Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2022, 9(3):155-160**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Bounds for Emanant of Lacunary Type Complex Polynomial

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Abstract Let $p(z)$ be a polynomial of degree n and let α be any real or complex number, then $D_\alpha p(z)$, the polar derivative of $p(z)$, is called by Laguerre [6] the "emanant" of $p(z)$, by Polya and Szeg ö [8] the "derivative of $p(z)$ with respect to the point α", and by Marden[7], simply the "polar derivative" of $p(z)$. It is obviously of interest to obtain estimates concerning growth of $D_\alpha p(z)$. In this paper we prove interesting results for the polar derivative of a lacunary type of polynomial which not only improve upon some earlier known results in the same area but also improve upon a result on ordinary derivative for polynomials in particular case.

Keywords Polynomials; Polar derivative; Inequalities; Zeros
(2020) Subject Classification: 30A10, 30D15

1. Introduction

Let α be a complex number. If $p(z)$ is a polynomial of degree *n*, then polar derivative of $p(z)$ with respect to point α, denoted by $D_\alpha p(z)$, is defined by

$$D_\alpha p(z) = n\, p(z) + (\alpha - z)p'(z). \tag{1.1}$$

and

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha p(z)}{\alpha}\right] = p'(z). \tag{1.2}$$

It is obviously of interest to obtain estimates concerning the growth of $D_\alpha p(z)$. In this direction Aziz [1] proved several sharp results concerning the maximum modulus of the polar derivative of a polynomial $p(z)$ with restricted zeros.

For the class of polynomials not vanishing in the disc $|z| < k$, $k \geq 1$, Aziz [1] proved the following result for the polar derivative of $p(z)$.

Theorem A. *If $p(z)$ is a polynomial of degree n having no zeros in the disc $|z| < k$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq 1$,*

$$\max_{|z|=1}|D_\alpha p(z)| \le n\left(\frac{k+|\alpha|}{1+k}\right)\max_{|z|=1}|p(z)|. \tag{1.3}$$

The result is best possible and equality in (1.3) *holds for the polynomial* $p(z)=(z+k)^n$, *with real* $\alpha \ge 1$.

As a refinement of Theorem A, Aziz and Shah [2] proved the following result, for the polar derivative of $p(z)$.

Theorem B. *If* $p(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n having no zero in* $|z|<k$, $k \ge 1$, *then for every real or complex number* α *with* $|\alpha| \ge 1$,

$$\max_{|z|=1}|D_\alpha p(z)| \le \frac{n}{1+k}\left\{(|\alpha|+k)\max_{|z|=1}|p(z)|-(|\alpha|-1)\min_{|z|=k}|p(z)|\right\}. \tag{1.4}$$

The result is best possible and extremal polynomial is $p(z)=\left(\frac{z+k}{1+k}\right)^n$ *for every real* $\alpha \ge k$.

For the class of polynomial $p(z)$ having all its zeros in $|z| \le 1$, Shah [9] proved the following

Theorem C. *If all the zeros of the* n^{th} *degree polynomial* $p(z)$ *lie in* $|z| \le 1$, *then for every real or complex number* α *with* $|\alpha| \ge 1$,

$$\max_{|z|=1}|D_\alpha p(z)| \ge \frac{n}{2}(|\alpha|-1)\max_{|z|=1}|p(z)|. \tag{1.5}$$

The result is sharp and extremal polynomial is $p(z)=(z-1)^n$ *with real* $\alpha \ge 1$.

For the class of polynomial $p(z)$ having all its zeros in $|z| \le 1$, Shah [9] extended Turán's Theorem to the polar derivative of the polynomial and proved the following.

Theorem D. *If all the zeros of the* n^{th} *degree polynomial* $p(z)$ *lie in* $|z| \le 1$, *then for every real or complex number* α *with* $|\alpha| \ge 1$,

$$\max_{|z|=1}|D_\alpha p(z)| \ge \frac{n}{2}(|\alpha|-1)\max_{|z|=1}|p(z)|. \tag{1.6}$$

The result is sharp and extremal polynomial is $p(z)=(z-1)^n$ *with real* $\alpha \ge 1$

As a generalization of Theorem D, Aziz and Rather [3] proved the following result for the polar derivative of a polynomial $p(z)$.

Theorem E. *If* $p(z)$ *is a polynomial of degree n having all its zeros in* $|z| \le k$, $k \le 1$, *then for every real or complex number* α *with* $|\alpha| \ge k$,

$$\max_{|z|=1}|D_\alpha p(z)| \ge n\left(\frac{|\alpha|-k}{1+k}\right)\max_{|z|=1}|p(z)|. \tag{1.7}$$

Inequality (1.7) *is best possible and equality occurs for* $p(z)=(z-k)^n$ *with real* $\alpha \ge k$.

