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Research Article

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An inequality on Fischer-type Determinant for accretive-dissipative matrices

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Abstract In this paper, we study the Fisher-type determinant inequality for accretive-dissipative matrix. Inspired by Ikramov and Lin, as a result, a new inequality about Fischer-type determinant inequality of accretive-dissipative matrix is given, at the same time, we extend the corresponding result.

Keywords Fischer-type Determinant; accretive-dissipative matrix; inequality

1. Introduction

Let I_n be an $n \times n$ unit matrix, $M^{m \times n}$ be the set of $m \times n$ complex matrices, if A be a $n \times n$ matrix, denote the eigenvalues of A by $\lambda_i(A)$, the conjugate transpose of A by A^H, A is called positive semi-definite matrix if A be a Herimite matrix and $\lambda_i(A) > 0.$ For any $A \in M^{n \times n}$, the following decomposition is called the Herimite decomposition of A, if A can be uniquely decomposed into:

$$A = B + iC. \tag{1.1}$$

where

$$B = \frac{A + A^H}{2}, \quad C = \frac{A - A^H}{2i},$$

and B, C are Herimite matrices. B , C are called the real part and the imaginary part of A respectively. For simplicity, (1.1) is always expressed in blocks as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} + i \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \tag{1.2}$$

say $k, l\left(k > 0, l > 0, k + l = n\right)$ the order of A_{11} and A_{22}, respectively, and let $m = \min\{k, l\}$. Since B , C are Herimite matrices, then

$B_{12} = B_{21}{}^H$, $C_{12} = C_{21}{}^H$.

If $B = I_n$ in (1.1), then A is called Buckley matrix. If B in (1.1) is a positive semi-definite matrix, then A is called accretive matrix. If C in (1.1) is a positive semi-definite matrix, then A is called dissipative matrix. If B, C in (1.1) are positive semi-definite matrices, then A is called accretive-dissipative matrix. In this definition, A is a dissipative matrix if and only if $-iA$ is a accretive matrix.

Let $A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M^{n\times n}$, if A_{11} is invertible, then the Schur complement of A_{11} in A is denoted by $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$. For a nonsingular matrix A, the condition number of A is denoted by

$$\kappa(A)=\sqrt{\frac{\lambda_{\max}(A^H A)}{\lambda_{\min}(A^H A)}},$$

which is the ratio of the largest and the smallest singular value of A.

In 1985, Horn and Johnson ([1], p478) gave the famous Fischer-type determinantal inequality

$$\det A \le \det A_{11} \det A_{22}. \tag{1.3}$$

In 2004, Ikramov [2] first proved the determinanatal inequality of accretive-dissipative matrix

$$|\det A| \le 3^m |\det A_{11}||\det A_{22}|. \tag{1.4}$$

In 2013, if $A\in M^{n\times n}$ is accretive-dissipative, Lin [3] got the result

$$|\det A| \le 2^{\frac{3}{2}m} |\det A_{11}||\det A_{22}|. \tag{1.5}$$

Fu and He [4] got a stronger result than (1.5):

$$|\det A| \le 2^{\frac{1}{2}m}\left(1+\left(\frac{1-\kappa}{1+\kappa}\right)^2\right)^m |\det A_{11}||\det A_{22}|. \tag{1.6}$$

where $\kappa = \max\{\kappa(\mathrm{A}), \kappa(\mathrm{B})\}$.

The purpose of this paper is to use the existing conclusions to present a new inequality on Fischer-type Determinant for accretive-dissipative matrix, and extend the result of Fu and He.

2. Main Result

In this section, we begin with the following lemmas.

Lemma 2.1[5] Let $A\in M^{n\times n}$ be accretive-dissipative as in (1.2), then A/A_{11} is also accretive –disspative.

Lemma 2.2[2] Let $A\in M^{n\times n}$ be accretive-dissipative as in (1.2), then

$$A^{-1} = E - iF,$$

where

$$E=\left(B+CB^{-1}C\right)^{-1},\quad F=\left(C+BC^{-1}B\right)^{-1}.$$

Lemma 2.3[2] Let $A = I_n + iC$ be Buckley matrix, then

$$A^{-1}=E+iF,\ E=\left(\mathrm{I}_n+C^2\right)^{-1},\ F=-\left(C+C^{-1}\right)^{-1}=-C\left(\mathrm{I}_n+C^2\right)^{-1}.$$

Lemma 2.4[5] Let $A\in M^{n\times n}$ be Herimite matrix and $B>0$, then

$$B+CB^{-1}C \ge 2C.$$

Lemma **2.5**[3] Let $B, C\in M^{n\times n}$ be positive semi-definite, then

$$\left|\det\left(B+iC\right)\right| \le \det\left(B+C\right) \le 2^{\frac{n}{2}}\left|\det\left(B+iC\right)\right|.$$

Lemma **2.6**[6] Let $A\in M^{n\times n}$ be positive definite, then

$$A_{12}A_{22}^{-1}A_{21} \le \left(\frac{\lambda_1-\lambda_n}{\lambda_1+\lambda_n}\right)^2 A_{11},$$

where λ_1 and λ_n are the maximum eigenvalue and minimum eigenvalue of A ,respectively.

