Available online www.jsaer.com

Journal of Scientific and Engineering Research, 2021, 8(7):1-11

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Boundedness theorems of non-autonomous stochastic differential systems driven by G-Brownian motion and mixed delays

Yuxin Zhu[1], Danhua He[1] *

[1] Department of Mathematics, Zhejiang International Studies University, China
*Corresponding author: danhuahe@126.com

Abstract Using Lyapunov functions, G-Ito formula and some relevant definitions and lemmas, the uniformly boundedness of non-autonomous stochastic differential systems driven by G-Brownian motion is analyzed. And we obtained some sufficient conditions of ultimately boundedness. Meanwhile, an example is given to verify the effectiveness of the obtained results.

Keywords stochastic differential systems; G-Brownian motion; mixed delays; boundedness

1. Introduction

Stochastic phenomena is ubiquitous and some inevitable stochastic factors even play a decisive role. Since deterministic dynamic systems cannot describe phenomena with stochastic factors accurately, stochastic dynamic systems have become an effective tool to make up for the deficiencies. Since 1990s, the Bremen research group which led by Ludwig Arnold in Germany consummated the linear theorem of finite-dimensional stochastic dynamic systems. And with Flandoli from Italy, who established the basic concept and framework of stochastic infinite dimensional dynamical systems, the research on stochastic dynamical systems has been developed rapidly. Correspondingly, in China, the research on stochastic dynamic systems has also achieved some leading results in the world. At present, the results of stochastic differential equations, stochastic partial differential equations and the generated stochastic dynamic systems involve Gaussian noise, Lévy noise, impulse noise, fractional Brownian motion and other non-continuous driving noise, see [1,2]. Based on the development of theory, a lot of significant results have been used in many fields, such as material mechanics, biology, nervous system, power system, control engineering and social sciences.

Time-delay is an indispensable factor while studying stochastic differential systems. And it’s an extremely common phenomenon in nature, whether the dynamic system under high-speed or low-speed motion, or under human-computer interaction. Time-delay system, which has delays at one or several places in the control system during the signal transmission, the evolution not only depends on the current condition, but the past. Some results show that even tiny time-delay can lead to complex dynamics of the system. Therefore, the study of time-delay systems has both theoretic and practical significance. In addition, time-delay systems often have discrete delays and distributed delays at the same time. Thus, it is necessary to consider the two cases together, which is called stochastic differential systems with mixed delays.

Boundedness is a quite important characteristic of stochastic differential systems. During the last decades, a lot of results have been reported, like the ultimate boundedness of nonlinear switched systems, see [3], the boundedness of non-autonomous delay stochastic differential systems in [4-6], the ultimate boundedness of non-autonomous impulsive dynamical complex network in [7], and the boundedness for a class of impulsive

Caputo fractional differential systems, see [8-9], the boundedness of stochastic differential systems with or without delays, see [10-12], and the boundedness of non-autonomous stochastic differential systems with Lévy noise and mixed delays, see [13].

However, there are few studies about boundedness of non-autonomous stochastic differential systems driven by G-Brownian motion with distributed delays. Therefore, based on above statement, this article aims to discuss the ultimate boundedness theorem of non-autonomous stochastic differential systems with mixed delays driven by G-Brownian motion. Using Lyapunov functions, G-Ito formula and some relevant definitions and lemmas, taking expectations, some sufficient conditions of globally exponentially ultimately boundedness are obtained.

2. Preliminaries

Followings are some relevant notations, definitions and lemmas.

$\mathbb{N}:=\{1,2,3,\ldots\}$. $\mathbb{R}_+:=[0,\infty)$. $\mathbb{R}_{t_0}:=[t_0,\infty)$. $\|u\|$:the Euclidean norm of a vector u . $\lambda_{\min}(\cdot)[\lambda_{\max}(\cdot)]$: the smallest (largest) eigenvalue of a symmetric matrix. $\overline{\mathrm{E}}$:the G-expectation of stochastic process. $C\left[[-\tau,0],\mathbb{R}^n\right]$:the family of bounded continuous functions ϕ from $[-\tau,0]$ to $\mathbb{R}^n$ with the norm $\|\phi\|_\infty=\sup_{-\tau\le\theta\le0}\|\phi(\theta)\|$. $C^b_{F_{t_0}}\left[[-\tau,0],\mathbb{R}^n\right]$:the family of bounded F_{t_0} -measurable, $C\left[[-\tau,0],\mathbb{R}^n\right]$ -valued random variable ϕ , satisfying $\overline{\mathrm{E}}\|\phi\|_\tau^p<\infty$. $C^{1,2}\left(\mathbb{R}_{t_0}\times\mathbb{R}^n,\mathbb{R}_+\right)$:the family of all nonnegative functions $V(t,u)$ from $\mathbb{R}_{t_0}\times\mathbb{R}^n$ to $\mathbb{R}_+$, which once continuously differentiable in $t\in\mathbb{R}_+$ and twice in $u\in\mathbb{R}^n$.

