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***Journal of Scientific and Engineering Research*, 2021, 8(5):16-18**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

An Extended Study of the Fourier Series Theorem

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Abstract This paper applies function Fourier series theorem to arbitrary interval by using transform method, which is called generalized Fourier series theorem.

Keywords Fourier series, generalized Fourier series theorem, arbitrary interval
Chinese library classification number: 0174.2

Introduction

Up to now, most of the discussions about the Fourier function develop Fourier series theorem through function f(x) on interval $[-\pi,\pi]$ and then generalize to symmetric interval$[-l,l]$. This paper uses transform method to extend function Fourier series theorem to arbitrary interval$[a,b]$, which f(x)is called generalized Fourier series theorem.

Generalized Fourier series theorem

Hypothesizing function $f(x)$on interval $[a,b]$ satisfies the condition of convergence theorem, then function $f(x)$on interval $[a,b]$ of generalized Fourier series:

1. When $x$is a point of discontinuity of function $f(x)$on interval $[a,b]$,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\left[\frac{2n\pi}{b-a}\left(x-\frac{a+b}{2}\right)\right]+b_n\sin\left[\frac{2n\pi}{b-a}\left(x-\frac{a+b}{2}\right)\right]\right\} \quad (1)$$

Coefficient

$$\begin{cases} a_n=\frac{2}{b-a}\int_a^b f(x)\cos\left[\frac{2n\pi}{b-a}\left(x-\frac{a+b}{2}\right)\right]dx \quad n=0,1,2,\cdots\cdots \\ b_n=\frac{2}{b-a}\int_a^b f(x)\sin\left[\frac{2n\pi}{b-a}\left(x-\frac{a+b}{2}\right)\right]dx \quad n=0,1,2,\cdots\cdots \end{cases} \quad (2)$$

In equation (1), when x is discontinuity point of the first kind of $f(x)$on interval $[a,b]$ and $x=a(\text{or } b)$,$\frac{f(x-0)+f(x+0)}{2}$should substitute the left $f(x)$of equation.

Demonstration

Making a variable substitution firstly, and make $Z=\frac{2\pi}{b-a}\left(x-\frac{a+b}{2}\right)$, then $x=\frac{b-a}{2\pi}Z+\frac{a+b}{2}$, hence, interval $a\le x\le b$ is substituted into interval $-\pi\le Z\le\pi$.

Again$f(x)=f\left(\frac{b-a}{2\pi}Z+\frac{a+b}{2}\right)=F(Z)$.

Thus $f(x)$is defined as a $b-a$periodic function, then $F(Z)$ is a periodic function cycles on 2π. In fact,

$$F(Z+2\pi)=f\left[\frac{b-a}{2\pi}(Z+2\pi)+\frac{a+b}{2}\right]$$
$$=f\left[\frac{b-a}{2\pi}Z+\frac{a+b}{2}+(b-a)\right]$$
$$=f\left[\frac{b-a}{2\pi}Z+\frac{a+b}{2}\right]=F(Z).$$

Through this transformation, function $f(x)$on interval $[a,b]$ converged as Fourier series is changed into function $F(Z)$ on interval $[-\pi,\pi]$.
Function $F(Z)$ on interval $[-\pi,\pi]$ expand into a Fourier series is as follows.
When Z is a point of discontinuity of function $F(Z)$on interval $[-\pi,\pi]$,

$$F(Z)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\{a_n\cos Z+b_n\sin Z\}. \quad (3)$$

Coefficient

$$\begin{cases} a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\pi F(Z)\cos nZdZ & n=0,1,2,\cdots\cdots \\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\pi F(Z)\sin nZdZ & n=0,1,2,\cdots\cdots \end{cases} \quad (4)$$

In equation (3), when Z is a discontinuity point of the first kind of $F(Z)$on interval $[-\pi,\pi]$or $Z=\pm\pi$, $\frac{F(Z-0)+F(Z+0)}{2}$ should substitute the left $F(Z)$of equation.
In equation (3) and (4), make $Z=\frac{2\pi}{b-a}\left(x-\frac{a+b}{2}\right)$ plug in, $F(Z)=f(x)$, thus,
equation (3) is converted to $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\left[\frac{2n\pi}{b-a}\left(x-\frac{a+b}{2}\right)\right]+b_n\sin\left[\frac{2n\pi}{b-a}\left(x-\frac{a+b}{2}\right)\right]\right\}$,
equation (4) is converted to

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\pi F(Z)\cos nZdZ$$

$$=\frac{2}{b-a}\int_a^b f(x)\cos\left[\frac{2n\pi}{b-a}\left(x-\frac{a+b}{2}\right)\right]dx \quad n=0,1,2,\cdots\cdots.$$

is equation (1).

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\pi F(Z)\sin nZdZ$$

$$=\frac{2}{b-a}\int_a^b f(x)\sin\left[\frac{2n\pi}{b-a}\left(x-\frac{a+b}{2}\right)\right]dx \quad n=0,1,2,\cdots\cdots.$$

is equation (2).

Demonstration complete.

Exceptionally, make $\boldsymbol{a}=-\boldsymbol{l},\boldsymbol{b}=\boldsymbol{l}$, thus,

Inference 1 Hypothesizing function $\boldsymbol{f(x)}$ on interval $[-\boldsymbol{l},\boldsymbol{l}]$ satisfies the condition of convergence theorem, then $\boldsymbol{f(x)}$on
interval $[-\boldsymbol{l},\boldsymbol{l}]$expands into Fourier series,

$$f(x)=\frac{a_0}{2}\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi}{l}\mathrm{x}+b_n\sin\frac{n\pi}{l}x\right] \quad n=1,2,\cdots\cdots$$

Coefficient

$$\begin{cases} a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos n\frac{n\pi}{l}xdx & n=0,1,2,\cdots\cdots \\ b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin n\frac{n\pi}{l}xdx & n=0,1,2,\cdots\cdots \end{cases}$$

Another exception, make $b=a+\lambda$, thus,

Inference 2 Hypothesizing function $\boldsymbol{f(x)}$on interval $[\boldsymbol{a},\boldsymbol{a}+\boldsymbol{\lambda}]$
satisfies the condition of convergence theorem, then $\boldsymbol{f(x)}$on
interval $[\boldsymbol{a},\boldsymbol{a}+\boldsymbol{\lambda}]$expands into Fourier series,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\left[\frac{2n\pi}{\lambda}\left(x-\frac{a+b}{2}\right)\right]+b_n\sin\left[\frac{2n\pi}{\lambda}\left(x-\frac{2a+\lambda}{2}\right)\right]\right\}$$

Coefficient

$$\begin{cases} a_n=\frac{2}{\lambda}\int_a^{a+\lambda}f(x)\cos\left[\frac{2n\pi}{\lambda}\left(x-\frac{2a+\lambda}{2}\right)\right]dx & n=0,1,2,\cdots\cdots \\ b_n=\frac{2}{\lambda}\int_a^{a+\lambda}f(x)\sin\left[\frac{2n\pi}{\lambda}\left(x-\frac{2a+\lambda}{2}\right)\right]dx & n=0,1,2,\cdots\cdots \end{cases}.$$

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