2. Main Theorem

For the class of Lacunary type of polynomials $p(z)=a_nz^n+\sum_{j=\mu}^{n}a_{n-j}z^{n-j}$, $1\le\mu\le n$, we prove the following result which gives a generalization as well as an improvement of Theorem E.

Theorem 2.1. *If* $p(z)=a_nz^n+\sum_{j=\mu}^{n}a_{n-j}z^{n-j}$, $1\le\mu\le n$, *is a polynomial of degree n having all its zeros in* $|z|\le k$, $k\le 1$, *then for every real or complex number* α *with* $|\alpha|\ge k^\mu$, *we have*

$$\max_{|z|=1}|D_\alpha p(z)|\ge n\left(\frac{|\alpha|-k^\mu}{1+k^\mu}\right)\max_{|z|=1}|p(z)|+\frac{n}{k^{n-\mu}}\left(\frac{|\alpha|+1}{1+k^\mu}\right)\min_{|z|=k}|p(z)| \tag{2.1}$$

Dividing both sides of (2.1) by $|\alpha|$, letting $|\alpha|\to\infty$, we obtain the following generalization of a result due to Govil [4] .

Corollary 2.2. *If* $p(z)=a_nz^n+\sum_{j=\mu}^{n}a_{n-j}z^{n-j}$, $1\le\mu\le n$, *is a polynomial of degree n having all its zeros in* $|z|\le k$, $k\le 1$, *then we have*

$$\max_{|z|=1}|p'(z)|\ge\frac{n}{1+k^\mu}\left\{\max_{|z|=1}|p(z)|+\frac{1}{k^{n-\mu}}\min_{|z|=k}|p(z)|\right\}. \tag{2.2}$$

The result is sharp and extremal polynomial is $p(z)=\left(z^\mu+k^\mu\right)^{\frac{n}{\mu}}$, *where n is a multiple of* μ .

For μ =1, inequality (2.1) reduces to the following result due to Govil and Mc Tume[5].

Corollary 2.3. *If* $p(z)=\sum_{j=0}^{n}a_jz^j$ *is a polynomial of degree n having all its zeros in* $|z|\le k$, $k\le 1$, *then for every real or complex number* α *with* $|\alpha|\ge k$, *we have*

$$\max_{|z|=1}|D_\alpha p(z)|\ge n\left(\frac{|\alpha|-k}{1+k}\right)\max_{|z|=1}|p(z)|+\frac{n}{k^{n-1}}\left(\frac{|\alpha|+1}{1+k}\right)\min_{|z|=k}|p(z)|. \tag{2.3}$$

The inequality is sharp and equality holds for $p(z)=(z-k)^n$ *with real* $\alpha\ge k$.

3. Lemmas

For the proofs of the theorems, we need the following lemmas.

Lemma 3.1. *Let* $p(z)$ *be polynomial of degree n and* α *is any real or complex number with* $|\alpha|\ne 0$ *.Then for* $|z|=1$,

$$|D_\alpha q(z)|=\left|n\bar{\alpha}\,p(z)+\left(1-\bar{\alpha}z\right)p'(z)\right|. \tag{3.1}$$

where $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

The above lemma is due to Aziz [1].

Lemma 3.2. *If all the zeros of an* n^{th} *degree polynomial* $p(z)$ *lie in* $|z|\le k$, *then for* $|\alpha|\ge k$, *the polar derivative* $D_\alpha p(z)$ *of* $p(z)$ *at point* α *also has all its zeros in* $|z|\le k$.
The above lemma is due to Marden [7].

Lemma 3.3. *Let* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$, $1\le\mu\le n$, *be a polynomial of degree n not vanishing in* $|z|<k$, $k\ge1$, *and* $q(z)=z^n\overline{p\left(\frac{1}{\bar z}\right)}$. *Then for* $|z|=1$ *and R ≥ 1,*

$$k^\mu|p(Rz)-p(z)|\le|q(Rz)-q(z)|-\left(R^n-1\right)\min_{|z|=k}|p(z)|. \tag{3.2}$$

The above lemma is due to Aziz and Rather [3].

Lemma 3.4. *Let* $p(z)=a_nz^n+\sum_{j=\mu}^{n}a_{n-j}z^{n-j}$, $1\le\mu\le n$, *be a polynomial of degree n having all its zeros in* $|z|\le k$, $k\le1$. *Then for* $|z|=1$ *and R ≥ 1*

$$k^\mu|p(Rz)-p(z)|\ge|q(Rz)-q(z)|+\frac{\left(R^n-1\right)}{k^{n-\mu}}\min_{|z|=k}|p(z)|. \tag{3.3}$$

where $q(z)=z^n\overline{p\left(\frac{1}{\bar z}\right)}$.

