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Research Article

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Eigen solutions of the Schrödinger equation in the presence of ring-shaped Hellmann potential

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Abstract In this paper, the Schrödinger equation is analytically solved for the Hellmann potential with a novel angle dependent part. The Nikiforov-Uvarov method is used to obtain energy eigenvalues and corresponding eigenfunctions. It is worthy to note that by employing the Greene and Aldrich approximation, we have been able to get a better and more accurate result for the Yukawa angle dependent Potential proposed in earlier studies. Numerical results were obtained for the Ring shaped Hellmann potential and the Hellmann potential respectively. It was found out that our results agree with existing literature.

Keywords: Schrödinger equation; Hellmann potential; Novel angle-dependent potential; Nikiforov-Uvarov method; Ring shaped potential.

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1 Introduction

It is well known that the exact solutions of the Schrödinger equation contain all the necessary information for a quantum system. This is attributed to the fact that the wave functions associated with these problems contain all the necessary information regarding the quantum systems under consideration [1-9].

The analytical solution with $l = 0$ and $l \neq 0$ for some potentials(central and non-central) has been addressed by many researchers in nonrelativistic quantum mechanics and relativistic quantum mechanics for bound and scattering States. These studies include the potentials include, Poschl-Teller potential [10-11], Coulomb ring-shaped potential [12], Yukawa-angle Dependent Potential [13], Hartman potential [14-15], non-central generalized inverse quadratic Yukawa potential [16], Pseudo-Coulomb Potential Plus a New Ring-Shaped Potential [17], Coulombicring-shaped potential [18] single ring-shaped oscillator potential [19] ring-shaped non-spherical harmonic oscillator potential [20-21] and spherically harmonic oscillatory ring-shaped potential [22-23] and Inversely Quadratic Hellmann Plus Ring-Shaped Potentials [24]. The methods which have been used to solve the differential equation arising from these considerations include; the asymptotic iteration method (AIM) [25-27], NUFA method [28-29], Nikiforov–Uvarov (NU) method [30–33], supersymmetric quantum mechanics (SUSYQM) [34], WKB [35] and the functional analysis approach (FAA) [36-37].

The Hellmann potential [38-41] is a superposition of Yukawa plus Coulomb potentials given thus

$$V(r) = -\left(\frac{V_0}{r}\right) + \left(\frac{V_1 e^{-\alpha r}}{r}\right) \tag{1}$$

where V_0 and V_1 the potential strength of Coulomb and Yukawa potentials respectively, α is the screening parameter and r is the distance between the two particles.

The Hellmann potential was first studied by Hellmann [38-40]. Thereafter, various authors worked on the potential, *e.g.*,[41] used the supersymmetric approach to study the approximate analytic solutions of the three-dimensional

Schrödinger equation with this potential by applying a suitable approximation scheme to the centrifugal term. Onate et al. [42] obtained approximate eigensolutions of the Duffin-Kemmer-Petiau and Klein-Gordon equations with the Hellmann potential. Hamzavi et al. [43] solved the approximate bound states solutions of the Hellmann potential using the generalized parametric Nikiforov-Uvarov method. Oluwadare and Oyewumi [44] investigated the scattering state solutions of the Klein–Gordon equation with equal scalar and vector Varshni, Hellmann and Varshni–Shukla potentials for any arbitrary angular momentum quantum number within the framework of the functional analytical method using a suitable approximation. Edet et al. [45] obtained the thermal and magnetic properties of the Hellmann potential.

The Hellmann potential is applied in the field of atomic and condensed matter physics, *e.g.*, electron-core [46,47], electron-ion [48] inner-shell ionization problem, alkali hydride molecules, solid state physics [49,50].

The Ring shaped potential have a wide range of applications in quantum chemistry and nuclear physics [51]. They have very important role in describing ring-shaped molecules like benzene and the interactions between deformed pair of nuclei [52-53]. They have also been used in demonstrating some of the pseudospin symmetry in nuclei physics [23]. The exact results can be used in accounting for some axial symmetric system in quantum chemistry. [22] proposed a non-central potential as;

$$V(r,\theta) = \frac{\hbar^2}{2\mu r^2}\left(\frac{C+B\cos^2\theta+A\cos^4\theta}{\sin^2\theta\cos^2\theta}\right) \quad (2)$$

Motivated by this potential, we attempt to propose a Ring shaped Hellmann potential by selecting $V(r)$as the Hellmann potential. The Ring shaped Hellmann potential is composed of Hellmann potential plus a Novel Angle Dependent (NAD) potential. It can be written as

$$V(r,\theta) = -\frac{V_0}{r} + \frac{V_1 e^{-\alpha r}}{r} + \frac{\hbar^2}{2\mu r^2}\left(\frac{C+B\cos^2\theta+A\cos^4\theta}{\sin^2\theta\cos^2\theta}\right) \quad (3)$$

The primary purpose of the present work is to solve the Schrodinger equation for the Ring shaped Hellmann potential and to calculate the energy eigenvalues and the corresponding wave functions which are expressed in terms of the Jacobi polynomials for any orbital quantum number *l*. We computed the energy spectrum numerically, this will enable us assess the effect of angle dependence on the energy eigenvalue of the Hellmann Potential. The Nikiforov–Uvarov (NU) is used in present calculations

The article is organized as follows: Section 2 gives a brief outline of the NU method used to solve the SE in the presence of the Ring shaped Hellmann potential. The separation of the radial and polar part of Schrodinger equation and analytical expressions for energy levels and corresponding wave functions are obtained for any *n, m* and *l* quantum numbers in Section 3. In Section 4, we shall discuss special cases of the potential under consideration. In Section 5, we discuss results and section 6, we give a brief concluding remark.

2. Review of Nikiforov-Uvarov Method

The Nikiforov-Uvarov (NU) method is based on solving the hypergeometric-type second-order differential equations by means of the special orthogonal functions [54]. The main equation which is closely associated with the method is given in the following form [55];

$$\psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z) = 0 \quad (4)$$

Where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials at most second-degree, $\tilde{\tau}(z)$ is a first-degree polynomial and $\psi(z)$ is a function of the hypergeometric-type.

The exact solution of Eq. (4) can be obtained by using the transformation

$$\psi(z) = \phi(z)y(z) \quad (5)$$

This transformation reduces Eq. (4) into a hypergeometric-type equation of the form

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0 \quad (6)$$

The function $\phi(s)$ can be defined as the logarithm derivative

$$\frac{\phi'(z)}{\phi(z)} = \frac{\pi(z)}{\sigma(z)} \quad (7)$$

where $\pi(z) = \frac{1}{2}[\tau(z) - \tilde{\tau}(z)]$ (8)

with $\pi(z)$ being at most a first-degree polynomial. The second $\psi(z)$ being $y_n(z)$ in Eq. (5), is the hypergeometric function with its polynomial solution given by Rodrigues relation

$$y^{(n)}(z) = \frac{B_n}{\rho(z)}\frac{d^n}{ds^n}[\sigma^n(z)\rho(z)] \quad (9)$$

Here, B_n is the normalization constant and $\rho(z)$ is the weight function which must satisfy the condition

$$\left(\sigma(z)\rho(z)\right)' = \sigma(z)\tau(z) \quad (10)$$

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z) \quad (11)$$

It should be noted that the derivative of $\tau(s)$ with respect to s should be negative. The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(s)$ and parameter λ, respectively:

$$\pi(z) = \frac{\sigma'(z)-\tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z)-\tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)} \quad (12)$$

where$k = \lambda - \pi'(z)$ (13)

The value of k can be obtained by setting the discriminant of the square root in Eq. (12) equal to zero. As such, the new eigenvalue equation can be given as

$$\lambda_n = -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z), n = 0,1,2, \ldots \quad (14)$$

3. Separation of Variables for the Schrodinger Equation

In spherical coordinates (r,θ,ϕ), the Schrodinger equation with potentials $V(r,\theta)$, respectively, can be written as follows [23]:

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r,\theta,\phi) + V(r,\theta)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi) \quad (15)$$

where E is the non-relativistic energy of the system, μ denotes the rest mass of the particle and $\hbar$ is the planck constant. The Schrodinger equation with potential is given by [56];

$$\left[-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] + V(r,\theta) - E\right]\psi(r,\theta,\phi) = 0 \quad (16)$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \quad (17)$$

Substituting Eq. (3) into Eq (16), we have

$$\left[-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] + \left(-\frac{V_0}{r} + \frac{V_1e^{-\alpha r}}{r} + \frac{\hbar^2}{2\mu r^2}\left(\frac{C+B\cos^2\theta+A\cos^4\theta}{\sin^2\theta\cos^2\theta}\right)\right) - \right.$$
$$E\psi r,\theta,\phi=0 \quad (18)$$

