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Exploration with Return of Constantly Connected Complete Cactus

Ahmed Mouhamadou Wade *, Ibrahima Gueye

* Associate Professor, LTISI laboratory GIT, Polytechnic School of Thiès (EPT), Thiès, Sénégal
awade@ept.sn, igueye@ept.sn

Abstract This paper studies the necessary and sufficient time to explore with return by a mobile agent the constantly connected dynamic graphs based on a balanced cactus-star. Exploring with return of a dynamic graph consists for a mobile agent of visiting all the vertices of the graph and returning to the starting vertex. This fundamental problem in distributed computing by mobile agents is increasingly studied in dynamic networks. This is motivated by the fact that nowadays networks are becoming more and more dynamic partly due to the very important increase of the number of communicating objects. We give in this paper a lower bound and an upper bound on the time complexity of the exploration of dynamic graphs based on cactus-stars.

Keywords Dynamic graph, Exploration, Mobile agent, Cactus-star

1. Introduction

The problem of exploration is fundamental in distributed computing by mobile agents. It has been extensively studied in static graphs since the seminal paper by Claude Shannon [9]. Concerning dynamic graphs, since a decade or so, researchers are increasingly interested in the study of the complexity of the exploration of dynamic networks by mobile agents. If we consider dynamic graphs in its generality, it is impossible to solve the problem of graph exploration. This forces researchers to make assumptions about the dynamics of the graph and to consider the problem for some family of underlying graph (see Section 2 for definition). During these last years, several families of underlying graphs are considered with assumptions on the dynamics of the graph. Constantly connected dynamic graphs was used in [8] to study the problem of information dissemination. A dynamic graph is constantly connected if at every time unit there is a stable spanning tree in the graph. Note that the spanning tree is not necessarily the same at each time unit. Several studies consider this assumption (constantly connected dynamic graphs) where the underlying graph of the dynamic graph is a ring of n vertices. The problem of exploration with termination by a mobile agent is considered in [3, 5, 6]. If the dynamics of the graph is known, [6] shows that a single agent can solve the problem, and $2n - 3$ time units are necessary and sufficient. If the dynamics is not known in advance, [3] shows that two agents knowing an upper bound N on the number of vertices can solve the problem, and $3N - 6$ time units are sufficient if all agents are active at each time step, and $O(N2\ )$ moves are sufficient if a subset of the agents might be active at each time step. The case when the agent has partial information about network changes is considered in [5]. The case where the agent must return to its initial position after having explored all the vertices of the graph is considered in [11]. The problem of perpetual exploration is considered in [1, 4]. In [1], the authors consider that all agents are active at each time step and show that to solve the problem, one agent is sufficient in the rings of size two, two agents are sufficient in the rings of size three, and three agents are sufficient for all other rings. The authors define a ring of size two as a two-node path if the graph is simple, or as two nodes linked by two bidirectional edges otherwise. In [4] the authors consider time varying graphs whose topology is arbitrary and unknown to the agents and investigates the number of agents that are necessary and sufficient to explore such graphs.

Recently, a generalization of the family of underlying graphs is being considered by researchers. The case where the underlying graphs is a cacti-path (path of cycles) is considered [10, 7]. This article is a contribution to this ongoing dynamic. We consider the problem of dynamic graph exploration by mobile agent assuming that the underlying graph is a cactus-star (**see Section 2** for definition).

Our results. We show that to explore with return constantly connected dynamic graphs based on a cactus-star, $2(N-\sqrt{N})$ time units are necessary and we give an algorithm which explores with return any dynamic graph based on cactus-star in $3N-4\sqrt{N}+1$ time units. With N the size on the dynamic graph.

2. Preliminaries

This section provides precise definitions of the concepts and models informally mentioned above. We also give some previous results from the literature on the problem studied. The proofs of the theorems mentioned in this section are given in [9].

Definition 1. (Dynamic graph)

A dynamic graph is a pair $\mathcal{G}=(V,\mathcal{E})$, where V is a set of n static vertices, and $\mathcal{E}$ is a function which maps every integer $i \geq 1$ to a set $\mathcal{E}(i)$ of undirected edges on V.

Definition 2. (Underlying graph)

Given a dynamic graph $\mathcal{G}=(V,\mathcal{E})$, the static graph $G=(V, \cup\ \mathcal{E}(i))$ is called the underlying graph of $\mathcal{G}$. Conversely, the dynamic graph $\mathcal{G}$ is said to be based on the static graph G.

