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On Lebesgue-like Corresponding Function Space of A Banach Space in Connection with Köthe-Toeplitz Duals of Some Generalized Cesàro Difference Sequence Spaces and Asymptotically Fixed Point Property

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Abstract In this study, we investigate the fixed point property for asymptotically nonexpansive mappings on some large classes of closed, bounded and convex subsets in some $L_1[0,1]$-like Banach spaces which are in connection with the corresponding function spaces of the Köthe-Toeplitz duals of some generalized Cesàro difference sequence spaces. In 1970, Cesàro Sequence Spaces was introduced by Shiue. In 1981, Kızmaz defined difference sequence spaces for ℓ^∞, c_0 and c. Then, in 1983, Orhan introduced Cesàro Difference Sequence Spaces. It has been noticed that their Köthe-Toeplitz duals are contained in ℓ^1 and fails the fixed point property for nonexpansive mappings. Very recently Nezir and Mustafa studied these spaces and they saw that there exist large classes of closed bounded and convex subsets with fixed point property for nonexpansive mappings as they wanted to do an analogue study of Goebel and Kuczumow's from 1979 where Goebel and Kuczomow studied the same question in larger space ℓ^1. It is notable that after Goebel and Kuczumow's study, Kaczor and Prus wanted to find large classes of closed bounded and convex subsets with fixed point property for asymptotically nonexpansive mappings; then indeed they gave positive answer in ℓ^1. In this study, we study Kaczor and Prus analogy in the Lebesgue-like corresponding function spaces of the spaces in connection with the Köthe-Toeplitz duals of generalized Cesàro difference sequence spaces and show that affine asypmtotically nonexpansive mappings on some large classes of closed, bounded and convex subsets of those spaces have fixed points.

Keywords Fixed point property, Asymptotically nonexpansive mappings, Köthe-Toeplitz duals, Cesaro difference sequences, Lebesgue-like Banach space

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1. Introduction and preliminaries

A Banach space $(X, \|.\|)$ said to have the fixed point property for nonexpansive mappings if every non-expansive self mappings defined on any non-empty closed, bounded and convex subset of the Banach space has a fixed point. As it is well known that nonexpansive mappings are the mappings which do not increase distances; that is, for a self mapping T defined on a subset C of a Banach space if $\| T(x) - T(y) \| \leq \| x - y \|, \text{for all } x, y \in C$ then we say T is nonexpansive mapping. It has been seen that most classical Banach spaces fail the fixed point property and especially there is a fact that if a Banach space is a non-reflexive Banach lattice then it fails the fixed point property if it contains either an isomorphic copy of c_0, Banach space of scalar sequences converging to 0, or an isomorphic copy of ℓ^1, Banach space of absolutely summable scalar sequences. It is a well known fact that there is a strong relation between reflexivity and fixed point property. Moreover, researchers have been interested in checking if a nonreflexive Banach space can be renormed to have the fixed point property to see how the fixed point property is related with reflexivity. In fact, the first example of a nonreflexive Banach space which is renormable to have the fixed point property was given by Lin [12]. Lin showed this fact by setting an equivalent norm on Banach space of absolutely summable scalar sequences, ℓ^1. Because of sharing many common properties, it is natural to ask if , Banach space of scalar sequences converging to 0, c_0 can be renormed to have the fixed point property for non-expansive mappings as another well known classical non-reflexive Banach space. Hernandes-Lineares and Japón [14] obtained an example for the class of nonreflexive Banach spaces which can be renormed to have the fixed point property for affine nonexpansive mappings and their space was the Banach space of Lebesgue integrable functions on [0,1], $L_1[0,1]$. It can be said that all these works are inspired by the work of Goebel and Kuczumow [9]. Goebel and Kuczumow showed that there exists very large class of non-weakly compact, closed, bounded and convex subsets of ℓ^1 respect to weak* topology of ℓ^1 with fixed point property for non-expansive mappings. Goebel and Kuczumow showed that there exists very large class of non-weakly compact, closed, bounded and convex subsets of ℓ^1 respect to weak* topology of ℓ^1 with fixed point property for non-expansive mappings. Later, Kaczor and Prus [10] investigated if similar result could be done for asymptotically nonexpansive mappings and they saw that there exists a large class of closed, bounded, convex subsets in ℓ^1 with fixed point property for affine asymptotically non-expansive mappings.