Substituting (17) into Eq. (18) and using the standard procedure of separating variables, we obtain the following differential equations:

$$\frac{d^2R_{nl}}{dr^2} + \left[\frac{2\mu E_{nl}}{\hbar^2} - \frac{2\mu}{\hbar^2}\left(-\frac{V_0}{r} + \frac{V_1e^{-\alpha r}}{r}\right) - \frac{\Lambda}{r^2}\right]R_{nl}(r) = 0 \quad (19)$$

$$\frac{d^2\Theta(\theta)}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta(\theta)}{d\theta} + \left(\Lambda - \left(\frac{\hbar^2}{2\mu r^2}\left(\frac{C+B\cos^2\theta+A\cos^4\theta}{\sin^2\theta\cos^2\theta}\right)\right) - \frac{m^2}{\sin^2\theta}\right)\Theta(\theta) = 0 \quad (20)$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + m^2\Phi(\phi) = 0 \quad (21)$$

where m^2 and Λ are separation constants, which are real and dimensionless. The solution of Eq. (21) is periodic and for bound state $\Phi(\phi)$ satisfies the periodic boundary condition $\Phi(\phi + 2\pi)$ and its solutions become,

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}}e^{-im\phi}, m = 0, \pm1, \pm2, \ldots \quad (22)$$

3.1 Exact Solutions of Ring-shaped Hellman potential

3.1.1 Solutions of the radial Schrodinger equation for ring-shaped Hellman potential

$$\frac{d^2R_{n\ell}(r)}{dr^2}+\left[\frac{2\mu E_{nl}}{\hbar^2}+\frac{2\mu V_0}{\hbar^2}\left(\frac{1}{r}\right)-\frac{2\mu V_1}{\hbar^2}\left(\frac{e^{-\alpha r}}{r}\right)-\frac{\Lambda}{r^2}\right]R_{n\ell}(r)=0 \tag{23}$$

The radial Schrödinger equation for this potential can be solved exactly for $l=0$ (s-wave) but cannot be solved for this potential for $l\neq 0$. To obtain the solution for $l\neq 0$, we employ the approximation scheme proposed by Greene and Aldrich [57] to deal with the centrifugal term, which is given as;

$$\frac{1}{r^2}\approx\frac{\alpha^2}{(1-e^{-\alpha r})^2} \tag{24}$$

It is noted that for a short-range potential, the relation (eqs. 24) is a good approximation to $\frac{1}{r^2}$, as proposed by Greene and Aldrich [58,59]. The implies that eq. (24) is not a good approximation to the centrifugal barrier when the screening parameter α becomes large. Thus, the approximation is valid when $\alpha << 1$. Substituting the approximation (eq. 24) into eq. (23), we obtain an equation of the form;

$$\frac{d^2R_{n\ell}(r)}{dr^2}+\left[\frac{2\mu E}{\hbar^2\alpha^2}+\frac{2\mu V_0}{\hbar^2}\left(\frac{\alpha}{1-e^{-\alpha r}}\right)-\frac{2\mu V_1}{\hbar^2}\left(\frac{\alpha e^{-\alpha r}}{1-e^{-\alpha r}}\right)-\frac{\alpha^2\Lambda}{(1-e^{-\alpha r})^2}\right]R_{n\ell}(r)=0 \tag{25}$$

Eq. (25) can be simplified into the form and introducing the following dimensionless abbreviations

$$\begin{Bmatrix}\varepsilon_n=-\frac{2\mu E}{\hbar^2\alpha^2}\\ \eta=\frac{2\mu V_1}{\hbar^2\alpha}\\ \beta=\frac{2\mu V_0}{\hbar^2\alpha}\end{Bmatrix} \tag{26}$$

$$\frac{d^2R_{n\ell}(r)}{dr^2}+\frac{1}{(1-e^{-\alpha r})^2}\left[-\varepsilon_n(1-e^{-\alpha r})^2+\beta(1-e^{-\alpha r})-\eta(1-e^{-\alpha r})-\Lambda\right]R_{n\ell}(r)=0 \tag{27}$$

Using a transformation $z=e^{-\alpha r}$ so as to enable us apply the NU method as a solution of the hypergeometric type

$$\frac{d^2R_{n\ell}(r)}{dr^2}=\alpha^2z^2\frac{d^2R_{n\ell}(z)}{dz^2}+\alpha^2z\frac{dR_{n\ell}(z)}{dz} \tag{28}$$

We obtain the differential equation

$$\frac{d^2R_{n\ell}}{dz^2}+\frac{(1-z)}{z(1-z)}\frac{dR_{n\ell}}{dz}+\frac{1}{z^2(1-z)^2}\left[-(\varepsilon_n-\eta)z^2+(2\varepsilon_n-\beta-\eta)z-(\varepsilon_n-\beta+\Lambda)\right]R_{n\ell}(z)=0 \tag{29}$$

Comparing Eq. (29) and Eq. (4), we have the following parameters

$$\begin{Bmatrix}\tilde{\tau}(z)=1-z\\ \sigma(z)=z(1-z)\\ \tilde{\sigma}(s)=-(\varepsilon_n-\eta)z^2+(2\varepsilon_n-\beta-\eta)z-(\varepsilon_n-\beta+\Lambda)\end{Bmatrix} \tag{30}$$

Substituting these polynomials into Eq. (12), we get $\pi(s)$ to be

$$\pi(z)=-\frac{z}{2}\pm\sqrt{(a-k)z^2+(b+k)z+c} \tag{31}$$

where

$$\begin{Bmatrix}a=\frac{1}{4}+(\varepsilon_n-\eta)\\ b=-(2\varepsilon_n-\beta-\eta)\\ c=\varepsilon_n-\beta+\Lambda\end{Bmatrix} \tag{32}$$

To find the constant k, the discriminant of the expression under the square root of Eq. (31) should be equal to zero. As such, we have that

$$k_\pm=-(\eta-\beta+2\Lambda)\pm 2\sqrt{\varepsilon_n-\beta+\Lambda}\sqrt{\frac{1}{4}+\Lambda} \tag{33}$$

Substituting Eq. (33) into Eq. (31) yields

$$\pi=-\frac{z}{2}\pm\begin{Bmatrix}(\sqrt{a_1}-\sqrt{a_3})z-\sqrt{a_1};\ \ for\ k_+=-(a_2)+2\sqrt{a_1}\sqrt{a_3}\\ (\sqrt{a_1}-\sqrt{a_3})z+\sqrt{a_1};\ \ for\ k_-=-(a_2)-2\sqrt{a_1}\sqrt{a_3}\end{Bmatrix} \tag{34}$$

where

$$\begin{Bmatrix} a_1 = \varepsilon_n - \beta + \Lambda \\ a_2 = \eta - \beta + 2\Lambda \\ a_3 = \frac{1}{4} + \Lambda \end{Bmatrix} \quad (35)$$

From the knowledge of NU method, we choose the expression $\pi(s)_-$ which the function $\tau(s)$ has a negative derivative. This is given by

$$k_- = -(\eta - \beta + 2\Lambda) \pm 2\sqrt{\varepsilon_n - \beta + \Lambda}\sqrt{\frac{1}{4} + \Lambda} \quad (36)$$

with $\tau(s)$ being obtained as

$$\tau(s) = 1 - 2z - 2\left(\sqrt{\varepsilon_n - \beta + \Lambda} + \sqrt{\frac{1}{4} + \Lambda}\right)z + 2\sqrt{\varepsilon_n - \beta + \Lambda} \quad (37)$$

Referring to Eq. (13), we define the constant λ as

$$\lambda = -(\eta - \beta + 2\Lambda) - 2\sqrt{\varepsilon_n - \beta + \Lambda}\sqrt{\frac{1}{4} + \Lambda} - \frac{1}{2} - \left(\sqrt{\varepsilon_n - \beta + \Lambda} - \sqrt{\frac{1}{4} + \Lambda}\right) \quad (38)$$

Taking the derivative of $\tau(s)$ from eq.(37), we have;

$$\tau'(z) = -2 - 2\left(\sqrt{\varepsilon_n - \beta + \Lambda} + \sqrt{\frac{1}{4} + \Lambda}\right) \quad (39)$$

and$\sigma(z)$ from eq.(30), we have;

$$\sigma''(z) = -2 \quad (40)$$

Substituting Eqs (39) into Eq. (40), we have

$$\lambda_n = n^2 + n + 2n\sqrt{\varepsilon_n - \beta + \Lambda} + 2n\sqrt{\frac{1}{4} + \Lambda} \quad (41)$$