Definition 3. (Constant connectivity)

A dynamic graph $\mathcal{G}$ is said to be constantly connected if, for any integer i, the static graph $Gi=(V,\ \mathcal{E}(i))$ is connected.

Definition 4. (Cactus)

A cactus is a simple graph $G=(V,E)$ in which two connected cycles have at most one vertex in common [2].

Definition 5. (Cactus-star)

A cactus-star is a cactus such that, if we represent the cycles by vertices and the connections between cycles by edges, we obtain a star (see Figure 1).

We say that a star cactus denoted C_k is of order *k* if the corresponding star is of order *k*. In the figure 1, we give an example of cactus-star of order 5.

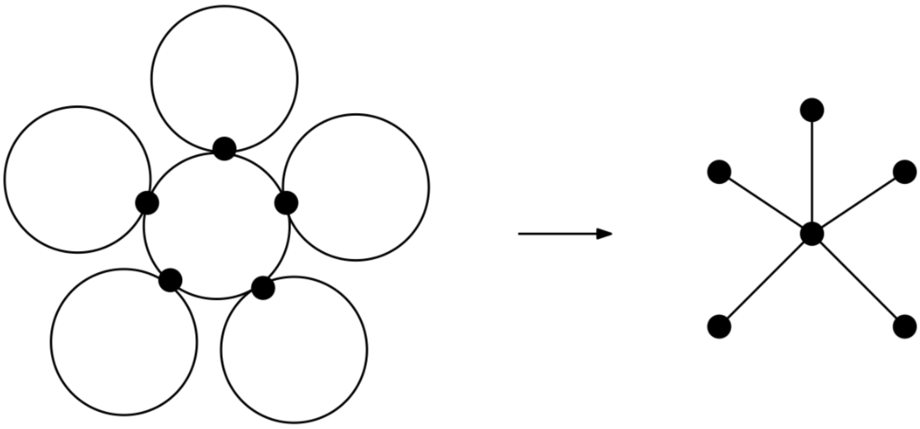

Figure 1: Example of Cactus-star of order 5

We give in the following definitions of different types of exploration of graphs by a mobile agent that exists, namely perpetual exploration, periodic exploration, exploration with stop and exploration with return.

Definition 6. (Perpetual exploration)

The agent must visit each vertex of the graph at least once. It does not have to stop after visiting all the vertices of the graph (it may not know if the graph is anonymous). The exploration time is the number of steps between the start of the exploration and the first moment when all the vertices have been visited.

Definition 7. (Periodic exploration)

In a periodic exploration, the agent must visit all the vertices of the graph periodically. In this type of exploration too, the agent does not have to stop after visiting all the vertices of the graph.