Thus, in this study, we work on Kaczor and Prus analogy for some Banach space contained in Lebesgue space $L_1[0,1]$. The spaces we consider are some spaces in structural connection with the corresponding function spaces of the Köthe-Toeplitz duals of some generalized Cesàro Difference Sequence Space which contained in Lebesgue space $L_1[0,1]$. We show that there exists a very large class of closed, bounded and convex subsets of those spaces with the fixed point property for asymptotically non-expansive mappings under affinity condition.

We recall that the Cesàro sequence spaces

$$\mathrm{ces}_p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

were introduced by Shiue [19] in 1970, where $1 \leq p < \infty$. It has been shown that $\ell^p \subset \mathrm{ces}_p$ for $1 < p \leq \infty$. Moreover, it has been shown that Cesàro sequence spaces ces_p for $1 < p < \infty$ are seperable reflexive Banach spaces. Furthermore, it was also proved by Cui [3], Cui, Hudzik and Li [4], and Cui, Meng, and Pluciennik [5] that Cesàro sequence spaces ces_p for $1 < p < \infty$ have the fixed point property.

Later, in 1981, Kızmaz [11] introduced difference sequence spaces for ℓ^∞, cand c_0 where they are the Banach spaces of bounded, convergent and null sequences $x = (x_n)_n$, respectively. As it is seen below, his definitions for these spaces were given using difference operator applied to the sequence x, $\triangle x = (x_k - x_{k+1})_k$.

$$\ell^\infty(\triangle) = \{x = (x_n)_n \subset \mathbb{R} \mid \triangle x \in \ell^\infty\},$$

$$\mathrm{c}(\triangle) = \{x = (x_n)_n \subset \mathbb{R} \mid \triangle x \in \mathrm{c}\},$$

$$\mathrm{c}_0(\triangle) = \{x = (x_n)_n \subset \mathbb{R} \mid \triangle x \in \mathrm{c}_0\}.$$

Kızmaz also investigated Köthe-Toeplitz Duals and some properties of these spaces. Furthermore, Cesàro sequence spaces X^p of non-absolute type were defined by NgPeng-Nung and LeePeng-Yee [17] as follows:

$$X^p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| < \infty \right\},$$

where $1 \leq p < \infty$. They prove that X^p is linearly isomorphic and isometric to ℓ^p for $1 \leq p \leq \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail. A survey study on Cesàro sequence spaces studying fixed point theory can be seen in the study by Chen, Cui, Hudzik, and Sims [2].

An important study by Orhan [18] was done, and he introduced Cesàro Difference Sequence Spaces by the following definitions:

$$C_p = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right| < \infty \right\},$$

where $1 \le p < \infty$ and $\triangle x_k = x_k - x_{k+1}$ for each $k \in \mathbb{N}$. He noted that their norms are given as below for any $x = (x_n)_n$:

$$\|x\|_p^* = |x_1| + \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p \right)^{1/p} \text{ and } \|x\|_\infty^* = |x_1| + \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|,$$

respectively.

Orhan showed that there exists a linear bounded operator $S: C_p \to C_p$ for $1 \le p \le \infty$ such that Köthe-Toeplitz β −Duals of these spaces are given respectively as follows:

$$S(C_p)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^q\} \text{ where } 1 < p < \infty \text{ and } q = \frac{p}{p-1},$$

$$S(C_1)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^\infty\} \text{ and}$$

$$S(C_\infty)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^1\}.$$

It is easily deduced that these spaces also have the similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail. Note also that Köthe-Toeplitz Dual for $p = \infty$ case in Orhan's study and ℓ^∞ case in Kızmaz study coincides. So for $m = 1$, considering the Köthe-Toeplitz Dual, we can write

$$\Upsilon_1 := S(C_\infty)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (\mathrm{n}a_n)_n \in \ell^1\}$$

$$= \left\{ a = (a_k)_k \subset \mathbb{R} \ : \ \|a\|_\Delta = \sum_{k=1}^{\infty} k|a_k| < \infty \right\}$$

such that $\Upsilon_1 \subset \ell^1$ and the corresponding function space for this dual can be given as below:

$$\Sigma_1 := \left\{ \begin{matrix} f:[0,1] \to \mathbb{R} \\ \text{measurable} \end{matrix} : \ \|f\| = \int_0^1 \mathrm{t}|f(t)|dt < \infty \right\}.$$

Here note that $L_1[0,1] \subset \Sigma_1$ and Υ_1 is the space when counting measure is used for Σ_1.

