1}^{\in_{k-1}}, r_k$ is the starting row r_1 of the column c_1 and the **ending row** of the scale *S* is the ending row r_k of the column c_{k-1} and we say that *S* is a scale from r_1 to r_k and *S* is a scale of length *k* for $1 \leq j \leq k-2$. If $s(S) = e(S)$, then the scale is called **closed scale**. If the scale S is reduced and closed , then S is called a **circuit** or a **cycle.** If $M_X(\Gamma)$ has no cycle, then $M_X(\Gamma)$ is called a forest incidence matrix of X – Labeled graph Γ. Two rows r_i and r_k in $M_X(\Gamma)$ are called **connected** if there is a scale *S* in $M_X(\Gamma)$ containing r_i and r_k. Moreover $M_X(\Gamma)$ is called **connected** if any two rows r_i and r_k in $M_X(\Gamma)$ are connected by a scale *S*. If $M_X(\Gamma)$ is a connected and forest , then $M_X(\Gamma)$ is called a **tree** incidence matrix of X – Labeled graph Γ.

A **component** of $M_X(\Gamma)$ is a maximal connected **subincidence matrix** of $M_X(\Gamma)$.

If $M_X(\Omega)$ is a **subincidence matrix** of $M_X(\Gamma)$, and every two rows r_i and r_k in $M_X(\Gamma)$ are joined by at least one scale *S* in $M_X(\Omega)$, then $M_X(\Omega)$ is called **spanning incidence matrix** of $M_X(\Gamma)$ and $M_X(\Omega)$ is called **spanning tree** of $M_X(\Gamma)$ if $M_X(\Omega)$ is spanning and tree incidence matrix . The inverse of $M_X(\Gamma)$ is incidence matrix of X^{-1} - labeled graph Γ.

Lemma 2.3.1: If Γ is a connected *X*- Labeled graph, then $M_X(\Gamma)$ is a connected incident matrix of *X*-Labeled graph.

Proof: Since each row and column in $M_X(\Gamma)$ represent a vertex and an edge of Γ respectively, and each edge of Γ with labeled $x \in X$ joins two vertices , so each column in $M_X(\Gamma)$ joins two rows at the non-zero entries x, x^{-1} respectively. Hence $M_X(\Gamma)$ is a connected incident matrix of *X*-Labeled graph.■

Now Let Γ and Δ be *X*-labeled graphs, then the incidence matrix of the product of two *X*-Labeled graphs Γ and Δ is denoted by $M_X(\Gamma \tilde{\times} \Delta)$ with the set of rows $\{(u,v): u \in V(\Gamma), v \in V(\Delta)\}$ and set of columns $\{(e_i, e_j): e_i \in E(\Gamma), e_j \in E(\Delta)$ & e_i, e_j have the same labeled$\}$ with the non-zero entries x_{kt} as in the definition of incidence matrices of *X*-labeled graphs.

3. Group Actions on Incidence Matrices of *X*- labeled graphs

Let *G* be a group and *X* be a subset of the group *G*, Γ be a connected graph and $M_X(\Gamma)$ be the incidence matrix of *X*- labeled connected graph.

Note: Henceforth we assume that the *X*-labeled graph is connected graph, and then $M_X(\Gamma)$ will be connected.

We now construct a tree $M_X(T)$ incidence matrix of *X*-labeled graph, to let the group *G* acts on it, as below,

For any closed reduced scale S_i of $M_X(\Gamma)$, choose a column c_j for some *j,* and then split the ending row $e(r_i)$ of c_j into two rows r_i and r_i^*, such that the ending row of c_j is r_i^* with same labeled of c_j, and the starting row of column c_{j+1} in S_i is r_i with the same labeled of c_{j+1} . Therefore we get a tree incidence matrix of *X*-labeled graph

3.1. Definition. For any group *G* and any incidence matrices of *X*- labeled graph $M_X(\Gamma(T))$ we say that a **group *G* acts** on the **tree incidence matrix** $M_X(\Gamma(T))$ **of *X*- labeled graph** Γ**, if it acts on rows and columns of** $M_X(\Gamma(T))$ **compactly,** as below:

i) for any $g \in G$ and any row $r \in M_X(\Gamma(T))$, there exists a row $r' \in M_X(\Gamma(T))$, such that $gr = r'$.

ii) for any $g \in G$ and any column $c \in M_X(\Gamma((T))$, there exists a column $c' \in M_X(\Gamma(T))$, such that $gc = c'$. That means $g(s(c)) = s(c')$ and $g(e(c)) = e(c')$, and c, c' have the same labeled of non-zero entries $x \in X$.