Definition 8. (Exploration with stop)
In this type of exploration, the agent must visit each vertex of the graph at least once. It must then stop once it has finished visiting all the vertices (not necessarily immediately after the last unknown vertex has been visited).
Definition 9. (Exploration with return)
The exploration with return is an exploration with stop with an additional constraint: the mobile entity (agent) must stop at its starting position.
In this paper, we consider the problem of exploration **with return** of dynamic graphs based on complete **cactus-star** of order n with N nodes. A cactus-star is complete if and only if all cycles of the cactus-star are same size n. This means that the number of vertices of the cactus-star is $N = n^2$. A mobile entity, called agent, operates on these dynamic graphs. The agent can traverse at most one edge per time unit. We say that the agent explores a dynamic graph if and only if it visits all its vertices. We also assume that the agent knows the dynamics of the graph, that is to say, the times of appearance and disappearance of the edges of the dynamic graph.
In this article, we will use the following result from the literature.
Theorem 1. [11] *For any integer $n \geq 3$ and for any constantly connected dynamic graph based on a ring with n vertices, there exists an agent (algorithm),* **Explore-with-return**, *exploring this dynamic graph and returning on its initial position in time at most $3n - 4$ (assuming that the agent knows the dynamics of the graph).*
Sketch of proof. Consider$n$ virtual agents placed on the n vertices (one agent on each vertex). Make all agents move in the clockwise direction for $n - 1$ time units from time t. Since at most one edge is removed at a time, it holds that, at each time, at most one such virtual agent is blocked at this time without having been blocked before. Thus, one of the n virtual agents is never blocked during the n –1 time units, and the starting vertex of this agent is the vertex (t) we are looking for. This can be done in at most n – 1 time units. Returning to the initial position takes also n – 1 other units of time. Which gives the claimed bound.
Theorem 2. [8] For any constantly connected dynamic graph on n vertices, at most $n - 1$ time units are sufficient for an agent to go from any vertex to any other vertex in the graph, when the agent knows the dynamics of the graph.
Sketch of proof. To prove the theorem, we will prove that for information dissemination in a constantly connected dynamic graph, $n - 1$ time units are sufficient. With *n* the size of the graph.
Suppose we want to disseminate information in a constantly connected dynamic graph in the assertion that, if a vertex receives the information, it broadcasts it to all those reachable neighbors. As the graph is connected to each time unit, so at least one new vertex is informed at each time unit. So after n time units, all vertices of the graph receive the information. Hence the proof that the temporal diameter is bounded by the number of vertices.
Lower bound
This section gives the lower bound on the exploration time of constantly connected dynamic graphs based on the complete cactus-star C_n of size *N*. We show that for any agent (algorithm), there exists a constantly connected dynamic graph based on C_n such that the agent must pay at least $2(N - \sqrt{N})$ time units to explore it with return. With C_n the cactus-star of order *n* in which all cycles have the same size *n*. We have the following Theorem.
Theorem 3. For any integer $N \geq 3$, there exists a constantly connected dynamic graph based on C_n such that any agent must pay at least $2(N - \sqrt{N})$ time units to explore with return the dynamic graph. This bound remains even if the agent knows the dynamics of the graph.
Proof. First of all, remind that C_n is composed of *n* cycles of size *n*. Let's now build the dynamic graph where any agent, whatever the algorithm it executes, will pay at least the announced bound. For each cycle of the dynamic graph, let $v_0, v_1, \dots, v_{n-1}$ be the vertices of the cycle in clockwise order. By construction, in each cycle, all the edges are present at the start of the exploration during the first time unit. Then the edge $\{v_1, v_2\}$, is absent until the end of the exploration. To make the analysis easier, let us now name the cycles of our dynamic graph C_n by $R_0, R_1, \dots, R_n$. Now suppose that R_n is the central cycle. To obtain our cactus-star C_n, the vertex v_0 of each cycle R_i is attached to the vertex v_i of the central cycle. **See Figure 2**. Note that the dynamic graph C_n is indeed constantly connected.
Consider any agent (algorithm). We will now prove that the time the agent uses to explore with return C_n is at least $2(N - \sqrt{N})$ time units. To explore the dynamic graph with return, the agent must visit all vertices of C_n

and return at the vertex v_0 of R_n. By construction, to explore with return any cycle of C_n, any agent pays at least $2n-2$ time units. Exploring a cycle requires visiting all of its vertices. In particular the vertices v_1, v_2 of each cycle and returning to v_0.

- ✓ **Suppose that the agent visits v_1, then v_2 before returning to v_0.**

To visit v_1, the agent pays at least one time unit, after that, visiting v_2 costs at least $n-1$ other time units because the edge $\{v_1, v_2\}$ is present only during the first time unit, so the agent must go around the cycle to visit v_2 for the first time. After visiting v_2 to return to the starting vertex (or attachment), for the same reasons, the agent will go around the cycle and pay $n-2$ other time units.

- ✓ **Suppose that the agent visits v_2, then v_1 before returning to v_0.**

To visit v_2 for the first time without passing through v_1, the agent pays at least $n-2$ time units. After that, to visit v_1, the agent has to go around the cycle and pay $n-2$ other time units, because the edge $\{v_1, v_2\}$ is absent after the first time unit. Returning to starting vertex (v_0), will cost one more unit of time.

This proves that the agent will pay at least $2n-2$ time units to explore with return each ring of the cactus-star. Since the star cactus is composed of n cycles, we will therefore have the announced bound to explore it with return. This concludes the proof.