Et and Çolak [6] generalized the spaces introduced in Kızmaz's work [11] in the following way for $m \in \mathbb{N}$.

$$\ell^\infty(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \ell^\infty\},$$

$$\mathrm{c}(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \mathrm{c}\},$$

$$\mathrm{c}_0(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \mathrm{c}_0\}$$

where $\triangle x = (\triangle x_k) = (x_k - x_{k+1})_k$, $\triangle^0 x = (x_k)_k$, $\triangle^m x = (\triangle^m x_k) = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1})_k$ and $\triangle^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$ for each $k \in \mathbb{N}$.

Also, Et [7] and Tripathy et. al. [20] generalized the space introduced by Orhan [18] in the following way for $m \in \mathbb{N}$.

$$X^p(\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle^m x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^\infty(\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle^m x_k \right| < \infty \right\},$$

Then, it is seen that that Köthe-Toeplitz Dual for $p = \infty$ case in Et's study [7] and ℓ^∞ case in Et and Çolak study [6] coincides such that Köthe-Toeplitz Dual was given as below for any $m \in \mathbb{N}$.

$$\Upsilon_m := \{a = (a_n)_n \subset \mathbb{R} : (n^m a_n)_n \in \ell^1\} = \left\{ a = (a_k)_k \subset \mathbb{R} : \ \|a\| = \sum_{k=1}^{\infty} k^m |a_k| < \infty \right\}.$$

Note that $\Upsilon_m \subset \ell^1$ for any $m \in \mathbb{N}$.

One can see that corresponding function space for these duals can be given as below:

$$\Sigma_m := \left\{ \begin{matrix} f:[0,1] \to \mathbb{R} \\ \text{measurable} \end{matrix} : \|f\| = \int_0^1 t^m |f(t)| dt < \infty \right\}.$$

Note that $L_1[0,1] \subset \Sigma_m$ and Υ_m is the space when counting measure is used for Σ_m.

We also note that under affinity condition, Goebel and Kuczumow analogy for Σ_1 was studied by Nezir and Oymak [16] but without the need for affinity condition, in the study by Nezir and Mustafa [15] study, Goebel and Kuczumow analogy was studied. As noted in the study by Nezir and Mustafa [15], one can consider another space which is in structural connection with Σ_1. Nezir and Mustafa answered some fixed point theory oriented questions for the Köthe-Toeplitz dual of a Cesàro difference sequence space, the aforementioned space which has structural connection with Σ_1 and the corresponding function spaces for both. In fact, Nezir and Mustafa showed those two function spaces fail the weak fixed point property for isometries and contractive mappings, that is they find weakly compact and convex subsets and fixed point free isometric and contractive self mappings defined on those subsets. Moreover, they study Goebel and Kuczumow analogy for the corresponding sequence spaces of those function spaces when the counting measure is used. Here we will be concentrated on the aforementioned structural related space with Σ_m for any $m \in \mathbb{N}$. The space we consider will be defined as below for any $m \in \mathbb{N}$.

$$\mathcal{M}_m := \left\{ \begin{matrix} f:[0,1] \to \mathbb{R} \\ \text{measurable} \end{matrix} : \|f\| = \int_0^1 \frac{|f(t)|}{t^m} dt < \infty \right\}.$$

As we have already stated, in this study, for any $m \in \mathbb{N}$, we consider Kaczor and Prus [10] analogy for $\mathcal{M}_m$ which is contained in the Lebesgue space $L_1[0,1]$. We show that there exists a large class of closed, bounded and convex subsets of $\mathcal{M}_m$ with fixed point property for affine asymptotically $\|\,.\,\|^{\sim}$-nonexpansive mappings.

Now we provide some preliminaries before giving our main results.

