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Research Article

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Stability of stochastic differential systems with Lévy noise

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Abstract In this paper, using multidimensional $It\hat{o}$ integrals and techniques of inequalities, a class of stochastic differential systems with Lévy noise is analyzed. Some sufficient conditions of exponential stability are obtained by reduction to absurdity.

Keywords Lévy noise; stochastic differential systems; stability

1. Introduction

Differential equation is an important tool to describe the change process of objective things. It can not only use known data to predict the possible development trend of the future, but also use the expected goal of the future to deduce the current conditions. It can be said that the mathematical model established by using differential equations as a tool is almost used in all fields of applied science. In real life, however, random interference is common. For example, environmental noise, accidental emergencies and so on. And sometimes such random factors may change the motion state of the original dynamic system. At this time, the deterministic process can not accurately describe its variation rules, so it is necessary to introduce stochastic differential equations to describe the dynamic system disturbed by such random factors. The neutral stochastic differential system is a very important kind of stochastic differential system. Compared with ordinary stochastic differential equation, neutral stochastic differential equation can reflect the law of system change more accurately and deeply, and most stochastic differential system can be deemed to its special circumstances.

In the theory of control system, the stability problem is particularly important, because the stability is a necessary condition to ensure the normal operation of the actual system. The stability of stochastic differential equations has attracted extensive attention of scientists for several decades. Scholars discussed in the literature [3] for a class of neutral by fractional Lévy noise disturbance in the stability of the mixed type of stochastic functional differential equations, using Lyapunov functionals, the negative half martingale convergence theorem and theory of M - matrix in the general equation of the solution for attenuation speed almost certain stability, and gives the conditions at any time the upper bound of the coefficients. For more results we can further refer the reference [4-9].

However, whether in nature or in engineering, many practical systems often suffer from sudden environmental disturbances that are not suitable to be described by Gaussian noise, such as earthquakes and hurricanes. Fortunately, Lévy noise, as an important non-Gaussian noise, can be used to describe these phenomena perfectly. Based on the mean square exponential stability of neutral stochastic differential system [9], Lévy noise was added to make it become stochastic differential system with Lévy noise, and its mean square exponential stability was studied. In comparison with literature [11], the author uses the method of Lyapunov function to obtain the conditions to ensure the asymptotic stability of p-th neutral stochastic differential equations with Lévy noise. In this paper, the mean square exponential stability condition of stochastic differential systems with Lévy noise is obtained by using reduction to absurdity, integral and inequality techniques.

2. Preliminaries

Let $\Re_+ := [0,\infty)$. $\|\cdot\|$: the Euclidean norm on $\Re^d$. (Ω, F, P): the complete probability space with a filtration $\{F_t\}_{t\geq 0}$. $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_m(t))^T$: the $m-$dimensional Wiener process defined on the probability space. $E\zeta$: the expectation of the random variable ζ. $C([-h,0],\Re^d)$: the space of continuous $\Re^d-$valued functions φ defined on $[-h,0]$ with the norm $\sup_{s\in[-h,0]}\|\varphi(s)\|$. $L^2_{F_t}([-h,0];\Re^d)$: the family of all F_t-measurable, $\Re^d$-valued random variables $\phi = \{\phi(\theta), -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0} E\|\phi(\theta)\|^2 < \infty$. $C^{2,1}(\Re^d \times \Re_+; \Re_+)$: the family of all nonnegative functions $W(u,t)$ from $\Re^d \times \Re_{t_0}$ to $\Re_+$,which are continuously twice differentiable in $u \in \Re^d$ and once differentiable in $t \in \Re_+$.

Consider the following stochastic differential systems with Lévy noise :

$$\begin{cases} d(x(t) - Dx(t-\tau)) = f(x_t,t)dt + g(x_t,t)dw(t) \\ \qquad\qquad + \int_{|v|<c} H(x_t,t,v)\tilde{N}(dt,dv)\,, t \geq t_0 > 0, \\ x_{t_0} = \xi, t = t_0. \end{cases} \tag{2-1}$$

where $x_t = \{x(t-\tau) : \tau \in [0,h]\}$ is a $L^2_{F_t}([-h,0];\Re^d)-$ valued stochastic process. The mappings $D : L^2_{F_t}([-h,0];\Re^d) \to \Re^d$, $f : \Re^d \times L^2_{F_t}([-h,0];\Re^d) \to \Re^d$, $g : \Re^d \times$ $L^2_{F_t}$ and $H : \Re^d \times L^2_{F_t}([-h,0]$ $;\Re^d) \times \Re^d \to \Re^d$ are $Borel$ measurable. The constant $c \in (0,+\infty]$ is the maximum allowable jump size, with the initial data $x(t_0) = \xi = \{\xi(t_0-\theta), -\tau \leq \theta \leq 0\} \in L^2_{F_{t_0}}([-h,0],\Re^d)$.