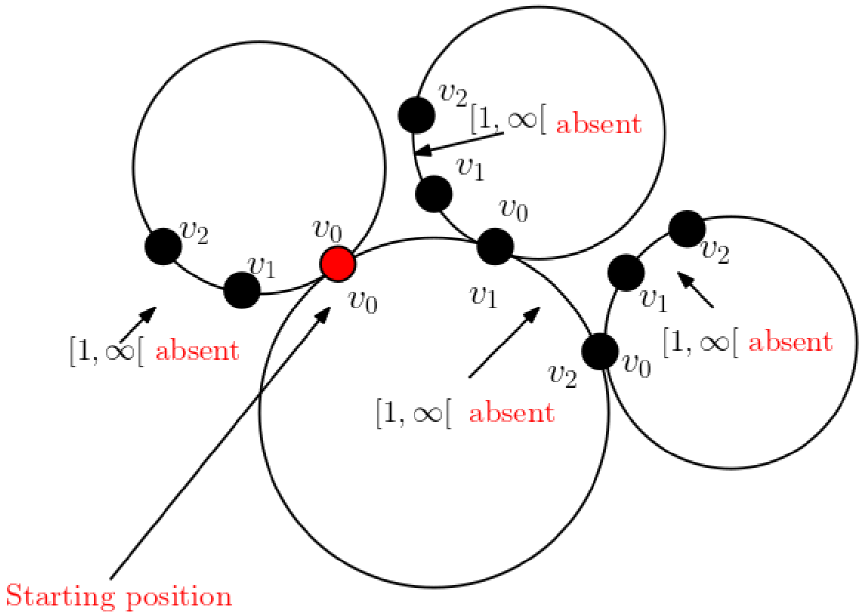

Figure 2: The difficult Cactus-star C_n

Upper bound

In this section, we will give an algorithm named **Cactus-Star-Exploration** which allows to explore any dynamic graph based on a cactus-star in at most $3N$ time units. Let us describe the algorithm before giving its complexity.

The algorithm **Cactus-Star-Exploration** is very simple. The agent will first go to a certain vertex of the central cycle and wait for $n-1$ time. Then it explores the leaf (using the **Explore-with-return** algorithm **c.f Theorem 1**), then goes to the next vertex on the central cycle in a clockwise direction (if the dynamics of the graph allows it), and so on until it visits all the vertices. The choice of the starting leaf is made according to the dynamics of the central cycle. The agent chooses the leaf such that it will never be blocked by the absence of an edge of the central ring. The existence of such a leaf is proved in the proof of the following theorem.

Theorem 5. For any integer $N \geq 3$, and for any contently connected dynamic graph based on a complete cactus-star C_n of size N, there exists an agent (algorithm) able to explore with return this dynamic graph in a time at most $3N - 4\sqrt{N} + 1$ time units, with $N = n^2$ the number of vertices of the dynamic graph and n the size of each cycle.

Proof. Let $N \geq 3$ and C_n be a constantly connected dynamic graph based on C_n. To know the best starting position of the agent, we assume that there is an agent on each vertex of the central cycle. The agents execute the same algorithm and the starting position of the agent that first finishes the exploration of the graph will be the starting position of our agent. To get to this vertex before starting the exploration, the agent pays at most *n-1* time units (**cf. Theorem 2**).

To prove the theorem, we show that the exploration time of the first agent who manages to explore the dynamic graph will respect the bound announced by the theorem. To determine the exploration time of the fastest agent, we assume that there is an agent on each vertex of the central ring and that they all start the exploration at the same time after $n-1$ time units. Note that the agents execute the same algorithm, and each is on a vertex (distinct from the others) of the central ring. After every $3n - 4$ (**cf. Theorem 1**) time units, the agents that have never been blocked end up visiting a new leaf and are all on a vertex of the central ring (we cannot have two agents never blocked on the same vertex). As the dynamic graph is constantly connected, so after every $3n - 4$ time units, at most one agent can be blocked for the first time, because at most one edge can be absent on the central cycle per time unit and that the agents never blocked move at the same time on the central ring. This means that there exists at least one agent who will succeed in exploring the dynamic graph without ever being blocked on the central cycle. Note that this agent pays $3n - 4$ units of time to explore each leaf. This proves that there is at least one agent who will succeed in exploring the dynamic graph in $(3n-4)*(n-1)+2n-2 = 3N - 5\sqrt{N} + 2$. To return to its starting position, this agent will pay at most *n-1* other time units. Which proves the announced bound and concludes the proof.

Conclusion

In this paper, we studied the time complexity for exploring with return constantly connected dynamic graphs based on complete cactus-star C_n, under the assumption that the agent knows the dynamics of the graph. We proved that to explore with return constantly connected dynamic graphs based on C_n, $2(N - \sqrt{N})$ time units are necessary and we give an algorithm which explores with return any dynamic graph based on cactus-star in $3N - 4\sqrt{N} + 1$ time units. With *N* the size on the dynamic graph. This study opens several perspectives. In the short term, it would be interesting to see if it is possible to find a dynamic graph more difficult to explore for any agent to have a better lower bound. This in order to reduce the gap between the two bounds. A further perspective is to consider the exploration problem of dynamic graphs using more than one agent, assuming standard models of communication between the agents. The objective would be to study whether dynamic graph exploration can be performed more efficiently by using more than one agent.

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