Definition 1.1. *Let $(X, \|\cdot\|)$ be a Banach space and C is a non-empty closed, bounded, convex subset.*

1. If $T: C \to C$ is a mapping such that for all $\lambda \in [0,1]$ and for all $x, y \in C$, $T\big((1-\lambda)x + \lambda\, y\big) = (1-\lambda)T(x) + \lambda\, T(y)$ then T is said to be an affine mapping.

2. If $T: C \to C$ is a mapping such that $\| T(x) - T(y) \| \leq \| x - y \|$, for all $x, y \in C$ then T is said to be a nonexpansive mapping. Also, if for every nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for nonexpansive mappings [fpp(ne)].

3. If $T: C \to C$ is a mapping such that there exists a sequence of scalars $(k_n)_{n\in N}$ decreasingly approach to 1 and $\| T^n(x) - T^n(y) \| \leq k_n \| x - y \|$, for all $x, y \in C$ and for all $n \in \mathbb{N}$ then T is said to be an asymptotically nonexpansive mapping.

Also, if for every asymptotically nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for asymptotically nonexpansive mappings [fpp(ane)].

Remark 1.1. *In 1979, Goebel and Kuczumow* [9] *showed there exists a large class of closed, bounded and convex subsets of ℓ^1 using a key lemma they obtained. Their lemma says that if $\{x_n\}$ is a sequence in ℓ^1 converging to x in weak-star topology, then for any $y \in \ell^1$,*

$$r(y) = r(x) + \|y - x\|_1 \quad where \quad r(y) = \limsup_n \|x_n - y\|_1 \ .$$

We will call this fact $\therefore$.

The analogue of this lemma for $L_1[0,1]$ is observed via the result in Brezis and Lieb [1]. Note that Hernández-Linares pointed this fact in his Ph.D. thesis, Hernández-Linares [13], written under supervision of Maria Japon Pineda. Now we provide the lemma which is deduced by their results and will be key for our results in this section.

Lemma 1.1. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued measurable functions which are uniformly bounded in $L_1[0,1]$. Assume that f_n converges to an $f \in L_1[0,1]$ pointwise almost everywhere (a.e.). Then for any $g \in L_1[0,1]$,*

$$S(g) = S(f) + \|f - g\|_1 \quad where \quad S(g) = \limsup_n \|f_n - g\|_1 \ .$$

Since $\mathcal{M}_m$ is contained in the Lebesgue space $L_1[0,1]$ and in fact $L_1[0,1]$ is isometrically isomorphic to $\mathcal{M}_m$, for any $m \in \mathbb{N}$, for $\mathcal{M}_m$ the following lemma can be given as straight and quick result.

Lemma 1.2. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued measurable functions which are uniformly bounded in $\mathcal{M}_m$ for any $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$* and *suppose that f_n converges to an $f \in \mathcal{M}_m$ pointwise almost everywhere (a.e.). Then for any $m \in \mathbb{N}$ and any $g \in \mathcal{M}_m$, $S(g) = S(f) + \|f - g\|$ where $S(g) = \limsup_n \|f_n - g\|$.*

2. Main Result

In this section, for any $m \in \mathbb{N}$, we consider Kaczor and Prus [10] analogy for $\mathcal{M}_m$, a space structural related with the corresponding function space of the Köthe-Toeplitz Dual of a generalized Cesàro Difference Sequence Space. We show that for any $m \in \mathbb{N}$, there exists a large class of closed, bounded and convex subsets of $\mathcal{M}_m$ with fixed point property for affine asymptotically nonexpansive mappings.

Now, we consider the following class of closed, bounded and convex subsets. Note that here we will be using the ideas similar to those in the section 2 of Ph.D. thesis of Everest [8], written under supervision of Chris Lennard, where Everest firstly provides Goebel and Kuczumow's proofs in detailed.

So we demonstrate examples of these subsets and provide a theorem related with each of them.