In order to prove the boundedness theorem, giving the following definitions and lemmas.

Definition 2.1 (*[13]*). Model (3-1) is said to be pth moment globally asymptotically stable if there exist positive constant λ, z_1 ,such for $\forall\phi\in C^b_{F_{t_0}}\left[[-\tau,0],\mathbb{R}^n\right]$,

$$\overline{\mathrm{E}}\left\|x(t;t_0,\phi)\right\|^p\le z_1\overline{\mathrm{E}}\|\phi\|_\tau^p\, e^{-\lambda\int_{t_0}^t\beta(s)ds}, p\ge2, t\ge t_0.$$

Definition 2.2 (*[14]*). Model (3-1) is said to be pth moment globally exponentially ultimately bounded if there exist positive constant λ, z_1, z_2 ,such for $\forall\phi\in C^b_{F_{t_0}}\left[[-\tau,0],\mathbb{R}^n\right]$,

$$\overline{\mathrm{E}}\left\|x(t;t_0,\phi)\right\|^p\le z_1\overline{\mathrm{E}}\|\phi\|_\tau^p\, e^{-\lambda\int_{t_0}^t\beta(s)ds}+z_2, p\ge2, t\ge t_0.$$

Lemma 2.3 (*[15]*). For $x_i\ge0,\alpha_i>0$ and $\sum_{i=1}^n\alpha_i=1$,

$$\prod_{i=1}^n x_i^{\alpha_i}\le\sum_{i=1}^n\alpha_i x_i.$$

Lemma 2.4 (*[16]*)**.** For each $\eta \in M_G^{p,0}([0,T])$, the Bochner integral and Ito integral are defined by $\int_0^T \eta_t(\omega)dt = \sum_{i=0}^{K-1} \xi_{t_i}(\omega)(t_{i+1}-t_i)$ and $I_{[0,T]}(\eta) = \int_0^T \eta_t dB_t := \sum_{i=0}^{K-1} \xi_i (B_{t_{i+1}} - B_{t_i})$, respectively.

Lemma 2.5 (*[17]*)**.** For each $\eta \in M_G^{1,0}([0,T])$, a map $Q_{[0,T]}(\eta): M_G^{1,0}([0,T]) \to L_G^1(\Omega_T)$ is defined by $Q_{[0,T]}(\eta) = \int_0^T \eta_t dB_t := \sum_{i=0}^{K-1} \xi_i \left(\langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i}\right)$.

Lemma 2.6 (*[18]*)**.** Let $k: \mathbb{R}^n \to \mathbb{R}^n$ borel measurable functions. Assumpt there is a constant $\rho \in (0,1)$, for all $y \in \mathbb{R}^n$, such that

$$|k(y(s))| \le \rho |y|.$$

3. Main Result

Consider the following stochastic differential systems driven by G-Brownian motion with mixed delays.

$$\begin{cases} du(t) = f(t,u(t),u(t-\tau),\int_{t-\tau}^{t} k(u(s))ds)dt + g(t,u(t),u(t-\tau))d\langle B\rangle_t + \sigma(t,u(t),u(t-\tau))dB_t, t \ge t_0 \\ u(t_0+s) = \phi(s), -\tau \le s < 0 \end{cases}, \tag{3-1}$$

where $(\langle B\rangle_t)_{t\ge t_0}$ denotes the quadratic variation process of the G-Brownian motion $(B_t)_{t\ge t_0}$, $f,g,\sigma \in M_G^p([0,T];\mathbb{R}^n)$; the initial value $\phi(s) \in C_{F_{t_0}}^b\left[[-\tau,0],\mathbb{R}^n\right]$. Assume that for any $\phi(s) \in C_{F_{t_0}}^b\left[[-\tau,0],\mathbb{R}^n\right]$, there exists at least one solution of system (3-1).

Theorem 3.1. Suppose that there exist a function $V(t,u) \in C^{1,2}(\mathbb{R}_{t_0} \times \mathbb{R}^n, \mathbb{R}_+)$ and constants $\beta > 0, A_6 \ge 0$ and $A_j > 0(j=0,1,\ldots,5)$ with $A_1A_3 > A_2A_4e^{\lambda\beta}$, such that