Denote by N the Poisson random measure defined on $\Re_+ \times (\Re^d - \{0\})$ with intensity measure υ and compensator $\tilde{N}$.In this article, we always suppose that N is independent of $w(t)$ and the Lévy measure υ satisfying $\tilde{N}(dt,dv) := N(dt,dv) - \upsilon(dv)dt$ and $\int_{\Re^d-\{0\}}(|v|^2 \wedge 1)\upsilon(dv) < \infty$.Generally, the pair (w,N) is called a Lévy noise.

According to the integral, it is not difficult to obtain the solution of the system (2-1) as follows

$$\begin{aligned} x(t) - Dx_t = \xi(0) - D\xi + \int_{t_0}^t f(x_s,s)ds + \int_{t_0}^t g(x_s,s)d\omega(s) \\ + \int_{t_0}^t \int_{|v|<c} H(x_s,t,v)\tilde{N}(ds,dv),\ t \geq t_0. \end{aligned} \tag{2-2}$$

To ensure existence and uniqueness of solution of the system (2-1), we assumed a constant $0 < l < 1$,such that

$$\| D\varphi_1 - D\varphi_2 \| \leq l \| \varphi_1 - \varphi_2 \|, \ \varphi_1, \varphi_2 \in C, \tag{2-3}$$

and f, g are locally Lipschitz continuous and satisfy the linear-growth condition.

If the solution of system (2-1) is denoted by $x(t,t_0,\xi)$, it is important to note that

$$E(\sup_{t\in[t_0,\infty)} \|x(t,t_0,\xi)\|^2) < \infty, \ t \geq t_0. \tag{2-4}$$

When $\varphi=0$ and $f(0,t)=0$ and $g(0,t)=0$ for any $t \in \Re_+$, we set $D\varphi=0$. Then $x(t)=0$ is the solution of system (2-1) with the zero initial data at $t_0 = 0$.

For the stability purpose of this paper, we need the following definition.

Definition 2.1 (*[9]*)**.** If exist positive constants δ and M such that

$$E\|x(t,t_0,\xi)\|^2 \leq Me^{-\delta(t-t_0)} \sup_{\theta\in[-h,0]} E\|\xi(\theta)\|^2,\ t \geq t_0,$$

for any $t_0 \geq 0$ and for any $\xi \in L^2_{F_{t_0}}([-h,0],\Re^d)$. Then the zero solution of system (2-1) is stable.

3. Main result

In order to prove the stability of neutral stochastic differential system, we need to define an operator by using the *Itô* formula, let $A \in C^{2,1}(\Re^d \times \Re_+; \Re_+)$, define an operator ℓA from $\Re_+ \times \Re^d$ to $\Re$ by

$$\begin{aligned}\ell A(t,\xi - Dx(t-\tau)) &= A_t(t,\xi - Dx(t-\tau)) + A_x(t,\xi - Dx(t-\tau)) f(\xi - x(t-\tau)) \\ &+ \frac{1}{2} trace[d^T(\xi, x(t-\tau)) A_{xx}(t,\xi - Dx(t-\tau)) d(\xi, x(t-\tau))] \\ &+ \int_{|v|<c} [A(t,\xi - Dx(t-\tau) + H(\xi, x(t-\tau), V)) \\ &- A(t,\xi - Dx(t-\tau)) - H(\xi, x(t-\tau), V) A_x(t,\xi - Dx(t-\tau))] v dv,\end{aligned} \tag{3-1}$$

The following lemma will be applied in the continuation.

Lemma 3.1 (*[9]*). Let $x(t) := x(t; t_0, \xi), t \ge t_0 - h$ be the solution of system(2-1). Set that exist $\sigma \in (0,1)$ and a *Borel* measurable function $r(\cdot): [-h,0] \to \Re_+, \int_{-h}^0 r(s) ds = 1$ such that

$$\| D\varphi \|^2 \le \sigma^2 \int_{-h}^0 r(s) \| \varphi(s) \|^2 \, ds, \varphi \in C. \tag{3-2}$$

And for some $\lambda \in (0, -\frac{2}{h} \ln \sigma)$, we have

$$E \| x(t) - Dx(t-\tau) \|^2 \le M e^{-\lambda(t-t_0)} E \| x(t_0 + \theta \|^2, t \ge t_0. \tag{3-3}$$

For some $M > 0$, then

$$E \| x(t) \|^2 \le \frac{M}{(1 - ke^{\frac{\lambda h}{2}})^2} e^{-\lambda(t-t_0)} \sup_{\theta \in [-h,0]} E \| x(t_0 + \theta) \|^2, t \ge t_0 - h. \tag{3-4}$$

The inequality in Lemma 3.1 will be used to prove Theorem 3.2, in which Theorem 3.2 is proved by contradiction and *Itô* formula, The mean square exponential stability condition for a class of stochastic differential systems with Lévy noise is obtained.