Example 2.1. *Fix $b \in (0,1)$ and $m \in \mathbb{N}$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1, f_2 := b\, e_2$, and $f_n := e_n$, for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n := (n+1)t^{n+m}$, $\forall n \in \mathbb{N}$. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of $\mathcal{M}_m$ by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \beta_n f_n : \forall n \in \mathbb{N}, \quad t_n \geq 0 \;\; and \;\; \sum_{n=1}^{\infty} \beta_n = 1 \right\}.$$

Example 2.2. *Fix $b \in (0,1)$ and $m \in \mathbb{N}$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1, f_2 := b\, e_2$, and $f_n := e_n$, for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n := \frac{t^m n e^{nt}}{(e^n - 1)}$, $\forall n \in \mathbb{N}$. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of $\mathcal{M}_m$ by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \beta_n f_n : \forall n \in \mathbb{N}, \quad t_n \geq 0 \;\; and \;\; \sum_{n=1}^{\infty} \beta_n = 1 \right\}.$$

Example 2.3. *Fix $b \in (0,1)$ and $m \in \mathbb{N}$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1, f_2 := b\, e_2$, and $f_n := e_n$, for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n := \frac{t^m n e^{nt}}{(e-1)} \chi_{[0,\frac{1}{n}]}$, $\forall n \in \mathbb{N}$, where χ is the characteristics funtion. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of $\mathcal{M}_m$ by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \beta_n f_n : \forall n \in \mathbb{N}, \quad t_n \geq 0 \;\; and \;\; \sum_{n=1}^{\infty} \beta_n = 1 \right\}.$$

Example 2.4. *Fix $b \in (0,1)$ and $m \in \mathbb{N}$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1, f_2 := b\, e_2$, and $f_n := e_n$, for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n := \frac{t^m 4n}{\pi(1+n^2t^2)} \chi_{[0,\frac{1}{n}]}$, $\forall n \in \mathbb{N}$, where χ is the characteristics funtion. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of $\mathcal{M}_m$ by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \beta_n f_n : \forall n \in \mathbb{N}, \quad t_n \geq 0 \;\; and \;\; \sum_{n=1}^{\infty} \beta_n = 1 \right\}.$$

Theorem 2.1. *For any $b \in (0,1)$ and any $m \in \mathbb{N}$, each set $E^{(m)}$ defined as in the examples above has the fixed point property for affine asymptotically $\|.\|$-nonexpansive mappings.*

Proof. *Fix $b \in (0,1)$ and $m \in \mathbb{N}$*. Let $T: E^{(m)} \to E^{(m)}$ be an affine asymptotically nonexpansive mapping. Then, since T is affine, by Lemma 1.1.2 in the Ph.D. thesis of Everest [8], there exists a sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}} \in E$ such that $\left\|Tx^{(n)} - x^{(n)}\right\| \underset{n}{\to} 0$. Without loss of generality, passing to a subsequence if necessary, there exists $x \in S(C_\infty)^\beta$ such that $x^{(n)}$ converges to x in weak* topology. Then, by Goebel Kuczumow analog fact ∴ given in the last part of the previous section, we can define a function $s: S(C_\infty)^\beta \to [0, \infty)$ by

$$s(y) = \limsup_n \|x^{(n)} - y\| \quad , \forall y \in \mathcal{M}_m \text{ and so } s(y) = s(x) + \|x - y\| \ , \ \forall y \in \mathcal{M}_m.$$

Now define the weak* closure of the set $E^{(m)}$ as it is seen below.

$$W := \overline{E^{(m)}}^{w^*} = \left\{ \sum_{n=1}^{\infty} t_n f_n \colon \text{each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n \leq 1 \right\}$$

Since T is asymptotically nonexpansive mapping, there exists a decreasing sequence $(\mathrm{k}_n)_{n\in N}$ in $[1,\infty)$ converging to 1 such that $\forall x, y \in E^{(m)}$ and $\forall n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \leq \mathrm{k}_n \| x - y \|.$$

Case 1: $x \in E^{(m)}$.

Fix $r \in \mathbb{N}$. Then, we have $s(T^r x) = s(x) + \|T^r x - x\|$ and $\forall n \in \mathbb{N}$,

$$\begin{aligned} s(T^r x) &= \limsup_n \|T^r x - x^{(n)}\| \\ &\leq \limsup_n \|T^r x - T^r(x^{(n)})\| + \limsup_n \|x^{(n)} - T^r(x^{(n)})\| \qquad (2.1) \\ &\leq \mathrm{k}_r \limsup_n \|x - x^{(n)}\| + \limsup_n \|x^{(n)} - T^r(x^{(n)})\| \\ &\leq \mathrm{k}_r \limsup_n \|x - x^{(n)}\| + \limsup_n \sum_{j=1}^{r} \|T^{j-1}(x^{(n)}) - T^j(x^{(n)})\| \\ &\leq \mathrm{k}_r \limsup_n \|x - x^{(n)}\| + \limsup_n \sum_{j=1}^{r} \mathrm{k}_{j-1} \|x^{(n)} - T(x^{(n)})\| \\ &= \mathrm{k}_r s(x). \end{aligned}$$