Comparing Eqs (38) and (41),and carrying out some algebraic manipulation. We have;

$$\varepsilon_n = -\lambda + \beta + \frac{1}{4}\left[\frac{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \Lambda}\right)^2 + \eta - \beta + \Lambda}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \Lambda}\right)}\right]^2 \quad (42)$$

Substituting Eqs. (17) and Eq. (32) into Eq. (31) yields the energy eigenvalue equation of the Hellman potential in the form

$$E_{n\ell} = \frac{\hbar^2\alpha^2\Lambda}{2\mu} - V_0\alpha - \frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \Lambda}\right)^2 + \frac{2\mu V_1}{\hbar^2\alpha} - \frac{2\mu V_0}{\hbar^2\alpha} + \Lambda}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \Lambda}\right)}\right]^2 \quad (43)$$

The corresponding wave functions can be evaluated by substituting $\pi(s)_-$ *and* $\sigma(s)$ from Eq. (34) and Eq. (30) respectively into Eq. (7) and solving the first order differential equation. This gives

$$\mathrm{A}(z) = z^{\sqrt{\varepsilon_n - \beta + \Lambda}}(1 - z)^{\frac{1}{2} + \sqrt{\frac{1}{4} + \Lambda}} \quad (44)$$

The weight function $\rho(s)$ from Eq. (10) can be obtained as

$$\rho(z) = z^{2\sqrt{\varepsilon_n - \beta + \Lambda}}(1 - z)^{2\sqrt{\frac{1}{4} + \Lambda}} \quad (45)$$

From the Rodrigues relation of Eq. (9), we obtain

$$y_n(z) \equiv \Omega_{n,l} P_n^{\left(2\sqrt{\varepsilon_n - \beta + \Lambda}, 2\sqrt{\frac{1}{4} + \Lambda}\right)}(1 - 2z) \quad (46)$$

where $P_n^{(\theta,\vartheta)}$ is the Jacobi Polynomial.

Substituting $\mathrm{A}(s)$ *and* $y_n(s)$ from Eq. (44) and Eq. (46) respectively into Eq. (5), we obtain

$$R_n(z) = \Omega_{n,l} z^{\sqrt{\varepsilon_n - \beta + \Lambda}}(1 - z)^{\frac{1}{2} + \sqrt{\frac{1}{4} + \Lambda}} P_n^{\left(2\sqrt{\varepsilon_n - \beta + \Lambda}, 2\sqrt{\frac{1}{4} + \Lambda}\right)}(1 - 2z) \quad (47)$$

where $\Omega_{n,l}$ is a normalization constant.

Using the normalization condition, we obtain the normalization constant as follows [60]:

$$\int_0^{\infty} R_{n,\ell}(r) \times R_{n,\ell}(r)^* dr = 1 \quad (48)$$

$$-\frac{1}{\alpha}\int_1^0 \left|R_{n,\ell}(z)\right|^2 \frac{dz}{z} = 1\,, z = e^{-\alpha r} \quad (49)$$

$$\frac{1}{2\alpha}\int_{-1}^{1} \left|R_{n,\ell}(y)\right|^2 \frac{2}{1-y} dy = 1, y = 1 - 2z \quad (50)$$

Substituting (47) into (50), we have

$$\frac{\Omega_{n\ell}^2}{2\alpha}\int_{-1}^{1} \left(\frac{1-y}{2}\right)^{2\varsigma-1} \left(\frac{1+y}{2}\right)^{2\xi} \left[P_n^{(2\varsigma,2\xi-1)}(y)\right]^2 dy = 1, \quad (51)$$

where

$$\xi = \frac{1}{2} + \sqrt{\frac{1}{4} + \Lambda}, \quad (52)$$

$$\varsigma = \sqrt{\varepsilon_n - \beta + \Lambda} \quad (53)$$

Comparing (51) with the integral of the form [60]

$$\int_1^{-1} \left(\frac{1-p}{2}\right)^x \left(\frac{1+p}{2}\right)^y \left[P_n^{(2x,2y-1)}(p)\right]^2 dp = \frac{2\Gamma(x+n+1)\Gamma(y+n+1)}{\mathrm{n!x}\Gamma(\alpha+\beta+n+1)} \quad (54)$$

We have the normalization constant as

$$\Omega_{n\ell} = \sqrt{\frac{\mathrm{n!}2\sqrt{\varepsilon_n-\beta+\Lambda}\alpha\Gamma\left(2\sqrt{\varepsilon_n-\beta+\Lambda}+\sqrt{1+4\Lambda}+n+2\right)}{\Gamma\left(2\left(\sqrt{\varepsilon_n-\beta+\Lambda}\right)+n+1\right)\Gamma\left(n+2+\sqrt{1+4\Lambda}\right)}} \quad (55)$$

3.2 Solutions of the angular Schrodinger equation for ring-shaped Hellman potential

In order to get the solution of equation Eq. (20), we introduce a coordinate transformation of the form, $z = cos^2\theta$ and Eq. (20) becomes

$$\frac{d^2\Theta(z)}{dz^2} + \frac{(1-3z)}{2z(1-z)}\frac{d\Theta(z)}{dz} + \frac{1}{\left(2z(1-z)\right)^2}\left(-(\Lambda+B)z^2 + (\Lambda - \mathrm{A} - m^2)z - C\right)\Theta(z) = 0 \quad (56)$$

Similarly, Comparing Eq. (56) and Eq. (4), we have the following parameters

$$\left\{\begin{matrix} \tilde{\tau}(s) = (1-3z) \\ \sigma(s) = 2z(1-z) \\ \tilde{\sigma}(s) = -(\Lambda+B)z^2 + (\Lambda - \mathrm{A} - m^2)z - C \end{matrix}\right\} \quad (57)$$

Substituting these polynomials into Eq. (12), we get $\pi(s)$ to be

$$\pi(z) = -\frac{1-z}{2} \pm \sqrt{(a-k)z^2 + (b+k)z + c} \quad (58)$$

where

$$\left\{\begin{matrix} a = \frac{1}{4} + (\Lambda + B) \\ b = -\frac{1}{2} - (\Lambda - \mathrm{A} - m^2) \\ c = \frac{1}{4} + C \end{matrix}\right\} \quad (59)$$

To find the constant k, the discriminant of the expression under the square root of Eq. (58) should be equal to zero. As such, we have that

$$k_{\pm} = -\frac{(\Lambda - \mathrm{A} - m^2 - C)}{2} \pm \frac{1}{2}\sqrt{1+4C}\sqrt{\mathrm{C} + \mathrm{A} + m^2 + B} \quad (60)$$

Substituting Eq. (60) into Eq. (58) yields

$$\pi = -\frac{z}{2} \pm \frac{1}{2}\left(\left(2\sqrt{1+4C} + \sqrt{\mathrm{C}+\mathrm{A}+m^2+B}\right)z - 2\sqrt{1+4C}\right) \quad (61)$$

From the knowledge of NU method, we choose the expression $\pi(s)_-$ which the function $\tau(s)$ has a negative derivative. This is given by

$$k_- = -\frac{(\Lambda - \mathrm{A} - m^2 - C)}{2} - \frac{1}{2}\sqrt{1+4C}\sqrt{\mathrm{C} + \mathrm{A} + m^2 + B} \quad (62)$$

with $\tau(s)$ being obtained as

$$\tau(s) = 2 - 4z - 2\left(\sqrt{1+4C} + \sqrt{\mathrm{C}+\mathrm{A}+m^2+B}\right)z + 2\sqrt{1+4C} \tag{63}$$

Referring to Eq. (13), we define the constant λ as

$$\lambda = -\frac{(\Lambda - \mathrm{A} - m^2 - C)}{2} - \frac{1}{2}\sqrt{1+4C}\sqrt{\mathrm{C}+\mathrm{A}+m^2+B} - \frac{1}{2} - \frac{1}{2}\left(2\sqrt{1+4C} + \sqrt{\mathrm{C}+\mathrm{A}+m^2+B}\right) \tag{64}$$

Taking the derivative of $\tau(s)$ from eq.(37), we have;

$$\tau'(z) = -4 - 2\left(\sqrt{1+4C} + \sqrt{\mathrm{C}+\mathrm{A}+m^2+B}\right) \tag{65}$$

and$\sigma(z)$ from eq.(57), we have;

$$\sigma''(z) = -4 \tag{66}$$

Substituting Eqs (65) into Eq. (66), we have

$$\lambda_{\tilde{n}} = 2\tilde{n}^2 + 2\tilde{n} + \tilde{n}\sqrt{1+4C} + \tilde{n}\sqrt{\mathrm{C}+\mathrm{A}+m^2+B} \tag{67}$$