Theorem 3.2. If there exists a constant $\eta > 0$,such that

$$\ell A(t, x - Dx(t-\tau)) \le -\eta \| \xi - Dx(t-\tau) \|^2, \tag{3-5}$$

and satisfy equation (3-2), then the system (2-1) is exponentially stable in mean square.

Proof：Assume $M > 2 + 2\sigma^2$ .Then we think the following two functions：

$$u(t) := E \| x(t) - Dx(t-\tau) \|^2, \ t \ge t_0 \tag{3-6}$$

and

$$v(t) := M e^{-\lambda(t-t_0)} \sup_{\theta \in [-h,0]} E \| \xi(\theta) \|^2, \ t \ge t_0, \tag{3-7}$$

where $\lambda > 0, \ \sigma \in (0,1)$.

It follows from system (2-1) and conditions (3-2) that

$$\begin{aligned} u(t_0) &= E \| x(t_0) - Dx_{t_0} \|^2 \le E(\| x(t_0) \| + \| Dx_{t_0} \|)^2 \le E(2 \| x(t_0) \|^2 + 2 \| Dx_{t_0} \|^2) \\ &\le 2E \| x(t_0) \|^2 + 2\sigma \int_{-h}^0 r(s) E \| x(t_0 + s) \|^2 \, ds = 2E \| \xi(0) \|^2 + 2\sigma^2 \int_{-h}^0 r(s) E \| \xi(s) \|^2 \, ds \\ &\le 2 \sup_{\theta \in [-h,0]} E \| \xi(\theta) \|^2 + 2\sigma^2 \int_{-h}^0 r(s) \sup_{\theta \in [-h,0]} E \| \xi(\theta) \|^2 \, ds \\ &= (2 + 2k^2) \sup_{\theta \in [-h,0]} E \| \xi(\theta) \|^2 < M \sup_{\theta \in [-h,0]} E \| \xi(\theta) \|^2 = v(t_0), \end{aligned}$$

Thus, $u(t_0) < v(t_0)$. It is shown that

$$u(t) \le v(t), \ \forall t \ge t_0, \tag{3-8}$$

Reduction to absurdity.
Then we have $\overline{t} > t_0$ such that $u(\overline{t}) \not\leq v(\overline{t})$ .Setting $t_* := \inf\{t > t_0 : u(t) \not\leq v(t)\}$,
such that

$$E[x(t_* - Dx(t_* - \tau)]^2 > e^{-\lambda(t_*-t_0)} E \| \xi - Dx(t_0 - \tau) \|^2, \tag{3-9}$$

Assume $\alpha > \lambda$, and let the function $V(x,t) := e^{\alpha t} \| x(t) - Dx(t-\tau) \|^2$, $(x,t) \in \Re^d \times \Re_+$, for any $m > \| \xi - Dx(t_0 - \tau) \|^2$,the stopping time is $\mu_m = \inf\{t > t_0 : \| x(t) \| \geq m\}$, Using the $It\hat{o}$ formula to $V(x,t)$,we acquire

$$\begin{aligned} & e^{\alpha(t\wedge\mu_m)} E \| x(t \wedge \mu_m) - Dx(t \wedge \mu_m - \tau) \|^2 \\ & = e^{\alpha t_0} E \| x(t_0) - Dx_{t_0} \|^2 + E \int_{t_0}^{t\wedge\mu_m} e^{\alpha s} \| x(s) - Dx_s \|^2 ds \\ & + E \int_{t_0}^{t\wedge\mu_m} e^{\alpha s} \ell \mathrm{A}(t, x(s) - Dx(s-\tau)) ds, \end{aligned} \tag{3-10}$$