Note: i) If $g(r_j) = r'$, then we write $r_j \sim r'$, for any rows r_j, r' in $M_X(\Gamma(T))$.

ii) If $g(c_i) = c'$, then we write $c_i \approx c'$, for any c_i, c' in $M_X(\Gamma(T))$.

Lemma 3.2: The relations $\sim$ and $\approx$ are equivalence relations.

Proof. i) Since $i_G(r) = r$, for any row *r* in $M_X(\Gamma(T))$ and i_G is the identity element of the group G, so $\sim$ is reflexive. If $g(r) = r'$, for any rows r, r' in $M_X(\Gamma(T))$ and some $g \in G$, then $r = g^{-1}(r')$, so $\sim$ is symmetric. Now for any rows r, r', r'' in $M_X(\Gamma(T))$ if $r \sim r'$, $r' \sim r''$, then there exist $g, g' \in G$, such that $g(r) = r', g'(r') = r''$, so $g'(g(r)) = g'g(r) = g'(r') = r''$. Therefore $\sim$ is transitive relation. Hence $\sim$ is an equivalence relation on rows of $M_X(\Gamma(T))$.■

ii) **Proof**: Since $i_G(c)=i_G(s(c),e(c))=(i_G(r),i_G(r'))=(r,r')=c$, for any column c in $M_X(\Gamma(T))$, so $\approx$ is reflexive. If $g(c)=c'$ for any columns $c,c'\in M_X(\Gamma(T))$,so $g(r_j,r_t)=(r',r'')$. Therefore $(r_j,r_t)=g^{-1}(r',r'')$, $c=g^{-1}(c'')$ and then $\approx$ is symmetric. Now for any columns c,c',c'' in $M_X(\Gamma(T))$, such that $c\approx c',c'\approx c''$, then $g(c)=c'$,
$g'(c')=c''$, for some $g,g'\in G$. Therefore $g'(g(c))=c'',g'g(c)=c''$ and then $\approx$ is transitive. Hence $\approx$ is an equivalence relation.■

Definition 3.3: The stabilizer of the row *r* is denoted by G_r and define by $G_r=\{g;g\in G,gr=r\}$. Also denote the stabilizer of the column *c* by G_c and define by $G_c=\{g;g\in G,gc=c\}$.

Lemma 3.4: The stabilizers of the rows *r* and the columns *c* in $M_X(\Gamma(T))$ are subgroups of *G*.

Proof: Since the identity element i_G of G, stabilize any row *r* or any column *c*, so $G_r(M_X(\Gamma(T))$ or $G_c(M_X(\Gamma))$ are non-empty sets.

Now For g and g' are elements in $G_r(M_X(\Gamma(T))$ and $G_c(M_X(\Gamma(T))$, so $g'g$ is an element in $G_r(M_X(\Gamma(T))$ and $G_c(M_X(\Gamma(T))$. Therefore the stabilizers of the rows and the columns are subgroups of G, because $g'g(r)=g'(g(r)=g'(r)=r$, for any row *r* in $M_X(\Gamma(T))$. Similarly for any column c in $M_X(\Gamma(T))$. Also if g is in stabilizers of rows or columns, so g^{-1} is an element in the stabilizer of row or column of $M_X(\Gamma(T))$.■

4. Incidence Matrix of Directed Graph of finite groups

In this section we will construct the incidence matrix of X- labeled graph of groups which is equivalent to the incidence matrix of directed graph of groups in [2].

Definition 4.1[2]: An incidence matrix of directed graph of finite groups consists of an incidence matrix of *X*- labeled graph with a spanning tree matrix of X-labeled graph $M_X(T)$, and a base row $r^*=r_1$, together with a finite group G_r for each row r, and a finite group G_c for each column c, such that :

1) The columns of $M_X(\Gamma)$ are directed away from $r^*=r_1$;

2) Each column group G_c is a subgroup of $G_{s(c)}$;

3) Each column group G_c is embedded in $G_{e(c)}$ by a fixed monomorphism ψ_c, defined by $\psi_c(g)=l_c^{-1}gl_c$, $g\in G_c$, and $l_c=s(c)$ is the non-zero entrance of c in $M_X(\Gamma)/M_X(T)$.It is denoted by $(G_r,G_c,l_c,M_X(T),M_X(W),r^*,\psi_c)$, $l_c=1$ if $l_c\in M_X(T)$ and $l_c\neq 1$, if $l_c\in M_X(\Gamma)/M_X(T)$.