(i) For all $(t,u) \in \mathbb{R}_{t_0} \times \mathbb{R}^n$,

$$A_1\|u\|^p \le V(t,u) \le A_2\|u\|^p,$$

(ii) For all $(t,u) \in \mathbb{R}_{t_0} \times \mathbb{R}^n$,

$$LV(t,u(t)) \le \beta(t)\left[-A_3V(t,u(t)) + A_4V(t,u(t-\tau)) + A_5\int_0^\tau \rho V(u(t-s))ds + A_6\right],$$

where $\beta(t)$ is a positive integral function and satisfies $\beta(t+\tau)\le A_0\beta(t)\beta(\tau)$, $\sup_{t\ge t_0}\int_{t-\tau}^{t}\beta(s)ds\le\bar{\beta}$ and $\lim_{t\to\infty}\int_{t_0}^{t}\beta(t)ds=\infty.$ Then system (3-1) is pth moment globally exponentially ultimately bounded with the bound $b=\frac{A_6}{\lambda A_1},$ where the constant λ is determined by the following inequality

$$\lambda A_2-A_1A_3+A_0A_2A_4\beta(\tau)e^{\lambda\bar{\beta}}+\rho A_0A_2A_5\int_0^{\tau}\beta(s)e^{\lambda\bar{\beta}}ds<0. \tag{3-2}$$

Proof. Applying the G-Ito formula to $e^{\lambda\int_{t_0}^{t}A(s)ds}V(t,u(t)),$

$$\begin{aligned}
&d\left(e^{\lambda\int_{t_0}^{t}\beta(s)ds}V(t,u(t))\right)\\
&=e^{\lambda\int_{t_0}^{t}\beta(s)ds}\left[\lambda\beta(t)V(t,u(t))+V_t(t,u(t))+\left\langle V_u(t,u(t)),f(t,u(t),u(t-\tau),\int_{t-\tau}^{t}k(u(s)))\right\rangle\right]dt\\
&+e^{\lambda\int_{t_0}^{t}\beta(s)ds}\langle V_u(t,u(t)),\sigma(t,u(t),u(t-\tau))\rangle dB_t\\
&+e^{\lambda\int_{t_0}^{t}\beta(s)ds}\langle V_u(t,u(t)),g(t,u(t),u(t-\tau))\rangle d\langle B\rangle_t\\
&+\frac{1}{2}e^{\lambda\int_{t_0}^{t}\beta(s)ds}\langle V_{uu}(t,u(t))\sigma(t,u(t),u(t-\tau)),\sigma(t,u(t),u(t-\tau))\rangle d\langle B\rangle_t.
\end{aligned}$$

Integral both sides from t_0 to t,

$$\begin{aligned}
e^{\lambda\int_{t_0}^{t}\beta(s)ds}V(t,u(t))-V(t_0,u(t_0))&=\int_{t_0}^{t}e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\left[\lambda\beta(\mu)V(\mu,u(\mu))+LV(\mu,u(\mu))\right]d\mu\\
&+\int_{t_0}^{t}e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\langle V_u(\mu,u(\mu)),\sigma(\mu,u(\mu),u(\mu-\tau))\rangle dB_\mu\\
&+M_t^{t_0},
\end{aligned}$$

where

$$\begin{aligned}
M_t^{t_0}&=\int_{t_0}^{t}e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\langle V_u(\mu,u(t)),g(\mu,u(t),u(\mu-\tau))\rangle d\langle B\rangle_\mu\\
&+\int_{t_0}^{t}\frac{1}{2}e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\langle V_{uu}(\mu,u(\mu))\sigma(\mu,u(\mu),u(\mu-\tau)),\sigma(\mu,u(\mu),u(\mu-\tau))\rangle d\langle B\rangle_\mu\\
&-\int_{t_0}^{t}e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}G(\langle V_u(\mu,u(\mu)),2g(\mu,u(\mu),u(\mu-\tau))\rangle\\
&+\langle V_{uu}(\mu,u(\mu))\sigma(\mu,u(\mu),u(\mu-\tau)),\sigma(\mu,u(\mu),u(\mu-\tau))\rangle)d\mu.
\end{aligned}$$

According to [19], it's easily to know that $\{M_t^{t_0}\}_{t\geq t_0}$ is a G-martingale and $\overline{\mathrm{E}}\left[M_t^{t_0}\,|F_t\right]=0$, where $\{F_t\}_{t\geq 0}$ is the filtration generated by the canonical process $(B_t)_{t\geq 0}$ as $F_t=\sigma\left(B_{t_0},0\leq t_0\leq t\right)$.

Therefore, taking expectation on the two sides yields

$$\overline{\mathrm{E}}\left[e^{\lambda\int_{t_0}^{t}\beta(s)ds}V(t,u(t))-V(t_0,u(t_0))\right]=\overline{\mathrm{E}}\left[\int_{t_0}^{t}e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\left[\lambda\beta(\mu)V(\mu,u(\mu))+LV(\mu,u(\mu))\right]d\mu\right].$$

(3-3)