Comparing Eqs (67) and (64)($\lambda = \lambda_{\tilde{n}}$),and carrying out some algebraic manipulation. We have;

$$\Lambda = \left(2\tilde{n} + 1 + \sqrt{\mathrm{C}+\mathrm{A}+m^2+B}\right)^2 + \sqrt{1+4C}\left(2\tilde{n} + 1 + \sqrt{\mathrm{C}+\mathrm{A}+m^2+B}\right) + C - B \tag{68}$$

or

$$\Lambda = \left(2\tilde{n} + 1 + \sqrt{\mathrm{C}+\mathrm{A}+m^2+B}\right)\left(2\tilde{n} + 1 + \sqrt{\mathrm{C}+\mathrm{A}+m^2+B} + \sqrt{1+4C}\right) + C - B \tag{69}$$

The corresponding wave functions can be evaluated by substituting $\pi(s)\_$ *and* $\sigma(s)$ from Eq. (57) and Eq. (61) respectively into Eq. (7) and solving the first order differential equation. This gives

$$\mathrm{E}(z) = z^{\frac{1}{4}+\frac{1}{2}\sqrt{\frac{1}{4}+C}}(1-z)^{\frac{1}{2}\sqrt{\mathrm{C}+\mathrm{A}+m^2+B}} \tag{70}$$

The weight function $\rho(s)$ from Eq. (10) can be obtained as

$$\rho(z) = z^{\sqrt{\frac{1}{4}+C}}(1-z)^{\sqrt{\mathrm{C}+\mathrm{A}+m^2+B}} \tag{71}$$

From the Rodrigues relation of Eq. (9), we obtain

$$y_{\tilde{n}}(z) \equiv \chi_{\tilde{n},m} P_{\tilde{n}}^{\left(\sqrt{\frac{1}{4}+C},\sqrt{\mathrm{C}+\mathrm{A}+m^2+B}\right)}(1-2z) \tag{72}$$

where $P_n^{(\theta,\vartheta)}$ is the Jacobi Polynomial.

Substituting $\mathrm{E}(s)$ *and* $y_{\tilde{n}}(s)$ from Eq. (70) and Eq. (72) respectively into Eq. (5), we obtain

$$\Theta_{\tilde{n}m}(z) = \chi_{\tilde{n},m} z^{\frac{1}{4}+\frac{1}{2}\sqrt{\frac{1}{4}+C}}(1-z)^{\frac{1}{2}\sqrt{\mathrm{C}+\mathrm{A}+m^2+B}} P_n^{\left(\sqrt{\frac{1}{4}+C},\sqrt{\mathrm{C}+\mathrm{A}+m^2+B}\right)}(1-2z) \tag{73}$$

Where $\chi_{\tilde{n},m}$is a normalization constant.

Now using Eq. (43), we obtain the discrete energy eigenvalues as

$$E_{n\,\tilde{n},m} = \frac{\hbar^2\alpha^2\kappa}{2\mu} - V_0\alpha - \frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\Lambda}\right)^2 + \frac{2\mu V_1}{\hbar^2\alpha} - \frac{2\mu V_0}{\hbar^2\alpha} + \kappa}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\kappa}\right)}\right]^2 \tag{74}$$

$$\kappa = \left(2\tilde{n} + 1 + \sqrt{\mathrm{C}+\mathrm{A}+m^2+B}\right)\left(2\tilde{n} + 1 + \sqrt{\mathrm{C}+\mathrm{A}+m^2+B} + \sqrt{1+4C}\right) + C - B \tag{75}$$

where $\tilde{n}$ is the number of nodes of the radial wave functions. The Λ is the contribution from the angle-dependent part of the potential and plays the role of centrifugal term.

$\psi(r,\theta,\phi) =$

$\frac{N_{\tilde{n}m}}{\sqrt{2\pi}} z^{\sqrt{\varepsilon_n - \beta + \Lambda}}(1-z)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\Lambda}} P_n^{\left(2\sqrt{\varepsilon_n-\beta+\Lambda},2\sqrt{\frac{1}{4}+\Lambda}\right)}(1-$

*2zcos2θ14+1214+C(sin2θ)12*C+A+*m2+BPn14+C,* C+A+*m2+B*−cos*2θe*−*imϕ*

(76)

where $\mathrm{N}_{n\,\tilde{n}m}$ is the new normalization constant

4. Special Cases

In this section, we take adjustments of some potential parameters in Eqs. (1) and (74) to have the following cases:

- **Yukawa-angle Dependent Potential**

$V_0 = 0$, Eq. (3) reduces to the Yukawa or modified coulomb potential

$$V(r,\theta) = \frac{V_1 e^{-\alpha r}}{r} + \frac{\hbar^2}{2\mu r^2}\left(\frac{C+B\cos^2\theta + A\cos^4\theta}{\sin^2\theta\cos^2\theta}\right) \tag{77}$$

and the energy equation(eq. 74) becomes

$$E_{n\,\tilde{n},m} = \frac{\hbar^2\alpha^2\Lambda}{2\mu} - \frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\Lambda}\right)^2 + \frac{2\mu V_1}{\hbar^2\alpha} + \Lambda}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\Lambda}\right)}\right]^2 \tag{78}$$

$$\Lambda = \left(2\tilde{n}+1+\sqrt{C+A+m^2+B}\right)\left(2\tilde{n}+1+\sqrt{C+A+m^2+B}+\sqrt{1+4C}\right)+C-B \tag{79}$$

Comment; The Authors in ref. [62] used Taylor's series expansion to deal with the exponential and didn't use the Greene and Aldrich approximation [56]. By employing the Greene and Aldrich approximation, we have been able to get a better and more accurate result.

- **Novel Angle Dependent Coulomb Potential**

$V_1 = 0$ and $\alpha \to 0$ Eq. (3) reduces to the NAD Coulomb potential(new Coulomb ring-shaped potential) [54]

$$V(r,\theta) = -\frac{V_0}{r} + \frac{\hbar^2}{2\mu r^2}\left(\frac{C+B\cos^2\theta + A\cos^4\theta}{\sin^2\theta\cos^2\theta}\right) \tag{80}$$

and the energy equation(eq. 74) becomes

$$E_{n\,\tilde{n},m} = -\frac{\mu V_0^2}{2\hbar^2\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\left(2\tilde{n}+1+\sqrt{C+A+m^2+B}\right)\left(2\tilde{n}+1+\sqrt{C+A+m^2+B}+\sqrt{1+4C}\right)+C-B}\right)^2} \tag{81}$$

After carrying out some algebraic manipulation, Eq.(81) agrees excellently with Eq. (28) of ref. [54]

- **Hartmann Potential**

$V_1 = 0$ $C = A = 0$ and $\alpha \to 0$ Eq. (3) reduces to the Hartmann Potential [54]

$$V(r,\theta) = -\frac{V_0}{r} + \frac{B}{r^2\sin^2\theta} \tag{82}$$

and the energy equation (eq. 74) becomes

$$E_{n\,\tilde{n},m} = -\frac{\mu V_0^2}{2\hbar^2\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\left(2\tilde{n}+1+\sqrt{m^2+B}\right)^2-B}\right)^2} \tag{83}$$

Eq. (83) agrees excellently with Eq. (29) of ref. [54]

- **Coulomb Potential Plus A New Ring-Shaped Potential**

$V_1 = 0 C = B = 0$ and $\alpha \to 0$ reduces to the Coulomb potential plus a new ring-shaped potential proposed by [61];

$$V(r,\theta) = -\frac{V_0}{r} + \frac{A\cos^2\theta}{r^2\sin^2\theta} \tag{84}$$

and the energy equation (eq. 74) becomes

$$E_{n\,\tilde{n},m} = -\frac{\mu V_0^2}{2\hbar^2\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\left(2\tilde{n}+1+\sqrt{A+m^2}\right)\left((2\tilde{n}+1)+\sqrt{A+m^2+1}\right)}\right)^2} \tag{85}$$

Eq. (83) agrees excellently with Eq. (19) of ref. [61]

5. Discussion

In this paper, we consider the eigensolutions of the Schrodinger equation with Ring-shaped Hellmann Potential. For verification sake, we compute the numerical eigenvalue of the Hellmann Potential (Table 1) using Eq.(43), our results are in agreement with literature. In Fig.(1-2), we plot the variation of the energy level for various n as a function of the $\tilde{n}$ and m. From the graph, it is evident that the energy increases as the quantum numbers (n, $\tilde{n}$ and

m) increases. Fig. 3;, we plot the variation of the energy level for various n as a function of the α. It's seen that the energy decreases as the screening parameter increases. Fig. (4-5), we plot the variation of the energy level for various n as a function of the V_0 and V_1. Similarly, the energy also decreases as parameters V_0 and V_1 increases. Figure 6-8 shows the variation of the energy level for various n as a function of the A, B and C. In these plots, it is clearly seen that as parameters A, B and C increases the eigenvalue also does same.