Definition 4.2: Let *G* be a group acts on the incidence matrices of *X*- labeled graph.

A subtree incidence matrix $M_X(\Gamma(T'))$ of a tree incidence matrix of X- labeled graph $M_X(\Gamma([T]))$. Therefore $M_X(\Gamma(T'))$ is called a tree of representative for the action of *G* on $M_X(\Gamma(T))$, if $M_X(\Gamma(T'))$ contains exactly one row from each row.

Lemma 4.3: Let *G* be a group acting on an incidence matrix of *X*-labeled graph. If there exists a row r_i , $i>1$, not in the row orbit of r_1, then there exists r_2 not in the row orbit of r_1.

Proof: Suppose that all rows of orbit r_2 are in the base orbit r_1 and there exists a row r_i not in the row orbit of r_1. Now choose the smallest i, such that the orbit r_i is not in the row orbit r_1. Therefore, the row orbit r_{i-1} is in the row orbit of r_1, i.e. $g(r_{i-1})=r_1$ say, and then $g'(r_i)=r_2$, for some $g'\in G$. Since by assuming that the row orbit r_2 is in the row orbit of the row orbit r_1, so $g''(r_i)=r_1$, i.e. the row orbit r_i is in the row orbit of r_1 which is a contradiction. Hence r_2 and r_1 are in different orbits.■

Lemma 4.4: If *h* and *f* are two columns of the same column orbit, then the starting rows $s(h)$ and $s(f)$ are of the same row orbits. Also the ending rows $e(h)$ and $e(f)$ are of the same row orbit.

Proof: Since the columns *h* and *f* are of the same column orbit, so there exists some g in *G*, such that $g(h)=f$. Since $g(s(h))=s(gh)$ and $g(e(h))=e(g(h))$, so $g(s(h))=s(f)$ and $g(e(h))=e(f)$.■

Lemma 4.5: If $s(h)$ and r_i ,(for some column *h* and row r_i in $M_X(\Gamma(T))$) are of the same row orbit, then there exists a column *f* for some column f in $M_X(\Gamma(T))$, such that *h* and *f* are of the same column orbit and $s(f)=r_i$.

Proof: Since $s(h)$ and r_i are of the same row orbit, so there exists some g in *G*, such that $g(s(h))=r_i$. Now let $f=g(h)$, then $r_i=g(s(h))=s(g(h))=s(f)$, for some column $f\in M_X(\Gamma(T))$.■

Lemma 4.6: Let *G* be a group acting on a tree incidence matrix of *X*- labeled graph, then a tree of representatives can always be exist.

Proof: Let us assume, there is more than one row orbit.

Therefore by assuming lemma 4.2, let r_1 be the base row and r_2 be of a different row orbits. Then there exists a column between them.

Now let *W* be the set of all subtree incidence matrix $M_X(\Gamma(T'))$ of the tree incidence matrix of *X*-labeled graph $M_X(\Gamma(T))$ containing the base row r_1, such that $M_X(\Gamma(T'))$ satisfies the following conditions;

1) All rows are of different row orbits;

2) i is minimal in the row orbit of r_i if case of r_i in $M_X(\Gamma(T'))$;

3) if r_i in $M_X(\Gamma(T'))$ and $j<i$, then every r_j is in the row orbit of some row of $M_X(\Gamma(T'))$. Since *W* is not empty set, because $M_X(\Gamma(T_1'))$ containing r_1 only is in *W*. Since *W* is a partially ordered by inclusion.