Following from (ii) and (3-3),

$$\begin{aligned}\overline{\mathrm{E}}\left[e^{\lambda\int_{t_0}^{t}\beta(s)ds}V(t,u(t))\right]&\leq A_2\overline{\mathrm{E}}\|\phi\|^p\\&+\overline{\mathrm{E}}\left[\int_{t_0}^{t}e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\left[A_2\lambda\beta(\mu)\|u(\mu)\|^p+\beta(\mu)\left(-A_1A_3\|u(\mu)\|^p+A_2A_4\|u(\mu-\tau)\|^p\right.\right.\right.\\&\left.\left.\left.+A_2A_5\int_0^{\tau}\rho\|u(\mu-s)\|^pds+A_6\right)\right]d\mu\right]\\&\leq A_2\overline{\mathrm{E}}\|\phi\|^p+\frac{A_6}{\lambda}\left(e^{\lambda\int_{t_0}^{t}\beta(s)ds}-1\right)+\overline{\mathrm{E}}\left[\int_{t_0}^{t}e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\left[\lambda A_2\beta(\mu)\|u(\mu)\|^p\right.\right.\\&\left.\left.+\beta(\mu)\left(-A_1A_3\|u(\mu)\|^p+A_2A_4\|u(\mu-s)\|^p+A_2A_5\int_0^{\tau}\rho\|u(\mu-s)\|^pds\right)\right]d\mu\right].\end{aligned}$$

(3-4)

On the other hand,

$$\begin{aligned}\overline{\mathrm{E}}\left[\int_{t_0}^{t}e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\beta(\mu)\|u(\mu-s)\|^pd\mu\right]&\leq e^{\lambda\bar{\beta}}A_0\beta(\tau)\overline{\mathrm{E}}\left[\int_{t_0}^{t}e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\beta(\mu)\|u(\mu)\|^pd\mu\right]\\&+\frac{1}{\lambda}\left(e^{\lambda\bar{\beta}}-1\right)\overline{\mathrm{E}}\|\phi\|^p.\end{aligned}$$

(3-5)

And

$$\overline{\mathrm{E}}\left[\int_{t_0}^{t} e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\beta(\mu)\int_0^{\tau}\rho A_2A_5\left\|u(\mu-s)\right\|^p dsd\mu\right]$$
$$=\overline{\mathrm{E}}\left[\int_0^{\tau}\rho A_2A_5\left(\int_{t_0}^{t} e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\beta(\mu)\left\|u(\mu-s)\right\|^p d\mu\right)ds\right]$$
$$=\overline{\mathrm{E}}\left[\int_0^{\tau}\rho A_2A_5\left(\int_{t_0}^{t-s} e^{\lambda\int_{t_0}^{\mu+s}\beta(s)ds}\beta(\mu+s)\left\|u(\mu)\right\|^p d\mu\right)ds\right]$$
$$+\overline{\mathrm{E}}\left[\int_0^{\mu}\rho A_2A_5\left(\int_{t_0-s}^{t_0} e^{\lambda\int_{t_0}^{\mu+s}\beta(s)ds}\beta(\mu+s)\left\|u(\mu)\right\|^p d\mu\right)ds\right]$$
$$\leq\overline{\mathrm{E}}\left[\int_0^{\tau}\rho A_2A_5e^{\lambda\overline{\beta}}\left(\int_{t_0}^{t-s} e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\beta(\mu+s)\left\|u(\mu)\right\|^p d\mu\right)ds\right]$$
$$+\overline{\mathrm{E}}\left[\int_0^{\tau}\rho A_2A_5\left(\int_{t_0-s}^{t_0} e^{\lambda\int_{t_0}^{\mu+s}\beta(s)ds}\beta(\mu+s)\left\|u(\mu)\right\|^p d\mu\right)ds\right]$$
$$\leq\overline{\mathrm{E}}\left[\int_0^{\tau}\rho A_0A_2A_5e^{\lambda\overline{\beta}}\beta(s)\left(\int_{t_0}^{t-s} e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\beta(\mu)\left\|u(\mu)\right\|^p d\mu\right)ds\right]$$
$$+\int_0^{\tau}\rho A_2A_5\left(\frac{1}{\lambda}\left(e^{\lambda\overline{\beta}}-1\right)\right)ds\overline{\mathrm{E}}\left\|\phi\right\|^p$$
$$\leq\left(\rho A_0A_2A_5\int_0^{\tau}\beta(s)e^{\lambda\overline{\beta}}ds\right)\overline{\mathrm{E}}\left[\int_{t_0}^{t} e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\beta(\mu)\left\|u(\mu)\right\|^p d\mu\right]$$
$$+\int_0^{\tau}\rho A_2A_5\left(\frac{1}{\lambda}\left(e^{\lambda\overline{\beta}}-1\right)\right)ds\overline{\mathrm{E}}\left\|\phi\right\|^p. \tag{3-6}$$