Therefore, $\|T^r x - x\| \leq (\mathrm{k}_r - 1) s(x)$ and so by taking limit as $r \to \infty$, we have $\lim_r \|T^r x - x\| = 0$ but then since $\lim_r \|T T^r x - Tx\| \leq \lim_r \mathrm{k}_1 \|T^r x - x\| = 0$, $\lim_r \|T^{r+1} x - Tx\| = 0$ and so $T^r x$ converges to x and Tx. Thus, by the uniqueness of limits $Tx = x$.

Case 2: $x \in W \backslash E^{(m)}$.

Then, x is of the form $\sum_{n=1}^{\infty} \gamma_n f_n$ such that $\sum_{n=1}^{\infty} \gamma_n < 1$ *and* $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$.

Define $\delta := 1 - \sum_{n=1}^{\infty} \gamma_n$ and next define

$$\mathrm{h} := (\gamma_1 + \delta) f_1 + \sum_{n=2}^{\infty} \gamma_n f_n.$$

Then, $\|\mathrm{h} - x\| = \|b\delta e_1\| = b\delta$.

Now fix $y \in E^{(m)}$ of the form $\sum_{n=1}^{\infty} t_n f_n$ such that $\sum_{n=1}^{\infty} t_n = 1$ with $t_n \geq 0$, $\forall n \in \mathbb{N}$.

Then,

$$\begin{aligned} \|\mathrm{y} - x\| &= \left\| \sum_{k=1}^{\infty} \beta_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k \right\| = \left\| \sum_{k=1}^{\infty} \beta_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k \right\| \\ &= \left\| \sum_{k=1}^{\infty} (\beta_k - \gamma_k) f_k \right\| \\ &= \int_0^1 \frac{1}{t^m} \left| \sum_{k=1}^{\infty} (\beta_k - \gamma_k) f_k \right| dm = \int_0^1 \left| \sum_{k=1}^{\infty} (\beta_k - \gamma_k) \frac{1}{t^m} f_k \right| dm \\ &\geq \left| \int_0^1 \sum_{k=1}^{\infty} (\beta_k - \gamma_k) \frac{1}{t^m} f_k dm \right| \\ &= \left| \sum_{k=1}^{\infty} (\beta_k - \gamma_k) \right| \\ &= \left| 1 - \sum_{k=1}^{\infty} \gamma_k \right| \\ &= \delta \end{aligned}$$

Hence,

$$\|y-x\| \geq b\delta \geq \|h-x\|.$$

Next, we have the following.

$$\begin{aligned} s(h) = s(x) + \|h-x\| \leq s(x) + \|T^r h - x\| = s(T^r h) \\ &= \limsup_n \|T^r h - x^{(n)}\| \text{ then similarly to the inequality (2.1)} \\ &\leq \limsup_n \|T^r h - T^m(x^{(n)})\| + \limsup_n \|x^{(n)} - T^r(x^{(n)})\| \\ &\leq k_r \limsup_n \|h - x^{(n)}\| + \limsup_n \|x^{(n)} - T^r(x^{(n)})\| \\ &\leq k_r \limsup_n \|h - x^{(n)}\| + \limsup_n \sum_{j=1}^{r} \|T^{j-1}(x^{(n)}) - T^j(x^{(n)})\| \\ &\leq k_r \limsup_n \|h - x^{(n)}\| + \limsup_n \sum_{j=1}^{r} k_{j-1} \|x^{(n)} - T(x^{(n)})\| \\ &= k_r s(h). \end{aligned}$$

Hence, $s(h) \leq s(T^r h) \leq k_r s(h)$ and so taking limit as $r \to \infty$, we have $\lim_m s(T^r h) = s(h)$ since $\lim_r k_r = 1$. That is, $\lim_r s(x) + \|T^r h - x\| = s(x) + \|h-x\|$ which means $\lim_r \|T^r h - x\| = \|h-x\|$. (2.2)