Now let $\{M_X(\Gamma(T_i'))\}$ be totally ordered subset of *W*. Therefore we show that there exists $M_X(\Gamma(T''))$ in *W*, $M_X(\Gamma(T''))\supseteq M_X(\Gamma(T_i'))$. Let $M_X(\Gamma(T''))=\cup M_X(\Gamma(T_i'))$

If $M_X(\Gamma(T''))$ is not a tree Incidence matrices of *X*- labeled graph, then there exists a finite reduced closed scale in $M_X(\Gamma(T''))$. Since $M_X(\Gamma(T''))=\cup M_X(\Gamma(T_i'))$, so there is a reduced closed scale in

$M_X(\Gamma(T_i'))$, for some *i* a contradiction. Therefore $M_X(\Gamma(T''))$ is a tree. Now suppose there exist r_i and r_j in $M_X(\Gamma(T''))$, such that $g(r_j) = r_i$. Since r_j in $M_X(\Gamma(T_j'))$ and r_i in $M_X(\Gamma(T_i'))$, $M_X(\Gamma(T_j'))$ is a subtree of $M_X(\Gamma(T_i'))$, so r_i and r_j in $M_X(\Gamma(T_i'))$ a contradiction. Therefore (1) holds.

Now let r_j in $M_X(\Gamma(T''))$, then r_j in $M_X(\Gamma(T_i'))$, for some i. Since (2) and (3) hold for $M_X(\Gamma(T_i'))$, so they also hold for $M_X(\Gamma(T''))$, i.e. $M_X(\Gamma(T''))$ is in *W*.

Hence by Zorn's Lemma, there exists a maximal $M_X(\Gamma(T'))$ in *W*. Now suppose that $M_X(\Gamma(T'))$ has set of rows $\{r_k\}$.

We now show that $M_X(\Gamma(T'))$ contain exactly one row from each row orbit.

Suppose not, so there exists r_j not in the orbit of any row orbit in $M_X(\Gamma(T'))$

Chosen such that j is as small as possible. Therefore $j \neq 0$ and by (2) $r_i \geq r_j$ by minimality of j, there exists a *g* in *G*, such that $g(r_{i-1}) = r_k$ in $M_X(\Gamma(T'))$.

Now $k \leq i-1$ by (b) and r_{i-1} is adjacent by a column to r_i, let c be the column joining r_{i-1} to r_i. Now let $r_t = g(r_i)$, then r_t is adjacent to r_k.

Let $M_X(\Gamma(T'''))$ be the tree incidence matrices of *X*-labeled graph consisting of $M_X(\Gamma(T'))$ with $g(c)$ and $g(r_i)$. Therefore (1) holds for r_i in $M_X(\Gamma(T'''))$.

However $g(r_i)$ is adjacent to r_k, so $g(r_i) = r_{k+1} = r_t$.

Therefore $t = k+1$. From above , we have $k+1 \leq i$ and so $t \leq i$. Since *i* is minimal , so $i \leq t$. Hence $i = t$, and by minimality of *i* , (2) holds for $M_X(\Gamma(T'''))$.

Now if $j < t$, then by the choice of *i*, r_j is in the row orbit of some row in $M_X(\Gamma(T'))$. Thus (3) holds. Therefore we have a larger subtree which satisfies (1), (2) and (3) a contradiction . Hence $M_X(\Gamma(T'))$ satisfies (1), (2) and (3). Therefore a tree incidence matrix of *X*- labeled graph of representatives always exists.■

Definition 4.7: A subtree incidence matrices of *X*-labeled graph $M_X(\Gamma(F))$ of $M_X(\Gamma(T))$ containing a tree incidence matrices of *X*-labeled graph of representatives $M_X(\Gamma(T'))$ (say) is called a **fundamental domain** for the action of *G* on $M_X(\Gamma(T))$, if each column in $M_X(\Gamma(F))$ has at least one end in $M_X(\Gamma(T'))$ and $M_X(\Gamma(F))$ contains exactly one column from each column orbit under *G*.

Lemma 4.8: Let *G* be a group acting on a tree incidence of *X*-labeled graph $M_X(\Gamma(T))$**,** then there is at most one column of $M_X(\Gamma(T'))$ in each column orbit.

Proof: Let c be a column in $M_X(\Gamma(T'))$, such that $s(c) = r_k$ **and** $e(c) = r_2$ and let c' be a column in $M_X(\Gamma(T'))$, such that $s(c') = r_m$, $e(c') = r_t$**.**

Now if c and c' are in same column orbit, then there exists a g in G, such $g(c)=c'$. Therefore $g(r_k)=r_m$ and $g(r_2)=r_t$. Since $M_X(\Gamma(T'))$ contains at most one row from each row orbit, so $r_k=r_m$ and hence $c'=c$. That mean, we have only one column of $M_X(\Gamma(T'))$ in each column orbit.