Substituting (3-5) and (3-6) into (3-4), yields

$$\overline{\mathrm{E}}\left[e^{\lambda\int_{t_0}^{t}\beta(s)ds}V(t,u(t))\right]\leq A_2\overline{\mathrm{E}}\left\|\phi\right\|^p$$
$$+\left(\lambda A_2-A_1A_3+A_0A_2A_4\beta(\tau)e^{\lambda\overline{\beta}}+\rho A_0A_2A_5\int_0^{\tau}\beta(s)e^{\lambda\overline{\beta}}ds\right)\overline{\mathrm{E}}\left[\int_{t_0}^{t} e^{\lambda\int_{t_0}^{\mu}\beta(s)ds}\beta(\mu)\left\|u(\mu)\right\|^p d\mu\right]$$
$$+\left[\frac{A_2A_4}{\lambda}\left(e^{\lambda\overline{\beta}}-1\right)+\int_0^{\tau}\frac{\rho A_2A_5}{\lambda}\left(e^{\lambda\overline{\beta}}-1\right)ds\right]\overline{\mathrm{E}}\left\|\phi\right\|^p+\frac{A_6}{\lambda}\left(e^{\lambda\int_{t_0}^{t}\beta(s)ds}-1\right).$$

Since $A_1A_3>A_0A_2A_4\beta(\tau)+A_0A_2A_5\int_0^{\tau}\beta(s)ds$, using the continuity, there is a positive constant λ such that (3-2) holds. Letting $n\to\infty$, one easily shows that

$$\overline{\mathrm{E}}\left[e^{\lambda\int_{t_0}^{t}\beta(s)ds}V(t,u(t))\right]\le A_2\overline{\mathrm{E}}\|\phi\|^p+\left[\frac{A_2A_4}{\lambda}\left(e^{\lambda\bar{\beta}}-1\right)+\int_0^{\tau}\frac{\rho A_2A_5}{\lambda}\left(e^{\lambda\bar{\beta}}-1\right)ds\right]\overline{\mathrm{E}}\|\phi\|^p$$
$$+\frac{A_6}{\lambda}\left(e^{\lambda\int_{t_0}^{t}\beta(s)ds}-1\right).$$

Together with the condition (i), implies that

$$\overline{\mathrm{E}}\|u(t)\|^p\le\left[\frac{A_2}{A_1}+\frac{A_2A_4}{\lambda A_1}\left(e^{\lambda\bar{\beta}}-1\right)+\frac{\rho A_2A_5}{\lambda A_1}\left(e^{\lambda\bar{\beta}}-1\right)\tau\right]\overline{\mathrm{E}}\|\phi\|^p e^{-\lambda\int_{t_0}^{t}\beta(s)ds}+\frac{A_6}{\lambda A_1}.$$

This completes the proof.

Theorem 3.2. Suppose that there exist a symmetric positive definite matrix P and constants satisfy $\bar{\alpha}>0, A_0>0, A_9\ge 0, A_{13}\ge 0$ and $A_j>0, j=6,7,8,10,11,12,$

(i) For all $(t,u)\in\mathbb{R}_{t_0}\times\mathbb{R}^n,$

$$u^TPf\le\alpha(t)\left[A_6u^TPu+A_7u^T(t-\tau)Pu(t-\tau)+A_8\int_0^{\tau}\rho u^T(t-s)Pu(t-s)ds+A_9\right],$$

where $\alpha(t)$ is a positive integral function satisfying $\alpha(t+\tau)\le A_0\alpha(t)\alpha(\tau),\ \sup_{t\ge t_0}\int_{t-\tau}^{t}\alpha(s)ds\le\bar{\alpha},$ and $\lim_{t\to\infty}\int_{t_0}^{t}\alpha(s)ds=\infty.$

(ii) For all $(t,u)\in\mathbb{R}_{t_0}\times\mathbb{R}^n,$

$$G\left(2p\left(u^TPu\right)^{\frac{p}{2}-1}\left(u^TPg+\sigma^TP\sigma\right)+p(p-2)\left(u^TPu\right)^{\frac{p}{2}-2}\left\|u^TP\sigma\right\|^2\right)$$
$$\le\alpha(t)\left[A_{10}\left(u^TPu\right)^{\frac{p}{2}}+A_{11}\left(u^T(t-\tau)Pu(t-\tau)\right)^{\frac{p}{2}}+A_{12}\int_0^{\tau}\rho\left(u^T(t-s)Pu(t-s)\right)^{\frac{p}{2}}ds+A_{13}\right],$$