Table 1: Comparison of energy spectrum from SUSY, Parametric Nikiforov-Uvarov (N.U) and Amplitude Phase method with $h = V_1 = 2\mu = 1$ and $V_0 = 4\mu$

State	α	Present	SUSY[42]	pNU[43]	APM[43]
1s	0.001	-0.25150025	-0.251 500	-0.251 500	-0.250 969
	0.005	-0.25750625	-0.257 506	-0.257 506	-0.254 933
	0.01	-0.26502500	-0.265 025	-0.265 025	-0.259 823
2s	0.001	-0.64001000	-0.064 001	-0.064 001	-0.063 243
	0.005	-0.07002500	-0.070 025	-0.070 025	-0.067 106
	0.01	-0.07760000	-0.077 600	-0.077 600	-0.071 689
2p	0.001	-0.06375025	-0.063 750	-0.064 000	-0.063 495
	0.005	-0.06875625	-0.068 756	-0.070 000	-0.067 377
	0.01	-0.07502500	-0.075 025	-0.077 500	-0.072 020
3s	0.001	-0.02928003	-0.029 280	-0.029 280	-0.028 283
	0.005	-0.03533403	-0.035 334	-0.035 334	-0.031 993
	0.01	-0.04300278	-0.043 003	-0.043 003	-0.036 142
3p	0.001	-0.02916803	-0.029 169	-0.029 279	-0.028 765
	0.005	-0.03475625	-0.034 756	-0.035 309	-0.032 480
	0.01	-0.04180278	-0.041 803	-0.042 903	-0.036 142
3d	0.001	-0.02894469	-0.028 945	-0.029 388	-0.028 767
	0.005	-0.03361736	-0.033 617	-0.035 817	-0.032 526
	0.01	-0.03946944	-0.039 469	-0.043 825	-0.036 613
4s	0.001	-0.01712900	-0.017 129	-0.029 280	-0.016 601
	0.005	-0.02322500	-0.023 225	-0.035 334	-0.020 077
	0.01	-0.03102500	-0.031 025	-0.043 003	-0.023 551
4p	0.001	-0.01706556	-0.017 066	-0.017 128	-0.016 602
	0.005	-0.02288906	-0.022 889	-0.023 200	-0.020 098
	0.01	-0.03030625	-0.030 306	-0.030 925	-0.023 641
4d	0.001	-0.01693906	-0.016 939	-0.017 189	-0.016 604
	0.005	-0.02222656	-0.022 227	-0.023 464	-0.020 098
	0.01	-0.02890625	-0.028 906	-0.031 356	-0.023 641
4f	0.001	-0.01675025	-0.016 750	-0.017 311	-0.016 607
	0.005	-0.02125625	-0.021 257	-0.024 024	-0.020 142
	0.01	-0.02690000	-0.026 900	-0.032 356	-0.024 056

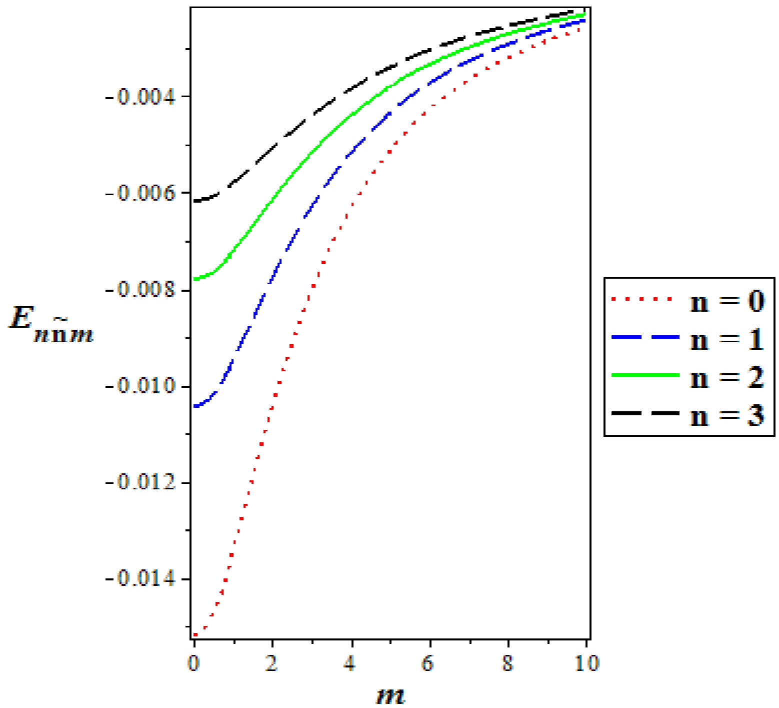

Figure 1: The variation of the energy level for various n as a function of the m. We choose $h = V_1 = 2\mu = 1, V_0 = 4\mu, \alpha = 0.001$, $\tilde{n} = 0$ and $A = B = C = 1$

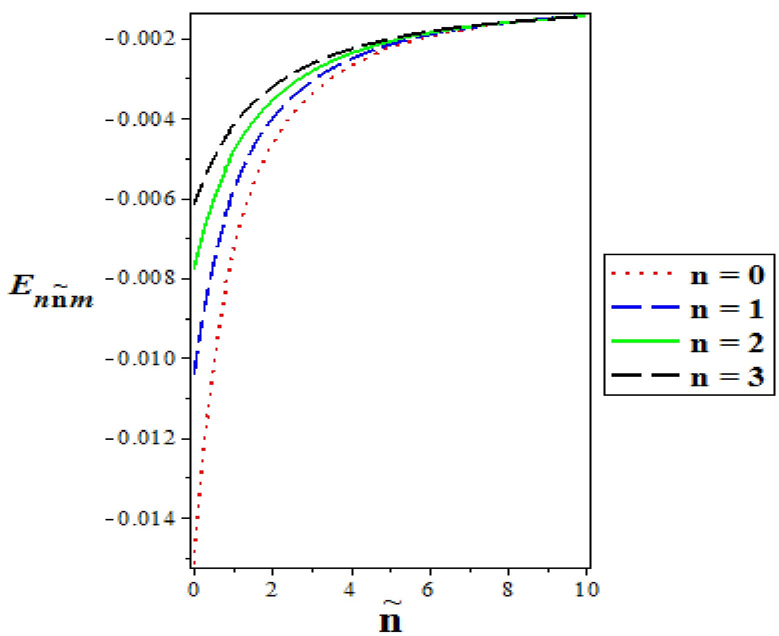

Figure 2: The variation of the energy level for various n as a function of the $\tilde{n}$. We choose $h = V_1 = 2\mu = 1, V_0 = 4\mu, \alpha = 0.001$, $m = 0$ and $A = B = C = 1$

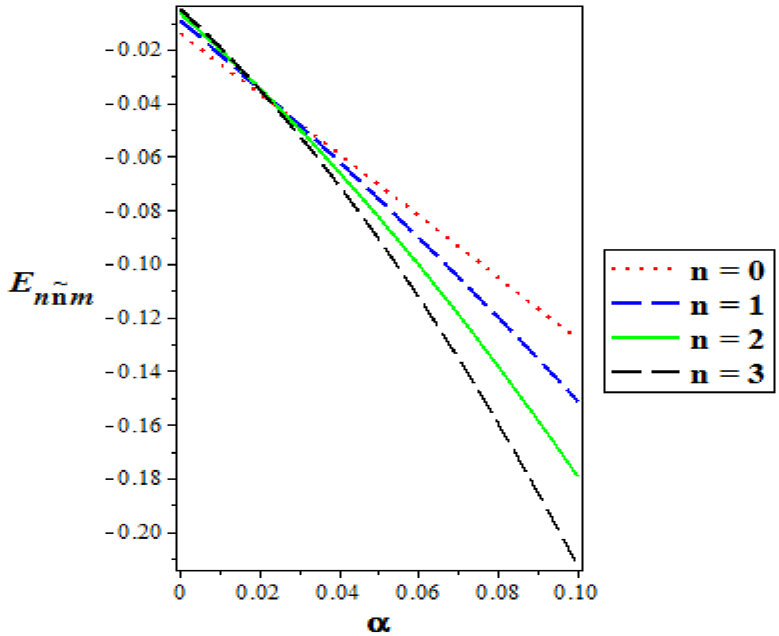

Figure 3: The variation of the energy level for various n as a function of the α. We choose $h = V_1 = 2\mu = 1, V_0 = 4\mu, \alpha = 0.001, \tilde{n} = m = 0$ and $A = B = C = 1$