Proof of Lemma 3.4. Since $p(z)$ has all its zeros in $|z|\le k$, $k\le1$, then $q(z)=z^n\overline{p\left(\frac{1}{\bar z}\right)}$ has all its zeros in $|z|\ge\frac{1}{k}$, $\frac{1}{k}\ge1$. On applying Lemma 3.3 to *q*(*z*), we get for $|z|=1$,

$$\frac{1}{k^\mu}|q(Rz)-q(z)|\le|p(Rz)-p(z)|-\left(R^n-1\right)\min_{|z|=\frac{1}{k}}|q(z)| \tag{3.4}$$

But

$$\min_{|z|=\frac{1}{k}}|q(z)|=\min_{|z|=\frac{1}{k}}\left|z^n\overline{p\left(\frac{1}{\bar z}\right)}\right|=\min_{|z|=1}\left|\frac{z^n}{k^n}\overline{p\left(\frac{k}{\bar z}\right)}\right|$$

$$=\frac{1}{k^n}\min_{|z|=1}|p(kz)|=\frac{1}{k^n}\min_{|z|=k}|p(z)|. \tag{3.5}$$

Combining inequalities (3.4) and (3.5), we get

$$\frac{1}{k^\mu}|q(Rz)-q(z)|\le|p(Rz)-p(z)|-\frac{\left(R^n-1\right)}{k^n}\min_{|z|=k}|p(z)|,$$

from which Lemma 3.4 follows.
From the above lemma, we deduce the following result, by dividing both sides of inequality (3.3) by (*R*-1) and let $R\to1$, which is also of independent interest .

Lemma 3.5. *Let* $p(z)=a_nz^n+\sum_{j=\mu}^{n}a_{n-j}z^{n-j}$, $1\le\mu\le n$, *be a polynomial of degree n having all its zeros In* $|z|\le k$, $k\le1$. *Then for* $|z|=1$

$$k^{\mu}|p'(z)| \geq |q'(z)| + \frac{n}{k^{n-\mu}} \min_{|z|=k} |p(z)| . \qquad (3.6)$$

where $q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}$.

4. Proof of the Main Theorem

Proof of Theorem 2.1. Let $q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}$. Then it can be easily verified that for $|z| = 1$,

$$|q'(z)| = |n\,p(z) - z\,p'(z)| \qquad (4.1)$$

Now for $|z| = 1$, we have from (4.1)

$$|n\,p(z)| = |n\,p(z) - z\,p'(z) + z\,p'(z)|$$

$$\leq |n\,p(z) - z\,p'(z)| + |p'(z)| = |q'(z)| + |p'(z)| . \qquad (4.2)$$

Inequality (4.2) when combined with Lemma 3.5, gives for $|z| = 1$,

$$|p'(z)| \geq \frac{n}{1+k^{\mu}} |p(z)| + \frac{n\,m}{k^{n-\mu}(1+k^{\mu})} . \qquad (4.3)$$

where $m = \min_{|z|=k} |p(z)|$.

Now for every real or complex number α, we have

$D_{\alpha} p(z) = n\,p(z) + (\alpha - z) p'(z)$.

This gives for $|z| = 1$,

$$|D_{\alpha} p(z)| \geq |\alpha|\,|p'(z)| - |n\,p(z) - z\,p'(z)|$$

$$= |\alpha|\,|p'(z)| - |q'(z)| . \qquad (4.4)$$

Now on combining inequality (4.4) with Lemma 3.5, we get for $|z| = 1$.

$$|D_{\alpha} p(z)| \geq \left(|\alpha| - k^{\mu}\right) |p'(z)| + \frac{n\,m}{k^{n-\mu}} . \qquad (4.5)$$

Inequality (4.3), when combined with above inequality (4.5), gives for $|z| = 1$,

$$|D_{\alpha} p(z)| \geq \left(|\alpha| - k^{\mu}\right) \left\{ \frac{n}{1+k^{\mu}} |p(z)| + \frac{n\,m}{k^{n-\mu}(1+k^{\mu})} \right\} + \frac{n\,m}{k^{n-\mu}} .$$

which on simplification gives for $|z| = 1$,

$$|D_{\alpha} p(z)| \geq n \left(\frac{|\alpha| - k^{\mu}}{1+k^{\mu}} \right) |p(z)| + \frac{n\left(|\alpha| + 1\right)}{k^{n-\mu}(1+k^{\mu})} m .$$

from which Theorem 2.1 follows easily.

Acknowledgement: The author is thankful to anonymous referee for valuable comments and suggestions.

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