(iii)

$$\left(\lambda_{\min}(P)\right)^{\frac{p}{2}}A_{14}>\left(\lambda_{\max}(P)\right)^{\frac{p}{2}}(2A_7+A_{11})A_0\alpha(\tau)e^{\lambda\bar{\alpha}}$$
$$+\left(\lambda_{\max}(P)\right)^{\frac{p}{2}}(2A_8+A_{12})A_0\int_0^{\tau}\rho\alpha(s)ds,$$

where

$$A_{14}=-\left(pA_6+(p-2)A_7+(p-2)A_9+A_{10}+(p-2)A_8\rho\tau\right),$$

Then system (3-1) is pth moment globally exponentially ultimately bounded with the bound $b=\dfrac{2A_9+A_{13}}{\left(\lambda_{\min}(P)\right)^{\frac{p}{2}}\lambda},$ where the constant λ is determined by the following inequality

$$\left(\lambda_{\max}(P)\right)^{\frac{p}{2}}\lambda-\left(\lambda_{\min}(P)\right)^{\frac{p}{2}}A_{14}+\left(\lambda_{\max}(P)\right)^{\frac{p}{2}}\square A_0\left(2A_7+A_{11}\right)\alpha(\tau)e^{\lambda\square}$$
$$+\left(\lambda_{\max}(P)\right)^{\frac{p}{2}}\square A_0\left(2A_8+A_{12}\right)\int_0^{\tau}\rho\alpha(s)e^{\lambda\square}ds<0. \tag{3-7}$$

Proof. Defined the Lyapunov function $V(t,u(t))=\left(u^T(t)Pu(t)\right)^{\frac{p}{2}}$,

$$\left(\lambda_{\min}(P)\right)^{\frac{p}{2}}\overline{\mathrm{E}}\|u\|^p\le\overline{\mathrm{E}}V(t,u(t))\le\left(\lambda_{\max}(P)\right)^{\frac{p}{2}}\overline{\mathrm{E}}\|u\|^p. \tag{3-8}$$

Using the conditions (i) and (ii), yields

$$\begin{aligned}
LV(t,u)&=p\left(u^TPu\right)^{\frac{p}{2}-1}u^TPf\\
&+G\left(2p\left(u^TPu\right)^{\frac{p}{2}-1}\left(u^TPg+\sigma^TP\sigma\right)+p(p-2)\left(u^TPu\right)^{\frac{p}{2}-2}\left\|u^TP\sigma\right\|^2\right)\\
&\le p\left(u^TPu\right)^{\frac{p}{2}-1}\left[\alpha(t)\left[A_6u^TPu+A_7u^T(t-\tau)Pu(t-\tau)+A_8\int_0^{\tau}\rho u^T(t-s)Pu(t-s)ds+A_9\right]\right]\\
&+\alpha(t)\left[A_{10}\left(u^TPu\right)^{\frac{p}{2}}+A_{11}\left(u^T(t-\tau)Pu(t-\tau)\right)^{\frac{p}{2}}+A_{12}\int_0^{\tau}\rho\left(u^T(t-s)Pu(t-s)\right)^{\frac{p}{2}}ds+A_{13}\right]\\
&\le\alpha(t)\Big[pA_6\left(u^TPu\right)^{\frac{p}{2}}+pA_7\left(u^T(t)Pu(t)\right)^{\frac{p}{2}-1}u^T(t-\tau)Pu(t-\tau)\\
&+pA_8\int_0^{\tau}\rho\left(u^TPu\right)^{\frac{p}{2}-1}u^T(t-s)Pu(t-s)ds+pA_9\left(u^TPu\right)^{\frac{p}{2}-1}\\
&+A_{10}\left(u^TPu\right)^{\frac{p}{2}}+A_{11}\left(u^T(t-\tau)Pu(t-\tau)\right)^{\frac{p}{2}}+A_{12}\int_0^{\tau}\rho\left(u^T(t-s)Pu(t-s)\right)^{\frac{p}{2}}ds+A_{13}\Big].
\end{aligned} \tag{3-9}$$

Using Lemma 2.3 and (3-9) produce,

$$\begin{aligned}
LV(t,u)&=\alpha(t)\Big\{pA_6\left(u^TPu\right)^{\frac{p}{2}}+(p-2)A_7\left(u^T(t)Pu(t)\right)^{\frac{p}{2}}+2A_7\left(u^T(t-\tau)Pu(t-\tau)\right)^{\frac{p}{2}}\\
&+(p-2)A_8\int_0^{\tau}\rho\left(u^T(t)Pu(t)\right)^{\frac{p}{2}}ds+2A_8\int_0^{\tau}\rho\left(u^T(t-s)Pu(t-s)\right)^{\frac{p}{2}}ds\\
&+pA_9\left[\frac{p-2}{p}\left(u^TPu\right)^{\frac{p}{2}}+\frac{2}{p}\right]+A_{10}\left(u^TPu\right)^{\frac{p}{2}}+A_{11}\left(u^T(t-\tau)Pu(t-\tau)\right)^{\frac{p}{2}}\\
&+A_{12}\int_0^{\tau}\rho\left(u^T(t-s)Pu(t-s)\right)^{\frac{p}{2}}ds+A_{13}\Big\}\\
&=\alpha(t)\Big[-A_{14}\left(u^TPu\right)^{\frac{p}{2}}+\left(2A_7+A_{11}\right)\left(u^T(t-\tau)Pu(t-\tau)\right)^{\frac{p}{2}}\\
&+\left(2A_8+A_{12}\right)\int_0^{\tau}\rho\left(u^T(t-s)Pu(t-s)\right)^{\frac{p}{2}}ds+2A_9+A_{13}\Big].
\end{aligned}$$