Lemma 4.9: Let G be a group acting on a tree incidence matrix of X-labeled graph $M_X(\Gamma(T))$ with $M_X(\Gamma(T'))$ a tree incidence matrix of X-labeled graph of representatives, then a fundamental domain of incidence matrix of X-labeled graph containing $M_X(\Gamma(T'))$ always exists.

Proof: Let **V** = $\{M_X(\Gamma(F_i))\}$ be the set of all subtrees incidence matrix of X-labeled graph $M_X(\Gamma(F))$ of $M_X(\Gamma(T))$ containing the chosen tree incidence matrix of X-labeled graph of representatives $M_X(\Gamma(T'))$, such that each c in $M_X(\Gamma(F))$ has at least one end in $M_X(\Gamma(T'))$ and $M_X(\Gamma(F))$ contains at most one column orbit c from each column orbit. Since $M_X(\Gamma(T'))$ contains at most one column orbit, so
$M_X(\Gamma(T'))$ in V. Now for any $M_X(\Gamma(F_i))$ and $M_X(\Gamma(F_{i+1}))$ in $M_X(\Gamma(F))$, and let $M_X(\Gamma(F_i))\subseteq M_X(\Gamma(F_{i+1}))$, so V be partially ordered set by inclusion.

Therefore let $M_X(\Gamma(F_i))$ be totally ordered subset of V= $\{M_X(\Gamma(F_i))\}$.

Now we show that there exists $M_X(\Gamma(F'))$ in V, such that $M_X(\Gamma(F'))\supseteq M_X(\Gamma(F_i))$.

Therefore let $M_X(\Gamma(F'))=\cup\{M_X(\Gamma(F_i))\}$. If it is not a tree incidence Matrix of X-labeled graph, then there exists a finite reduced closed scale in $M_X(\Gamma(F_i))$ for some i a contradiction . Hence $M_X(\Gamma(F'))$ is a tree incidence Matrix of X-labeled graph.

Now let c and c' in $M_X(\Gamma(F'))$, Since c in $M_X(\Gamma(F_i))$ **and** c' in $M_X(\Gamma(F_j))$. But $M_X(\Gamma(F_i))\subseteq M_X(\Gamma(F_j))$, and $M_X(\Gamma(F_i)), M_X(\Gamma(F_j))$ are in V, then c and c' are in different column orbit. Since each column orbit has at least one end in $M_X(\Gamma(T'))$, so $M_X(\Gamma(F'))$ in V. Now by Zorn's lemma, there exists a maximal $M_X(\Gamma(F))$ in V. It remains to show that $M_X(\Gamma(F'))$ contains exactly one column from each column orbit. Suppose not, so there is a column c'' in $M_X(\Gamma(F'))$ and there is no g in G, such that $g(c'')=c$ for any c in $M_X(\Gamma(F'))$. Since $R(M_X(\Gamma(T')))\cap s(c'')\neq\phi$, so the row orbit of $s(c'')$ has one row r_m say in $R(M_X(\Gamma(T')))$. By lemma 4.3, there is a column c''' in $C(M_X(\Gamma(T)))$, such that $g(c'')=c'''$, for some g in G, with $s(c''')=r_m$ in $M_X(\Gamma(T'))$. Also $g(e(c''))=e(c''')$ .Therefore We have a larger subtree incidence Matrices of X-labeled graph $M_X(\Gamma(\overline{F}))$, with
$R(M_X(\Gamma(\overline{F})))=R(M_X(\Gamma(F)))\cup\{e(c''')\}$ and
$C(M_X(\Gamma(\overline{F})))=C(M_X(\Gamma(F)))\cup\{c''',\overline{c'''}\}$, such that each column in $M_X(\Gamma(F))$ has

at least one end in $M_X(\Gamma(T'))$ and $M_X(\Gamma(\overline{F}))$ contains at most one column from each column orbit, a contradiction to the maximality of $M_X(\Gamma(F))$. Therefore a fundamental domain of a tree incidence matrix of *X*-labeled graph always exists.■

We now use $M_X(\Gamma(F))$ to construct a directed graph of groups of incidence matrix of *X*-labeled graph, that by changing the direction of all columns in $M_X(\Gamma(T'))$ to be a way from the base row r_1, and then dentified the rows of the same row orbits in $M_X(\Gamma(F))$, we have a directed incidence matrix of *X*- labeled graph $M_X(\Gamma(\vec{F}))$, rows corresponding to rows of $M_X(\Gamma(T'))$ and columns to columns of $M_X(\Gamma(F))$. Let the row groups of $M_X(\Gamma(\vec{F}))$ be the stabilizers of the row groups G_r in $M_X(\Gamma(T'))$ and the columns groups G_c be the stabilizers of the columns *c* in $M_X(\Gamma(F))$.