Moreover, for any $y \in E^{(m)}$,

$$\begin{aligned} \|y-h\| &= \left\| \sum_{k=1}^{\infty} t_k f_k - (\gamma_1+\delta) f_1 - \sum_{k=2}^{\infty} \gamma_k f_k \right\| \\ &= \left\| \sum_{k=2}^{\infty} (t_k - \gamma_k) f_k - (\gamma_1 + \delta - t_1) f_1 \right\| \\ &= \int_0^1 \frac{1}{t^m} \left| \sum_{k=2}^{\infty} (t_k - \gamma_k) f_k - (\gamma_1 + \delta - t_1) f_1 \right| dm \\ &\leq \sum_{k=2}^{\infty} |t_k - \gamma_k| + b|\gamma_1 + \delta - t_1| \\ &= \sum_{k=2}^{\infty} |t_k - \gamma_k| + b\left|\gamma_1 + 1 - \sum_{k=1}^{\infty} \gamma_k - 1 + \sum_{k=2}^{\infty} t_k\right| \\ &\leq \sum_{k=2}^{\infty} |t_k - \gamma_k| + b \sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &= (1+b) \sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &= \frac{1+b}{1-b}(1-b) \sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &= \frac{1+b}{1-b}\left[b\delta - b\delta + (1-b)\sum_{k=2}^{\infty} |t_k - \gamma_k|\right] \\ &= \frac{1+b}{1-b}\left[b\big(1-(1-\delta)\big) - b\delta + (1-b)\sum_{k=2}^{\infty} |t_k - \gamma_k|\right] \\ &= \frac{1+b}{1-b}\left[b\big(1-(1-\delta)\big) + (1-b)\sum_{k=2}^{\infty} |t_k - \gamma_k| - b\delta\right] \end{aligned}$$

$$= \frac{1+b}{1-b}\left[b\left(\sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \gamma_k\right) + (1-\mathrm{b})\sum_{k=2}^{\infty} |t_k - \gamma_k| - b\delta\right]$$

$$\leq \frac{1+b}{1-b}\left[b\sum_{k=1}^{\infty} |t_k - \gamma_k| + (1-\mathrm{b})\sum_{k=2}^{\infty} |t_k - \gamma_k| - b\delta\right]$$

Hence,

$$\|\mathrm{y} - h\| \leq \frac{1+b}{1-b}\left[b|t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k| - b\delta\right] = \frac{1+b}{1-b}[\|\mathrm{y} - x\| - \|\mathrm{h} - x\|]$$

Now, fix $\varepsilon > 0$ and recall that $b \in (0,1)$. Then, we can choose $\mu(\varepsilon) := \frac{1-b}{1+b}\varepsilon \in (0, \infty)$ such that for any $y = \sum_{k=1}^{\infty} t_k f_k \in E^{(m)}$,

$$|\|\mathrm{y} - x\| - \|\mathrm{h} - x\|| \leq \|\mathrm{y} - x\| - \|\mathrm{h} - x\| < \mu.$$

Then, $\|\mathrm{y} - h\| < \frac{1+b}{1-b}\mu = \varepsilon$.

Hence, for every $\varepsilon > 0$, there exists $\mu = \mu(\varepsilon)$ such that if $|\|\mathrm{y} - x\| - \|\mathrm{h} - x\|| < \mu$ then $\|\mathrm{y} - h\| < \varepsilon$ so this implies for any sequence $(z_n)_n$ in $E^{(m)}$ with $\lim_n \|z_n - x\| = \|\mathrm{h} - x\|$ implies $\lim_n \|z_n - h\| = 0$. But then since in (2.2) we obtained $\lim_r \|T^r\mathrm{h} - x\| = \|\mathrm{h} - x\|$, we have $\lim_r \|T^r\mathrm{h} - h\| = 0$.

Furthermore,

$$\|h - Th\| \leq \lim_r \|T^r\mathrm{h} - h\| + \lim_r \|T^r\mathrm{h} - Th\|$$

$$\leq \mathrm{k}_1 \lim_r \|T^{r-1}\mathrm{h} - h\| = 0$$

Hence, $Th = h$ and so $E^{(m)}$ has fpp(ane.) as desired.

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