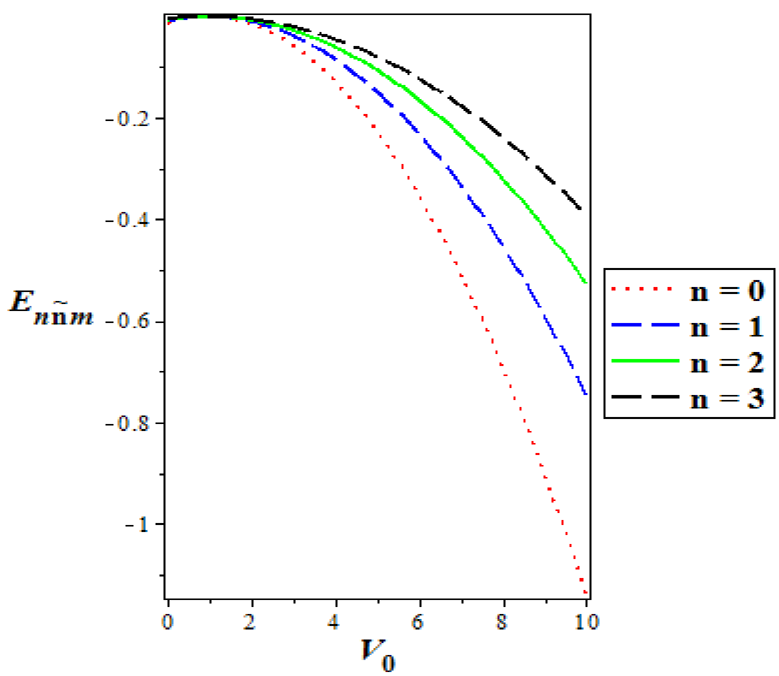

Figure 4: The variation of the energy level for various n as a function of the V_0. We choose $h = V_1 = 2\mu = 1, \alpha = 0.001, \tilde{n} = m = 0$ and $A = B = C = 1$

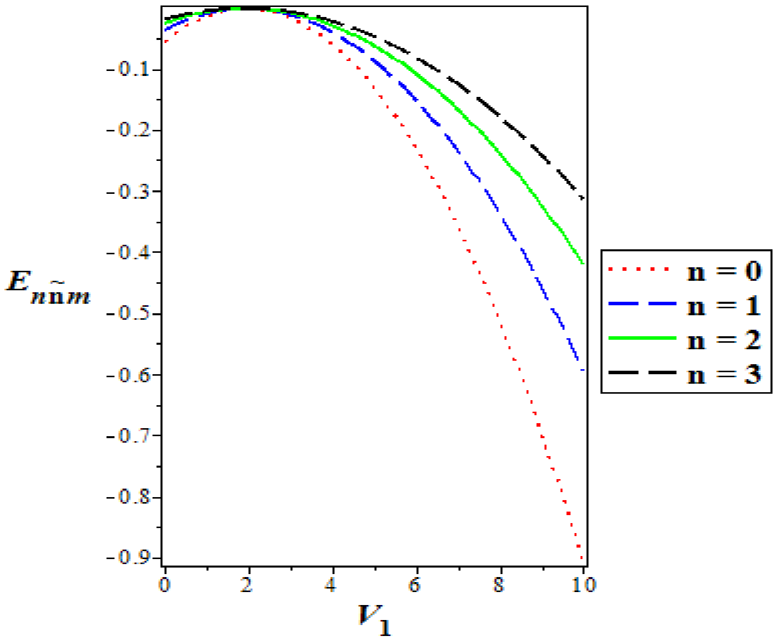

Figure 5: The variation of the energy level for various n *as a function of the* V_1*. We choose* $h = 2\mu = 1, V_0 = 4\mu, \tilde{n} = m = 0$ *and* $A = B = C = 1$

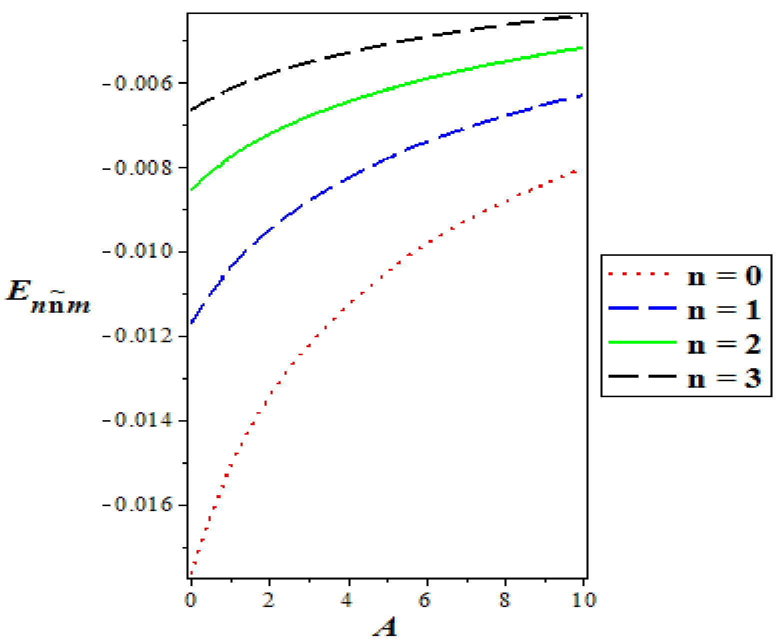

Figure 6: The variation of the energy level for various n *as a function of the* A*. We choose* $h = V_1 = 2\mu = 1, V_0 = 4\mu, \alpha = 0.001$, $\tilde{n} = m = 0$ *and* $B = C = 1$

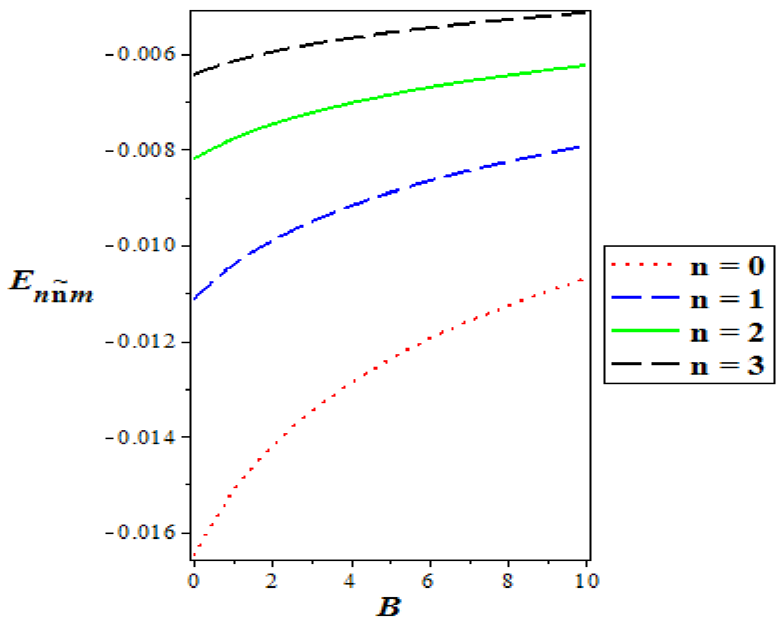

Figure 7: The variation of the energy level for various n *as a function of the* B. *We choose* $h = V_1 = 2\mu = 1, V_0 = 4\mu, \alpha = 0.001$, $\tilde{n} = m = 0$ *and* $A = C = 1$

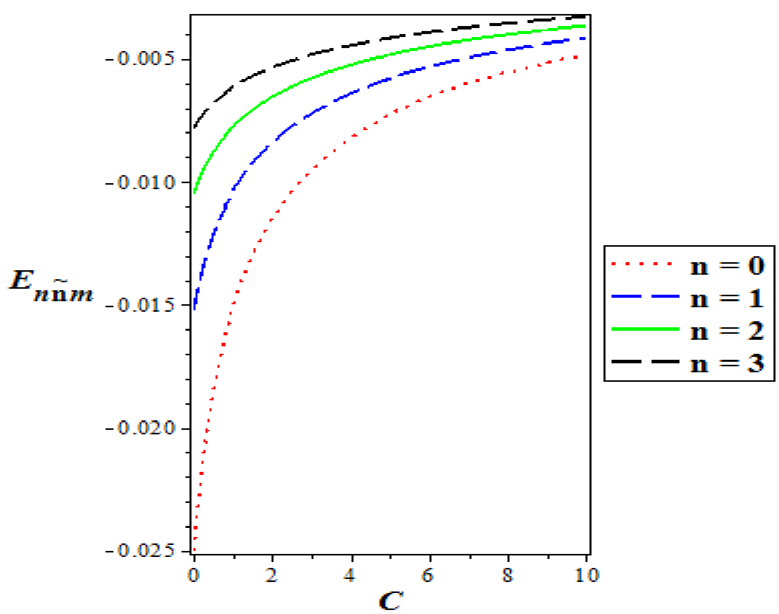

Figure 8: The variation of the energy level for various n *as a function of the* C. *We choose* $h = V_1 = 2\mu = 1, V_0 = 4\mu, \alpha = 0.001$, $\tilde{n} = m = 0$ *and* $A = B = 1$

6. Conclusion and Remarks

We have proposed a new approximate exactly solvable non-central potential, ring-shaped Hellmann potential and solved the Schrödinger equation with this potential by the NU method analytically. It is shown that in spherical coordinate, the Schrödinger equation with this non-central potential could be separated into the angular and radial components. Our results may have interesting applications in the study of various quantum mechanical systems and atomic physics. We presented the effect of the angle-dependent part on radial solutions and some special cases are also discussed. Numerical results were obtained for the Ring shaped Hellmann potential and the Hellmann potential respectively. It was found out that our results agree with existing literature. The thermal and magnetic properties of this model should be considered as an extension [63].