By the condition,

$$\left(\lambda_{\min}\left(P\right)\right)^{\frac{p}{2}} A_{14} > \left(\lambda_{\max}\left(P\right)\right)^{\frac{p}{2}}\left(2A_7 + A_{11}\right)\square A_0 \alpha\left(\tau\right)e^{\lambda \square}$$
$$+\left(\lambda_{\max}\left(P\right)\right)^{\frac{p}{2}}\left(2A_8 + A_{12}\right)\square A_0 \int_0^{\tau} \rho\alpha\left(s\right)ds,$$

using the continuity, there exists a positive constant λ satisfying (3-7). Thus, it follows from (3-8) and Theorem 3.1 that

$$\bar{\mathrm{E}}\left\|u\left(t\right)\right\|^p \le \frac{\left(\lambda_{\max}\left(P\right)\right)^{\frac{p}{2}}}{\left(\lambda_{\min}\left(P\right)\right)^{\frac{p}{2}}}\left[1+\left(\frac{2A_7 + A_{11}}{\lambda}+\frac{2A_8 + A_{12}}{\lambda}\rho\tau\right)\left(e^{\lambda \square}-1\right)\right]\bar{\mathrm{E}}\left\|\phi\right\|^p e^{-\lambda\int_{t_0}^{t}\alpha(s)ds}$$
$$+\frac{2A_9 + A_{13}}{\left(\lambda_{\min}\left(P\right)\right)^{\frac{p}{2}}\lambda},$$

where the positive constant λ is determined by (3-7).

That's the completed proof.

From the above results, followings are Corollaries 3.1-3.5.

Corollary 3.1. The system will degenerate to (3-10) without distributed delays

$$\begin{cases} du(t) = f(t,u(t),u(t-\tau))dt + g(t,u(t),u(t-\tau))d\left\langle B\right\rangle_t + \sigma(t,u(t),u(t-\tau))dB_t, t \ge t_0 \\ u(t_0+s) = \phi(s), -\tau \le s < 0 \end{cases}, \quad (3\text{-}10)$$

while (3-10) satisfying the condition (i) in Theorem 3.1, and $A_5 \ge 0$, the condition (ii) changes to

$$LV(t,u(t)) \le \beta\left(t\right)\left[-A_3V(t,u(t)) + A_4V(t,u(t-\tau)) + A_5\right],$$

the system is pth moment globally exponentially ultimately bounded identically. For the detailed certification, see [20].

Corollary 3.2. Suppose all the conditions in Theorem 3.1 are satisfied, if $\beta\left(t\right)=1$, the system (3-1) is pth moment globally exponentially ultimately bounded.

Corollary 3.3. Suppose all the conditions in Theorem 3.2 are satisfied, if $\alpha\left(t\right)=1$, the system (3-1) is pth moment globally exponentially ultimately bounded.

Corollary 3.4. Suppose all the conditions in Theorem 3.1 are satisfied, if $A_6 = 0$, the system (3-1) is pth moment globally asymptotically stable.

Corollary 3.5. Suppose all the conditions in Theorem 3.2 are satisfied, if $A_9 = A_{13} = 0$, the system (3-1) is pth moment globally asymptotically stable.

4. Illustrative example

Consider the following 1-D stochastic differential systems driven by G-Brownian motion with mixed delays:

$$dX(t)=(4+\cos t)\left(-13u(t)+2u(t-2)+\int_{t-2}^{t}u(s)ds+3\right)dt+(4+\cos t)u(t)d\langle B\rangle_t$$
$$+\sqrt{(4+\cos t)}u(t)dB_t, t\geq t_0, \tag{4-1}$$

where B is a one dimension G-Brownian motion and $B=N(\{0\}\times[\frac{1}{4}\times\frac{1}{2}])$. Taking Lyapunov function as $V(t,u(t))=u^2(t)$,

$$V_u(t,u(t))f(t,u(t))=(4+\cos t)\left(-26u(t)+4u(t)u(t-2)+2u(t)\int_{t-2}^{t}u(s)ds+6u(t)\right)$$
$$\leq(4+\cos t)\left(-26u(t)+2u^2(t)+2u^2(t-2)+\int_{t-2}^{t}u^2(t)+u^2(s)ds+6u(t)\right)$$
$$\leq(4+\cos t)\left(-19u^2(t)+2u^2(t-2)+\int_0^2u^2(t-s)ds+3\right).$$

$$V_u(t,u(t))g(t,u(t))=2(4+\cos t)u^2(t), V_{uu}(t,u(t))\sigma^2(t,u(t))=2(4+\cos t)u^2(t).$$

