Available online www.jsaer.com

Journal of Scientific and Engineering Research, 2022, 9(4):88-90

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Discussion on Solutions of a Class of Cubic Diophantine Equations

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Abstract This paper uses the basic theory of Pell equation to discuss the following contents: hypothesizing $p = 3r^2 - 2$ or $3p = r^2 + 2$, and p is odd prime in the $p \equiv 1(mod\ 6)$, r is positive integer, the cubic Diophantine equation $x^3 - 3py^2 = 1$ has no positive integer solution (x, y); when $p < 100$, the equation has positive integer solution (x, y) if $p = 37$.

Keywords Cubic Diophantine equation; Pell equation; Petrr group

Cubic Diophantine Equation

$x^3 - 3py^2 = 1\ x, y \in \mathbb{N}$, p is an odd prime greater than 3 (1)

The solution of this equation has attracted public attention in number theory, since the literature [1] had proven: when $p \equiv 5(mod\ 6)$, equation (1) has no solution(x, y), hence it comes to the discussion about $p \equiv 1(mod\ 6)$, which only some conclusions have been reached.

In this paper, the following results are discussed according to the basic properties of Pell equation:

Theorem If $p = 3r^2 - 2, r \in \mathbb{N}$ or $3p = r^2 + 2$ (2)

then equation (1) has no solution(x, y).

In order to prove the theorem, this paper first proves the following conclusions.

Lemma 1 If (x, y) is a set of solutions to equation (1), thus,

$x - 1 = 9q^2, x^2 + x + 1 = 3pb^2, y = 3ab, a, b \in \mathbb{N}$ (3)

Demonstration Hypothesizing (x, y) is a set of solutions to equation (1), because $x^3 \equiv 1(mod\ 3)$ according to (1), then $x \equiv 1(mod\ 3)$, $x^2 + x + 1 = 0(mod\ 3)$ and $gcd\ (x^2 - 1, x^2 + x + 1) = 3$. Since p is an odd prime greater than 3, thus $x - 1 = 9pa^2, x^2 + x + 1 = 3b^2$ according to (1),

$y = 3ab, a, b \in \mathbb{N}$ (4)

Or (2). According to the literature [1], (4) is not valid.

Lemma 2 [2] Hypothesizing D is a given positive integer with no square factor, there exists a unique set of positive integer (D_1, D_2, λ) that satisfy

$D_1D_2 = D$,$\lambda \in \{1,2\}$, $gcd(D, \lambda) = 1$, $(D_1, D_2, \lambda) \neq (1, D, 1)$ (5)

thus equation

$D_1u^2 - D_2v^2 = \lambda, u, v \in \mathbb{N}$ (6)

has a solution (u, v).

For a given D, the group of positive integers defined by Lemma 2 is called the Petrr group of D, marked as $p(D)$.

Lemma 3 Hypothesizing D_1 , D_2 are positive integer satisfy $D_1 > 1$, if equation

$D_1u^2 - D_2v^2 = 1, u, v \in \mathbb{N}$ (7)

has a solution (u,v), then it must have a unique solution (u_1,v_1)that satisfy $u_1\sqrt{D_1}+v_1\sqrt{D_2}\le u\sqrt{D_1}+v\sqrt{D_2}$, (u,v) is all the solutions to this equation.
(u_1,v_1) is called the smallest solution to the equation (7). Hence any set of solutions (u,v) to the equation can be expressed as

$$u\sqrt{D_1}+v\sqrt{D_2}=\left(u_1\sqrt{D_1}+v_1\sqrt{D_2}\right)^k, k\in\mathbb{N} \tag{8}$$

Demonstration Refer to literature [3]
Lemma 4 If the solution (u,v) of equation (7) satisfies

$$u\sqrt{D_1}+v\sqrt{D_2}<\left(\sqrt{D_1}+\sqrt{D_2}\right)^3 \tag{9}$$

Then (u,v) must be the smallest solution to the equation.
Demonstration According to the lemma 3: there exists positive odd number k which makes (u,v) satisfies (8). Hypothesizing (u,v) is not the smallest solution of equation (7), then $k\ge 3$, according to (8),

$$u\sqrt{D_1}+v\sqrt{D_2}\ge\left(u_1\sqrt{D_1}+v_1\sqrt{D_2}\right)^3\ge\left(\sqrt{D_1}+\sqrt{D_2}\right)^3 \tag{10}$$

is in contradiction with (9). Therefore, if (u,v) satisfies (9), then it must be the smallest solution of equation (7).
Lemma 5 Any set of solutions (u,v) to the equation (7) can satisfy u_1/u and v_1/v, and (u_1,v_1) is the smallest solution of the equation.
Demonstration According to the lemma 3: there exists positive odd number K which makes the equation (8) true, then,

$$u=u_1\sum_{i=0}^{(k-1)/2}\binom{k}{2i}(D_1u_1^2)^{\left(\frac{k-1}{2}-i\right)}(D_2v_1^2)^i$$

$$v=v_1\sum_{i=0}^{(k-1)/2}\binom{k}{2i+1}(D_1u_1^2)^{\left(\frac{k-1}{2}-i\right)}(D_2v_1^2)^i \tag{11}$$

u_1/u and v_1/v can be proven from (11).

Demonstration of lemma
Hypothesizing (x,y) is a set solution of equation (1), according to the lemma 1, there exists positive integer a and b which make x and y satisfy (3), then

$$p(2b)^2-3(a^2+1)^2=1 \tag{12}$$

when x is eliminated in (3).
From (12), equation

$$pu^2-3v^2=1, u,v\in\mathbb{N} \tag{13}$$

has solutions

$$(u,v)=(2b,6a^2+1) \tag{14}$$

Hypothesizing $D=3p$, since p is an odd prime greater than 3, therefore, D is a positive integer with no square factor. Since equation (12) has solutions, according to lemma 2, $p(D)=(p,3,1)$, thus,

$$3u^2-pv^2=2, u,v\in\mathbb{N} \tag{15}$$

and

$$3pu^2-v^2=2, u,v\in\mathbb{N} \tag{16}$$

have no solutions (x,y).
If p satisfies (2), then equation (15) has solutions $(u,v)=(r,1)$ Nevertheless, it can be told that from above: it is inapprehensible when equation (1) has solutions. Therefore, there must be no solution to equation (1) if p satisfies (2).
Similarly, if p satisfies $3p=r^2+2, r\in\mathbb{N}$, since equation (16) has solutions $(u,v)=(1,r)$, therefore, equation (1) has no solution.

Reference

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