Now, we are ready to give our result.

Theorem 1 Let $A \in M^{n\times n}$ be accretive-dissipative as in (1.1), then

$$|\det A| \le \left(2x^2+2y^2\right)^{\frac{1}{2}m}\left(1+\left(\frac{1-\kappa}{1+\kappa}\right)^2\right)^m |\det A_{11}|\left|\det\left(\frac{B_{22}}{x}+i\frac{C_{22}}{y}\right)\right|. \quad (2.1)$$

where $\kappa=\max\{\kappa(\mathrm{B}),\kappa(\mathrm{C})\}$ and $x, y \in R$ are positive.

Proof. By Lemmas 2.1-2.2, we have

$$\begin{aligned} A/A_{11} &= A_{22}-A_{21}A_{11}^{-1}A_{12} \\ &= B_{22}+iC_{22}-\left(B_{12}^H+iC_{12}^H\right)\left(B_{11}+iC_{11}\right)^{-1}\left(B_{12}+iC_{12}\right) \\ &= B_{22}+iC_{22}-\left(B_{12}^H+iC_{12}^H\right)\left(E_k-iF_k\right)\left(B_{12}+iC_{12}\right). \end{aligned}$$

Obviously,

$$E_k=\left(B_{11}+C_{11}B_{11}^{-1}C_{11}\right)^{-1},\ F_k=\left(C_{11}+B_{11}C_{11}^{-1}B_{11}\right)^{-1},$$

where E_k and F_k are positive definite.

By Lemma 2.4, and the monotonicity of invertible operator inverse, we obtain

$$E_k \le \frac{1}{2}C_{11}^{-1}, \quad F_k \le \frac{1}{2}B_{11}^{-1}. \quad (2.2)$$

Let $A/A_{11}=R+iS$, $R=R^H$, $S=S^H$, by lemma 2.1, R and S are positive definite, by computation, we get

$$R=B_{22}-B_{12}^HE_kB_{12}+C_{12}^HE_kC_{12}-B_{12}^HF_kC_{12}-C_{12}^HF_kB_{12},$$
$$S=C_{22}+B_{12}^HF_kB_{12}-C_{12}^HF_kC_{12}-C_{12}^HE_kB_{12}-B_{12}^HE_kC_{12}.$$

Since

$$\left(B_{12}^H \pm C_{12}^H\right)F_k\left(B_{12}\pm C_{12}\right)\ge 0, \quad \left(B_{12}^H \pm C_{12}^H\right)E_k\left(B_{12}\pm C_{12}\right)\ge 0,$$

it follows that

$$\pm\left(B_{12}^HF_kC_{12}+C_{12}^HF_kB_{12}\right)\le B_{12}^HF_kB_{12}+C_{12}^HF_kC_{12},$$
$$\pm\left(C_{12}^HE_kB_{12}+B_{12}^HE_kC_{12}\right)\le B_{12}^HE_kB_{12}+C_{12}^HE_kC_{12}.$$

Hence

$$R+S \le B_{22}+2B_{12}^HF_kB_{12}+C_{22}+2C_{12}^HE_kC_{12}. \quad (2.3)$$

Recall B , C are positive definite, by lemma 2.6, we have

$$B_{12}B_{22}^{-1}B_{12}^H \le \left(\frac{\lambda_1-\lambda_n}{\lambda_1+\lambda_n}\right)^2 B_{11}, \quad C_{12}C_{22}^{-1}C_{12}^H \le \left(\frac{\lambda_1'-\lambda_n'}{\lambda_1'+\lambda_n'}\right)^2 C_{11}. \quad (2.4)$$

where λ_1 and λ_n (λ_1' and λ_n') are the maximum eigenvalue and minimum eigenvalue of B (C).