References

[1]. A.N. Ikot, U.S. Okorie, G. Osobonye, P.O. Amadi, C.O. Edet, M.J. Sithole, G.J. Rampho, R. Sever, *Superstatistics of Schrödinger equation with pseudo-harmonic potential in external magnetic and Aharanov-Bohm fields*, Heliyon6 (2020) e03738, https://doi.org/10.1016/j.heliyon.2020.e03738

[2]. Okorie, U. S., Ikot, A. N., Edet, C. O., Rampho, G. J., Horchani, R., &Jelassi, H. (2021). Bound and scattering states solutions of the Klein–Gordon equation with generalized Mobius square potential in D-dimensions. *The European Physical Journal D*, *75*(2), 1-8.

[3]. H. Louis, B.I. Ita, T.O. Magu, O.U. Akakuru, N.A. Nzeata-Ibe, A.I Ikeuba, A.I. Pigweh and C.O. Edet, *Solutions to the Dirac Equation for Manning-Rosen Plus Shifted Deng-Fan Potential and Coulomb-Like Tensor Interaction Using Nikiforov-Uvarov Method*, Intl. J. Chem. 10 (2018) 99, https://doi.org/10.5539/ijc.v10n3p99

[4]. U. S. Okorie, A. N. Ikot, C. O. Edet, I. O. Akpan, R. Sever and R. Rampho, *Solutions of the Klein Gordon equation with generalized hyperbolic potential in D-dimensions*, J. Phys. Commun. 3 (2019) 095015, https://doi.org/10.1088/2399-6528/ab42c6

[5]. C.O. Edet, U.S. Okorie, G. Osobonye, A.N. Ikot, G.J. Rampho and R. Sever, *Thermal properties of Deng–Fan–Eckart potential model using Poisson summation approach*, J. Math. Chem. 58 (2020) 989. https://doi.org/10.1007/s10910-020-01107-4

[6]. H. Louis, B.I. Ita, O.U. Akakuru, N.A. Nzeata-Ibe, A.I. Ikeuba, T.O. Magu, P.I. Amos and C.O. Edet, *l-state Solutions of the Relativistic and Non-Relativistic Wave Equations for Modified Hylleraas-Hulthen Potential Using the Nikiforov-Uvarov Quantum Formalism*, Oriental J. Phys. Sci. 3 (2018) 1. http://www.orientjphysicalsciesnces.org/

[7]. C.O. Edet, P.O. Amadi, U.S. Okorie, A. Tas, A.N. Ikotand G. Rampho, Solutions of Schrödinger equation and thermal properties of generalized trigonometric P¨oschl-Teller potential. (2020). Rev. Mex. Fıs. 66 (2020) 824. https://doi.org/10.31349/RevMexFis.66.824

[8]. A. N. Ikot, U. S. Okorie, G. J. Rampho, P. O. Amadi, C. O. Edet, I. O. Akpan, H. Y. Abdullah & R. Horchani, Klein–Gordon Equation and Nonrelativistic Thermodynamic Properties with Improved Screened Kratzer Potential. Journal of Low Temperature Physics, 1-21. (2021). https://doi.org/10.1007/s10909-020-02544-w

[9]. A. N. Ikot, C. O. Edet, P. O. Amadi, U. S. Okorie, G. J. Rampho, and H. Y. Abdullah, *Thermodynamic properties of Aharanov-Bohm (AB) and magnetic fields with screened Kratzer potential*, Eur. Phys. J. D 74 (2020) 159, https://doi.org/10.1140/epjd/e2020-10084-9

[10]. Y. Xu, S. He and C. S. Jia. Approximate analytical solutions of the Klein–Gordon equation with the Pöschl–Teller potential including the centrifugal term *Phys. Scripta* 81, 045001 (2010).

[11]. F. L. Lu and C. Y. Chen, Bound states of the Schrödinger equation for the Pöschl—Teller double-ring-shaped Coulomb potential *Chin. Phys. B*19, 100309 (2010).

[12]. M. C. Zhang, B. A., G H Huang-Fu. Exact solutions of a new Coulomb ring-shaped potential. *J Math Chem* 48, 876 (2010).

[13]. A. D. Antia and O. P. AkpanYukawa-angle Dependent Potential and its Applications to Diatomic Molecules under Schrodinger wave Equation *JAplTheol*1 , 9(2017)

[14]. C-Y Chen and S H Dong. Exactly complete solutions of the Coulomb potential plus a new ring-shaped potential. *PhysLett. A*335, 374

[15]. A. de Souza Dutra and M. Hott, Dirac equation exact solutions for generalized asymmetrical Hartmann potentials. Phys. Lett. A 356, 215 (2006).

[16]. C. O. Edet, P. O. Okoi, S. O. Chima, *Analytic solutions of the Schrödinger equation with non-central generalized inverse quadratic Yukawa potential*, Rev. Bras. Ens. Fís. 42 (2019) e20190083. https://doi.org/10.1590/1806-9126-RBEF-2019-0083.

[17]. Xiao Yong-Jun and Long Zheng-Wen. Solution of Klein–Gordon Equation for Pseudo-Coulomb Potential Plus a New Ring-Shaped Potential. Commun. Theor. Phys. 53, 54 (2010)

[18]. S. H. Dong and M. Lozada-Cassou,Exact solutions of the Klein–Gordon equation with scalar and vector ring-shaped potentials. Phys. Scr. 74285(2006)

[19]. Wei Li, Chang-Yuan Chen, and Shi-Hai Dong, "Ring-Shaped Potential and a Class of Relevant Integrals Involved Universal Associated Legendre Polynomials with Complicated Arguments," Advances in High Energy Physics, vol. 2017, Article ID 7374256, 4 pages, 2017. https://doi.org/10.1155/2017/7374256

[20]. H. X. Quan, L. Guang , W. Z. Min, N. L. Bin, and M. Yan. Solving Dirac Equation with New Ring-Shaped Non-Spherical Harmonic Oscillator Potential. Commun. Theor. Phys.53, 242 (2010)

[21]. A. N. Ikot, I. O. Akpan, T.M. Abbey and H.Hassanabadi. Exact Solutions of Schrödinger Equation with Improved Ring-Shaped Non-Spherical Harmonic Oscillator and Coulomb Potential. Commun. Theor. Phys. 65 569 (2016)

[22]. M C Zhang, G. H. Sun, and S. H. Dong, Exactly complete solutions of the Schrödinger equation with a spherically harmonic oscillatory ring-shaped potentialPhys. Lett. A 374, 704 (2010).

[23]. B. J.Falaye The Klein-Gordon equation with ring-shaped potentials: Asymptotic iteration method Journal of Mathematical Physics 53, 082107 (2012)

[24]. A. D.Antia, E.E. ItuenNonrelativistic Treatment of Schrodinger Particles under Inversely Quadratic Hellmann Plus Ring-Shaped Potentials. Ukr. J. Phys. 62, 633 (2017)

[25]. A.N. Ikot, U.S. Okorie, A.T. Ngiangia, C.A. Onate, C.O. Edet, I.O. Akpan, P.O. Amadi, *Bound state solutions of the Schrödinger equation with energy dependent molecular Kratzer potential via asymptotic iteration method*, Eclet. Quím. J. 45 (2020) 65, https://doi.org/10.26850/1678-4618eqj.v45.1.p65-76

[26]. Edet, C. O., Ikot, A. N., Onyeaju, M. C., Okorie, U. S., Rampho, G. J., Lekala, M. L., & Kaya, S. (2021). Thermo-Magnetic Properties of the Screened Kratzer potential with Spatially varying mass under the influence of Aharanov-Bohm (AB) and Position-Dependent Magnetic fields. *Physica E: Low-dimensional Systems and Nanostructures*, 114710.