Letting $m \to \infty$ on both sides of (3-10) yields

$$\begin{aligned} e^{\alpha t} E \| x(t) - Dx(t-\tau) \|^2 \leq & e^{\alpha t_0} E \| \xi - Dx_{t_0} \|^2 \\ & + \int_{t_0}^{t} \alpha e^{\alpha s} E \| x(s) - Dx_s \|^2 ds \\ & + \int_{t_0}^{t} e^{\alpha s} E \| -\eta(x(s) - Dx(s-\tau)) \|^2 ds, \end{aligned} \tag{3-11}$$

It follows from (3-9) and (3-11) that

$$\begin{aligned} & e^{\alpha t_*} E \| x(t_*) - Dx(t_* - \tau) \|^2 \leq e^{\alpha t_0} E \| \xi - Dx(t_0 - \tau) \|^2 \\ & \quad + \int_{t_0}^{t_*} \alpha e^{\alpha s} [E \| \xi - Dx(t_0 - \tau) \|^2 e^{-\lambda(s-t_0)}] ds \\ & \quad - \eta \int_{t_0}^{t_*} e^{\alpha s} [E \| \xi - Dx(t_0 - \tau) \|^2 e^{-\lambda(s-t_0)}] ds \\ & = e^{\alpha t_0} E \| \xi - Dx(t_0 - \tau) \|^2 + \int_{t_0}^{t_*} (\alpha - \eta) e^{(\alpha-\lambda)s + \lambda t_0} E \| \xi - Dx(t_0 - \tau) \|^2 ds \\ & = e^{\alpha t_0} E \| \xi - Dx(t_0 - \tau) \|^2 + E \| \xi - Dx(t_0 - \tau) \|^2 \frac{e^{\alpha t_*} e^{-\lambda(t_*-t_0)} - e^{\alpha t_0}}{\alpha - \lambda} (\alpha - \eta), \end{aligned} \tag{3-12}$$

Considering $-\eta < -\lambda$, we acquire

$$\begin{aligned} & e^{\alpha t_*} E \| x(t_*) - Dx(t_* - \tau) \|^2 \\ & < e^{\alpha t_0} E \| \xi - Dx(t_0 - \tau) \|^2 + E \| \xi - Dx(t_0 - \tau) \|^2 \frac{e^{\alpha t_*} e^{-\lambda(t_*-t_0)} - e^{\alpha t_0}}{\alpha - \lambda} (\alpha - \lambda) \\ & = e^{\alpha t_0} [E \| \xi - Dx(t_0 - \tau) \|^2 - E \| \xi - Dx(t_0 - \tau) \|^2] \\ & + e^{\alpha t_*} e^{-\lambda(t_*-t_0)} E \| \xi - Dx(t_0 - \tau) \|^2 \\ & = e^{\alpha t_*} e^{-\lambda(t_*-t_0)} E \| \xi - Dx(t_0 - \tau) \|^2 . \end{aligned} \tag{3-13}$$

Thus,

$$E \| x(t_*) - Dx(t_* - \tau) \|^2 < e^{-\lambda(t_*-t_0)} E \| \xi - Dx(t_0 - \tau) \|^2 . \tag{3-14}$$

Which is in conflict with (3-9).
Therefore,

$$E\|x(t)-Dx(t-\tau)\|^2\le Me^{-\lambda(t-t_0)}E\|\xi(\theta)\|^2, t\ge t_0. \tag{3-15}$$

Form Lemma 3.1, we know

$$E\|x(t)\|^2\le \frac{M}{(1-ke^{\frac{\lambda h}{2}})^2}e^{-\lambda(t-t_0)}\sup_{\theta\in[-h,0]}E\|\xi(\theta)\|^2, t\ge t_0. \tag{3-16}$$

So system (2-1) is exponentially stable in mean square.

Corollary 3.3. Suppose that (3-5) is satisfied. And condition (3-2) is changed to

$$E\|D\xi\|^2\le\sigma^2\sup_{\theta\in[-h,0]}E\|\xi(\theta)\|^2, \xi\in L^b_{F_t}([-h,0],\Re^d), t\in\Re_+, \tag{3-17}$$

Then system (2-1) is exponentially stable in mean square.

Proof：The proof procedure is similar to the Theorem 3.2.

Corollary 3.4. Suppose that (3-2) is satisfied. And condition (3-5) is changed to

$$\ell A(t,x-Dx(t-\tau))\le-\eta(t)\|\xi-Dx(t-\tau)\|^2, \tag{3-18}$$

Where $\eta(t)>0,\ t\ge t_0.$. Then system (2-1) is exponentially stable in mean square.

Proof：Let $\eta:=\sup_{t\ge t_0}\eta(t)$, the proof procedure is similar to the Theorem 3.2.

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