Now we define a map λ_c be $x \mapsto l_c^{-1} x l_c$, for $x \in G_c$, $l_c = 1$ if $c \in M_X(\Gamma(T'))$ and $l_c(r_m) = e(c)$ if $c \in M_X(\Gamma(F)) \setminus M_X(\Gamma(T'))$, $r_m \in M_X(\Gamma(T'))$.

Lemma 4.10: $\lambda_c(x)$ is in G_r, where λ_c is defined as above and $x \in G_{r_t}$.

Proof: Since G_c is a subset of $G_{e(c)}$, so $x(r_t) = r_t = x l_c(r_m) = l_c(r_m)$, and then $l_c^{-1} x l_c r_m = r_m$. Hence $\lambda_c(x) = l_c^{-1} x l_c$, and in G_{r_m}.

Since we identified the rows of same orbit, r_m in $M_X(\Gamma(T'))$ and $e(c)$ in $M_X(\Gamma(F)) \setminus M_X(\Gamma(T'))$ in forming $M_X(\Gamma(\vec{F}))$, so $e(c)$ corresponding to a row r_m in $M_X(\Gamma(\vec{F}))$. Since λ_c is a map from G_c in to $G_{e(c)}$, where c in $M_X(\Gamma(\vec{F}))$. So λ_c is a monomorphism. Hence, we have a directed incidence matrix of *X*-labeled graph of groups $(G_r, G_c, l_c, M_X(\Gamma(T')), M_X(\Gamma(F)))$.■

5. Conclusion

In this work, we show that Bass- Serre theory can apply on Incidence Matrix of *X*- labeled graph and we have the same results with computer Program as in [1], [2], [3], [4] and this work.

6. Example

In this example we will construct the directed incidence matrix of graph of groups of *X*-labeled graph which the same as in [2]. In the following example we will define an action on incidence matrix of *X*-labeled graph and construct the directed incidence matrix of graph of finite groups of *X*-labeled graph.

Let $\Gamma^*(H)$ be a core graph of finitely generated subgroup *H* of free group *F* generated by $X = \{a, b\}$.

$\Gamma^*(H)$:

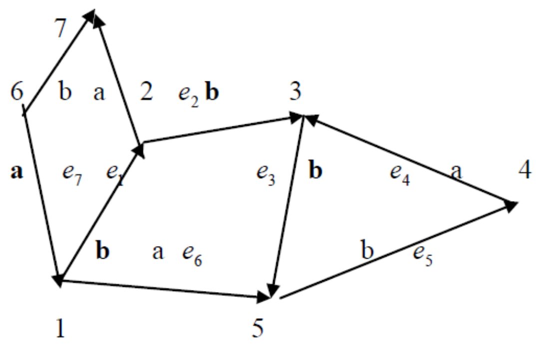

Figure 1

Therefore the incidence Matrix of core graph of the above graph is as follows:

$$M_X(\Gamma^*(H)):$$

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
r_1	b	0	0	0	0	a	a^{-1}	0	0
r_2	b^{-1}	b	0	0	0	0	o	0	a
r_3	0	b^{-1}	b	a^{-1}	0	0	0	0	0
r_4	0	0	0	a	b^{-1}	0	0	0	0
r_5	0	0	b^{-1}	0	b	a^{-1}	0	0	0
r_6	0	0	0	0	0	0	a	b	0
r_7	0	0	0	0	0	0	0	b^{-1}	a^{-1}

Figure 2

Since $M_X(\Gamma^*(H))$ is a connected incidence matrix of core graph and columns c_2 and c_3 are adjacent by the row r_3, so split the row r_3 into two rows r_3, r_3', such that $e(c_2)=r_3'$ and $s(c_3)$, by similarity for rows r_4 and r_7, so we have, $e(c_5)=r_4'$,

$s(c_4)=r_4$, and $e(c_8)=r_7'$, $e(c_9)=r_7$ .Therefore we get a tree incidence matrix of core graph which is $M_X(T\,(H))$ as below:

$M_X(T\,(H))$:

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
r_1	b	0	0	0	0	a	a^{-1}	0	0
r_2	b^{-1}	b	0	0	0	0	0	0	a
r_3	0	0	b	a^{-1}	0	0	0	0	0
r_3'	0	b^{-1}	0	0	0	0	0	0	0
r_4	0	0	0	a	0	0	0	0	0
r_4'	0	0	0	0	b^{-1}	0	0	0	0
r_5	0	0	b^{-1}	0	b	a^{-1}	0	0	0
r_6	0	0	0	0	0	0	a	b	0
r_7	0	0	0	0	0	0	0	0	a^{-1}
r_7'	0	0	0	0	0	o	0	b^{-1}	0

Figure 3

The tree representative of incidence matrix of X-labeled graph is $M_X(T'(H))$:

	c_1	c_3	c_4	c_6	c_7	c_9
r_1	b	0	0	a	a^{-1}	0
r_2	b^{-1}	0	0	0	0	a
r_3	0	b	a^{-1}	0	0	0
r_4	0	0	a	0	0	0
r_5	0	b^{-1}	0	a	a^{-1}	0
r_6	0	0	0	0	a	0
r_7	0	0	0	0	0	a

Figure 4

The fundament domain of incidence matrix of *X*-labeled graph is $M_X(F(H))$:

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
r_1	b	0	0	0	0	a	a^{-1}	0	0
r_2	b^{-1}	b	0	0	0	0	0	0	a
r_3	0	0	b	a^{-1}	0	0	0	0	0
r_3'	0	b^{-1}	0	0	0	0	0	0	0
r_4	0	0	0	a	0	0	0	0	0
r_4'	0	0	0	0	b^{-1}	0	0	o	0
r_5	0	0	b^{-1}	0	b	a^{-1}	0	0	0
r_6	0	0	0	0	0	0	a	b	0
r_7	0	0	0	0	0	0	0	0	a^{-1}
r_7'	0	0	0	0	0	0	0	b^{-1}	0

Figure 5

By the action of H on $M_X(F(H))$, we get that $h'r'_7 = r_7 = h_1r_7$, $h''r'_3 = r_3 = h_2r_3$ and $h'''r'_4 = r_4 = h_3r_4$, because $a^{-1}ba^{-1}b^{-1} \in H$, $bbba^{-1} \in H$ and $abab^{-1}a^{-1} \in H$ are reduced closed scales. Also the stabilizers of the rows and the columns are the trivial subgroups of *H*.

$M_X(H_r, H_c, M_X(T'(H)), l_X)$:

		$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
		c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
$\{1\}$	r_1	b	0	0	0	0	a	a^{-1}	0	0
$\{1\}$	r_2	b^{-1}	b	0	0	0	0	0	0	a
$\{1\}$	r_3	0	b^{-1}	b	a^{-1}	0	0	0	0	0
$\{1\}$	r_4	0	0	0	a	b^{-1}	0	0	0	0
$\{1\}$	r_5	0	0	b^{-1}	0	b	a^{-1}	0	0	0
$\{1\}$	r_6	0	0	0	0	0	0	a	b	0
$\{1\}$	r_7	0	0	0	0	0	0	0	b^{-1}	a^{-1}

Figure 6

References

[1]. W. S. Jassim; "Incidence matrices of *X*-labeled graphs and an application", SQU Journal for Science, 14(1):61-69, 2009.

[2]. -------------; "Incidence matrices of directed graph of groups and their up-down pregroups", SQU Journal for Science, 2017,22(1), 40 – 47.

[3]. W.S. Jassim & M.M. Farman; "On incidence matrix of *X*- labeled graph", JUSEJ, June 2018,Vol. 05, No. 01, 18-28.

[4]. W.S. Jassim;" New applications of Incidence Matrices of X-labeled graphs and it's absorbs", accepted for publishing in JMCS, Jun 2019.

[5]. --------------;"Directed Graphs of Groups and their Up-Down Pregroups", Faculty of Science Bulletin, 17 (2004)137-154, Sana'a University.

[6]. J.P. Serre; " Tree ", Book, Published by Springer –Verlag, Berlin, Heidelberg, Newyork, 1980

[7]. R.C. Lyndon and P.E. Schupp; "Combinatorial group theory", Engebniss, Vol. 89, Berlin-Heidelberg, New-york, Springer, 1977.