And,

$$LV(t,u(t))=(4+\cos t)\left(-19u(t)+2u^2(t-2)+\int_0^2u^2(t-s)ds+3\right)+G\left(4(4+\cos t)u^2(t)\right)$$
$$\leq(4+\cos t)\left(-19u(t)+2u^2(t-2)+\int_0^2u^2(t-s)ds+3\right)+(4+\cos t)u^2(t)$$
$$\leq(4+\cos t)\left(-18u(t)+2u^2(t-2)+\int_0^2u^2(t-s)ds+3\right), t\geq t_0.$$

Obviously, when taking $p=2, A_1=A_2=1, A_3=18, A_4=2, A_5=1, A_6=3, \beta(t)=4+\cos t, \rho=1,\quad A_0=1,$ conditions (i) and (ii) in Theorem 3.1 are satisfied. And

$$A_0A_2A_4\beta(\tau)+A_0A_2A_5\int_0^\tau\beta(s)ds=2(4+\cos 2)+\int_0^2(4+\cos s)ds$$
$$<A_1A_3=18,$$

meanwhile, inequality (3-2) is satisfied by taking $\lambda=0.05$. According to Theorem 3.1, the system (4-1) is pth moment globally exponentially ultimately bounded with the bound $b=\dfrac{A_6}{\lambda A_1}=60.$

References

1. J. Huang, Y. Li, Y. Zheng. (2012). Introduction to Stochastic Dynamic Systems. *Science Press.*
2. G. Gong. (1987). Introduction to Stochastic Differential Equations. *Peking University Press.*
3. A.V. Platonov. (2018). On the global asymptotic stability and ultimate boundedness for a class of nonlinear switched systems. *Nonlinear Dynamics*, *92*, 1555–1565.
4. D. Xu, X. Wang. (2009). A new nonlinear integro-differential inequality and its application. *Appl. Math. Lett*, *22*, 1721–1726.
5. D. Xu, L. Xu. (2010). New results for studying a certain class of nonlinear delay differential systems. *IEEE Trans. Automat. Control*, *55* (7), 1641–1645.

6. D. Xu, S. Long. (2012). Attracting and quasi-invariant sets of non-autonomous neural networks with delays. *Neurocomputing*, *77*, 222–228.
7. D. He, L. Xu. (2015). Ultimate boundedness of non-autonomous dynamical complex networks under impulsive control. *IEEE Trans. Circuits Syst. II*, *62*, 997–1001.
8. L. Xu, H. Hu, F. Qin. (2016). Ultimate boundedness of impulsive fractional differential equations. *Appl. Math. Lett*, *62*, 110–117.
9. L. Xu, J. Li, S.S. Ge. (2017). Impulsive stabilization of fractional differential systems. *ISA Trans*, *70*, 125–131.
10. H. Hu, L. Xu. (2018). Existence and uniqueness theorems for periodic Markov process and applications to stochastic functional differential equations. *Math. Anal. Appl*, *466*, 896–926.
11. L. Xu, S.S. Ge. (2015). The pth moment exponential ultimate boundedness of impulsive stochastic differential systems. *Appl. Math. Lett*, *42*, 22–29.
12. L. Xu, Z. Dai, D. He. (2018). Exponential ultimate boundedness of impulsive stochastic delay differen-tial equations. *Appl. Math. Lett*, *85*, 70–76.
13. W. Yin, J. Cao, Y. Ren. (2019). Quasi-sure exponential stabilization of stochastic systems induced by G-Brownian motion with discrete time feedback control. *Appl. Math. Lett*, *474*, 276–289.
14. D. He, L. Xu. (2020). Boundedness analysis of non-autonomous stochastic differential systems with Lévy noise and mixed delays. *AIMS Mathematics*, *5*(6), 6169-6182.
15. E. Beckenbach, R. Bellman. (1961). Inequalities. *Springer-Verlag, New York*.
16. S. Peng. (2007). G-expectation, G-Brownian motion and related stochastic calculus of Itˆo type, in: Stochastic Analysis and Applications. *Abel Symp. Springer, Berlin*, *2*, 541–567.
17. S. Peng. (2007). G-Brownian motion and dynamic risk measures under volatility uncertainty, *arXiv*: *0711.2834*.
18. X. Liu, F. Deng, L.Liu, S. Luo, X. Zhao. (2020). Mean-square stability of two classes of θ-methods for neutral stochastic delay integro-differential equations. *Appl. Math. Lett*, *109*.
19. S. Peng, (2010). Nonlinear expectations and stochastic calculus under uncertainty, *arxiv:1002.4546.*
20. D. He, Q. Yin. (2019). Boundedness theorems for non-autonomous stochastic delays differential systems driven by G-Brownian motion. *Appl. Math. Lett*, *91*, 83-89.