[27]. Edet, C. O., &Ikot, A. N. (2021). Effects of Topological Defect on the Energy Spectra and Thermo-magnetic Properties of $$ CO $$ CO Diatomic Molecule. *Journal of Low Temperature Physics*, *203*(1), 84-111.

[28]. G. J. Rampho , A. N. Ikot , C. O. Edet& U. S. Okorie, *Energy spectra and thermal properties of diatomic molecules in the presence of magnetic and AB fields with improved Kratzer potential*, Mol. Phys, (2020) https://doi.org/10.1080/00268976.2020.1821922

[29]. Ikot, A. N., Okorie, U. S., Amadi, P. O., Edet, C. O., Rampho, G. J., & Sever, R. (2021). The Nikiforov–Uvarov-Functional Analysis (NUFA) Method: A New Approach for Solving Exponential-Type Potentials. *Few-Body Systems*, *62*(1), 1-16.

[30]. C.O. Edet and P.O. Okoi. Any l-state solutions of the Schr¨odinger equation for q-deformed Hulthen plus generalized inverse quadratic Yukawa potential in arbitrary dimension Revista Mexicana de F´ısica 65, 333. (2019)

[31]. Edet, C. O., Okorie, U. S., Ngiangia, A. T., &Ikot, A. N. (2020). Bound state solutions of the Schrodinger equation for the modified Kratzer potential plus screened Coulomb potential. *Indian journal of Physics*, *94*(4), 425-433.

[32]. Edet, C. O., Okorie, K. O., Louis, H., &Nzeata-Ibe, N. A. (2020). Any l-state solutions of the Schrodinger equation interacting with Hellmann–Kratzer potential model. *Indian Journal of Physics*, *94*(2), 243-251.

[33]. A. D.Antia, A. N.Ikot, H. Hassanabadi, E. Maghsoodi, Indian J. Phys. 87, 1133 (2013)

[34]. A. N. Ikot, H. Hassanabadi, H. P.Obong, and Y. C. Umoren. The Relativistic Screened Coulomb plus Ringed-Shaped-Like Potential via Shape-Invariance Approach. Z. Naturforsch. 69a, 659 (2014)

[35]. B.I. Ita, H. Louis, O.U. Akakuru, N.A. Nzeata-Ibe, A.I. Ikeuba, T.O. Magu, P.I. Amos and C.O. Edet, *Approximate Solution to the Schrödinger Equation with Manning-Rosen plus a Class of Yukawa Potential via WKBJ Approximation Method*, Bulg. J. Phys. 45 (2018) 323. https://www.bjp-bg.com/paper1.php?id=914

[36]. Edet, C. O., &Ikot, A. N. (2021). Shannon information entropy in the presence of magnetic and Aharanov–Bohm (AB) fields. *The European Physical Journal Plus*, *136*(4), 1-11.

[37]. U S Okorie, C O Edet, A N Ikot, G J Rampho and R Sever, Thermodynamic functions for diatomic molecules with modified Kratzer plus screened Coulomb potential, Indian J Phys (2020) https://doi.org/10.1007/s12648-019-01670-w

[38]. H Hellmann,. A New Approximation Method in the Problem of Many Electrons J. Chem. Phys. 3 61 (1935)

[39]. H Hellmann, Acta Physicochim. U.R.S.S. 1, 913 (1934/1935)

[40]. H. Hellmann, W. Kassatotchkim, Metallic Binding According to the Combined Approximation Procedure J. Chem. Phys. 4, 324 (1936)

[41]. C. A. Onate, M. C. Onyeaju, A. N.Ikot, and O. Ebomwonyi, Eigen solutions and entropic system for Hellmann potential in the presence of the Schrödinger equation. Eur. Phys. J. Plus 132, 462 (2017)

[42]. C. A. Onate, J. O. Ojonubah, A. Adeoti, E. J. Eweh and M. Ugboja, Approximate Eigen Solutions of D.K.P. and Klein-Gordon Equations with Hellmann Potential. Afr. Rev. Phys. 9, 0062 (2014)

[43]. M. Hamzavi, K E. Thylwe and A. A.Rajabi,Approximate Bound States Solution of the Hellmann Potential. Commun. Theor. Phys. 60, 1 (2013)

[44]. O.J. Oluwadare and K.J. Oyewumi Scattering states solutions of Klein–Gordon equation with three physically solvable potential models Chinese J. of Phys.55, 2422 (2017)

[45]. Edet, C. O., Amadi, P. O., Onyeaju, M. C., Okorie, U. S., Sever, R., Rampho, G. J., ...& Ikot, A. N. (2021). Thermal Properties and Magnetic Susceptibility of Hellmann Potential in Aharonov–Bohm (AB) Flux and Magnetic Fields at Zero and Finite Temperatures. *Journal of Low Temperature Physics*, *202*(1), 83-105. https://doi.org/10.1007/s10909-020-02533-z

[46]. J. Callaway and P. S. Laghos, Application of the Pseudopotential Method to Atomic Scattering. Phys. Rev. 187, 192 (1969).

[47]. G McGinn, Atomic and Molecular Calculations with the Pseudopotential Method. VII OneValence-Electron Photoionization Cross Sections J. Chem. Phys. 53, 3635 (1970)

[48]. N. Kandirmaz. The Arbitrary l-state Solutions of the Hellmann Potential by Feynman Path Integral Approach. Physical Science International Journal 21, 1, (2019)

[49]. J. C. Phillips and L. Kleinman. New Method for Calculating Wave Functions in Crystals and Molecules Phys. Rev. 116, 287. (1959)

[50]. A. J. Hughes and J. Callaway, Energy Bands in Body-Centered and Hexagonal Sodium Phys. Rev. 136, A1390,(1964)

[51]. S. H. Dong, G.H. Sun, M. Lozada-Cassou. An algebraic approach to the ring-shaped non-spherical oscillator. Phys. Lett. A 328, 299 (2005).

[52]. C. Berkdemir, A novel angle-dependent potential and its exact solution. J. Math. Chem. 46, 139 (2009)

[53]. C. Berkdemir and Y. F. Cheng, On the exact solutions of the Dirac equation with a novel angle-dependent potential. Phys. Scr. 79, 034003 (2009)

[54]. A. A. Rajabi and M. Hamzavi. A new Coulomb ring-shaped potential via generalized parametric Nikiforov-Uvarov method Journal of Theoretical and Applied Physics 2013, 7, 1 (2013)

[55]. G Sezgo *Orthogonal Polynomials*. American Mathematical Society, New York (1939)

[56]. P.O. Okoi, C.O. Edet and T.O.Magu, *Relativistic treatment of the Hellmann-generalized Morse potential*, Rev. Mex. Fis. 66 (2020) 1, https://doi.org/10.31349/RevMexFis.66.1

[57]. Edet, C. (2020). Effects of Magnetic AndAharanov-Bohm (AB) Fields on the Energy Spectra of the Yukawa Potential. *arXiv preprint arXiv:2012.08644*.

[58]. M. R. Setare and S. Haidari, Spin symmetry of the Dirac equation with the Yukawa potential. Phys. Scr. 81, 065201 (2010).

[59]. W. C. Qiang, K. Li and W. L. Chen, New bound and scattering state solutions of the Manning–Rosen potential with the centrifugal term. J. Phys. A: Math. Theor. 42, 205306 (2009)

[60]. O. Ebomwonyi, C. A. Onate , M. C. Onyeaju and A. N. Ikot, Any l states solutions of the Schrödinger equation interacting with Hellmann-generalized Morse potential model, Karbala Int. J. Modern, 3, 59 (2017).

[61]. C. Y. Chen and S. H. Dong, Exactly complete solutions of the Coulomb potential plus a new ring-shaped potential. Phys. Lett. A 335, 374(2005).

[62]. A. D.Antia and O. P. Akpan Nonrelativistic Studies of Diatomic Molecules of Schrödinger Particles with Yukawa Plus Ring-Shaped Potential Model .Appl. and Comp. Math. 6, 54(2017)

[63]. Ikot, A. N., Edet, C. O., Okorie, U. S., Abdel-Aty, A. H., Ramantswana, M., Rampho, G. J., ... & Kaya, S. (2021). Solutions of the 2D Schrodinger equation and its thermal properties for improved ultra-generalized exponential hyperbolic potential (IUGE-HP). The European Physical Journal Plus, 136(4), 1-18