Then

$$B_{12}^H B_{11}^{-1} B_{12} \le \left(\frac{\lambda_1-\lambda_n}{\lambda_1+\lambda_n}\right)^2 B_{22}, \quad C_{12}^H C_{11}^{-1} C_{12} \le \left(\frac{\lambda_1'-\lambda_n'}{\lambda'+\lambda_n'}\right)^2 C_{22}. \qquad (2.5)$$

Let $f(z)=\left(\frac{z-1}{z+1}\right)^m (m\ge 1)$, $f(z)$ is increasing on $[1,\infty)$. In general, we let $m=l$, λ_j, $j=1,\cdots,n$ are the eigenvalues of $B_{22}^{-\frac{1}{2}} C_{22} B_{22}^{-\frac{1}{2}}$, and $B_{22}^{\frac{1}{2}}$ is the unique positive definite square root of B_{22}, then, we get

$$|1+i\lambda_j| \le |1+\lambda_j| \le \sqrt{x^2+y^2}\left|\frac{1}{x}+i\frac{\lambda_j}{y}\right|. \qquad (2.6)$$

Set

$$\rho = 2^{\frac{1}{2}m}\left(1+\left(\frac{1-\kappa}{1+\kappa}\right)^2\right)^m,$$

by lemma 2.5 and (2.4), we have

$$\begin{aligned}
\left|\det\frac{A}{A_{11}}\right| &= |\det R+iS| \le \det(R+S) \le \det\left(B_{22}+2B_{12}^H F_k B_{12}+C_{22}+2C_{12}^H E_k C_{12}\right)\\
&\le \det\left(B_{22}+B_{12}^H B_{11}^{-1}B_{12}+C_{22}+C_{12}^H C_{11}^{-1}C_{12}\right)\\
&\le \det\left(B_{22}+C_{22}+\left(\frac{\lambda_1-\lambda_n}{\lambda_1+\lambda_n}\right)^2 B_{22}+\left(\frac{\lambda_1'-\lambda_n'}{\lambda_1'+\lambda_n'}\right)^2 C_{22}\right)\\
&= \det\left(B_{22}+C_{22}+\left(\frac{\frac{\lambda_1}{\lambda_n}-1}{\frac{\lambda_1}{\lambda_n}+1}\right)^2 B_{22}+\left(\frac{\frac{\lambda_1'}{\lambda_n'}-1}{\frac{\lambda_1'}{\lambda_n'}+1}\right)^2 C_{22}\right)\\
&\le \left(1+\left(\frac{\kappa-1}{\kappa+1}\right)^2\right)^m \det(B_{22}+C_{22}) \le \rho\left|\det(B_{22}+iC_{22})\right| \le \rho\left|\det(B_{22}+C_{22})\right|\\
&= \rho\left|\det B_{22}^{\frac{1}{2}}\left(I+B_{22}^{-\frac{1}{2}}C_{22}B_{22}^{-\frac{1}{2}}\right)B_{22}^{\frac{1}{2}}\right|\\
&= \rho\left|\det B_{22}^{\frac{1}{2}}\right|\left|\det B_{22}^{\frac{1}{2}}\right|\left|\det\left(I+B_{22}^{-\frac{1}{2}}C_{22}B_{22}^{-\frac{1}{2}}\right)\right|\\
&= \rho|\det B_{22}|\prod_{j=1}^{m}|1+\lambda_j|\\
&\le \rho|\det B_{22}|\prod_{j=1}^{m}\sqrt{x^2+y^2}\left|\frac{1}{x}+i\frac{\lambda_j}{y}\right|\\
&= \rho\left(x^2+y^2\right)^{\frac{m}{2}}|\det B_{22}|\left|\det\left(\frac{I}{x}+\frac{i}{y}B_{22}^{-\frac{1}{2}}C_{22}B_{22}^{-\frac{1}{2}}\right)\right|\\
&= \rho\left(x^2+y^2\right)^{\frac{m}{2}}\left|\det\left(\frac{B_{22}}{x}+i\frac{C_{22}}{y}\right)\right|.
\end{aligned}$$

where

$$\kappa = \max\left(\frac{\lambda_1}{\lambda_n}, \frac{\lambda_1'}{\lambda_n'}\right) \geq 1,$$

it is the maxium condition number of B , C . Since

$$|\det A| = |\det A_{11}|\left|\det \frac{A}{A_{11}}\right|,$$

then

$$|\det A| \leq \left(2x^2 + 2y^2\right)^{\frac{1}{2}m} \left(1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right)^m |\det A_{11}|\left|\det\left(\frac{B_{22}}{x} + i\frac{C_{22}}{y}\right)\right|.$$

Therefor, we complete Theorem 1.

Remark 1 It is clear that inequality (2.1) is an extension of inequality (1.6).

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