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Research Article

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An Estimate for Upper Bound of Maximum Modulus of Complex Polynomial

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Abstract Let $p(z)$ be a polynomial of degree n. In this paper we have obtained an inequality for the maximum modulus of a polynomial involving the coefficients of polynomial having all its zeros outside a disk of prescribed radius. Our result not only improves upon some well known results but also gives generalizations of some well known earlier proved inequalities.

Keywords Polynomials; complex domain; Inequalities; Zeros
AMS Subject Classification: 30A10, 26A10

1. Introduction and Statement of Results

THEOREM 1.1.*If $p(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

The result is best possible and equality holds for $p(z) = \lambda z^n$, $\lambda (\neq 0)$ being a complex number.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequality (1.1) can be sharpened. In fact in this case the following result was conjectured by Erdös and later verified by Lax [7].

THEOREM 1.2.*If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

The result is best possible and equality in (1.2) *holds for $p(z) = \lambda + \mu z^n$ where $|\lambda| = |\mu|$.*

Simple proofs of this theorem were given by de-Bruijn [4] and Aziz and Mohammad [2]. For other proofs see Boas [3] and Rahman [9].

Inequality (1.2) was further improved by Aziz and Dawood [1] under the same hypothesis by proving the following result.

THEOREM 1.3.*If $p(z)$ is a polynomial of degree n, which does not vanish in $|z| < 1$, then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{2}\left\{\max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)|\right\}. \tag{1.3}$$

The result is best possible and equality holds for $p(z) = \lambda + \mu z^n$, $|\lambda| = |\mu|$.

As a generalization of Theorem 1.2, Malik [8] proved the following

THEOREM 1.4.*If* $p(z)$ *is a polynomial of degree n having no zeros in,* $|z| < k,\ k \ge 1,$ *then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.4}$$

The result is sharp and extremal polynomial is $p(z) = (z+k)^n$.

Various other results in the same sphere could be seen in literature (for references see [10], [11], [12]).

In the present paper, we prove the following result which provides improvement of the Theorem 1.4 due to Malik [8]. The theorem is also of independent interest and could be generalized into many other results. This result paves the path to other results too.

THEOREM 1.5. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree n, not vanishing in* $|z| < k,\ k \ge 1,$ *then for* $0 \le r \le \rho \le k$ *, we have*

$$\max_{|z|=\rho} |p(z)| \le \left(\frac{\rho+k}{r+k}\right)^n \left[1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|} \times \left\{1 - \left(\frac{k+r}{k+\rho}\right)^n\right\}\right] \max_{|z|=r} |p(z)| - \frac{(\rho^n - r^n)}{k^{n-2}}\left\{\frac{n|a_0| + \rho|a_1|}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|}\right\} \min_{|z|=k} |p(z)| \tag{1.5}$$

2. Lemmas

LEMMA 2.1. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree n, having no zeros in,* $|z| < k,\ k \ge 1,$ *then*

$$\max_{|z|=1} |p'(z)| \le n \frac{n|a_0| + k^2|a_1|}{(1+k^2)n|a_0| + 2k^2|a_1|} \max_{|z|=1} |p(z)|. \tag{2.1}$$

The above lemma is due to Govil, Rahman and Schmeisser [5].

LEMMA 2.2. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree n, having no zeros in,* $|z| < k,\ k \ge 1,$ *then*

$$\max_{|z|=1}|p'(z)| \le n\frac{n|a_0|+k^2|a_1|)}{(1+k^2)n|a_0|+2k^2|a_1|}\max_{|z|=1}|p(z)| - \frac{n}{k^n}\left\{1-\frac{n|a_0|+k^2|a_1|)}{(1+k^2)n|a_0|+2k^2|a_1|}\right\}\min_{|z|=k}|p(z)| \tag{2.2}$$

PROOF OF LEMMA 2.2. The lemma can be easily proved by replacing $p(z)$ by $F(z)=p(z)+\lambda m\frac{z^n}{k^n}$ in Lemma 2.1 and applying Rouche's theorem.

LEMMA 2.3. *If* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree n, not vanishing in* $|z|<k$, $k>0$, *then for* $0\le r\le \rho\le k$,

$$\max_{|z|=\rho}|p(z)| \le \left(\frac{\rho+k}{r+k}\right)^n \max_{|z|=1}|p(z)| . \tag{2.3}$$

The result is best possible and equality occurs for $p(z)=(z+k)^n$.

The above result is due to Jain [6].

3. Proof of the Main Theorem

PROOF OF THEOREM 1.5. Since $p(z)$ has no zeros in $|z|<k$, $k\ge 1$, therefore the polynomial $F(z)=p(tz)$, where $0\le t\le k$, has no zeros in $|z|<k/t$, where $k/t\ge 1$. Applying Lemma 2.2 to the polynomial $F(z)$, we get

$$\max_{|z|=1}|F'(z)| \le n\left\{\frac{n|a_0|+\frac{k^2}{t^2}t|a_1|}{\left(1+\frac{k^2}{t^2}\right)n|a_0|+2\frac{k^2}{t^2}t|a_1|}\right\}\max_{|z|=1}|F(z)| - \frac{n}{k^n/t^n}\left\{1-\frac{n|a_0|+\frac{k^2}{t^2}t|a_1|}{\left(1+\frac{k^2}{t^2}\right)n|a_0|+2\frac{k^2}{t^2}t|a_1|}\right\}\min_{|z|=\frac{k}{t}}|F(z)|,$$

$$\max_{|z|=1}|p'(tz)|t \le nt\left\{\frac{n|a_0|t+k^2|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|}\right\}\max_{|z|=1}|p(tz)| - \frac{nt^n}{k^n}\left\{1-\frac{n|a_0|t^2+k^2t|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|}\right\}\min_{|z|=k}|p(z)|$$

which implies

$$\max_{|z|=t}|p'(z)| \le n\left\{\frac{n|a_0|t+k^2|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|}\right\}\max_{|z|=t}|p(z)|$$
$$-\frac{nt^{n-1}}{k^n}\left\{1-\frac{n|a_0|t^2+k^2t|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|}\right\}\min_{|z|=k}|p(z)|$$

Now, for $0\le r\le\rho\le k$ and $0\le\theta<2\pi$, we have

$$\left|p(\rho e^{i\theta})-p(re^{i\theta})\right| \le \int_r^\rho \left|p'(te^{i\theta})\right|dt$$

which implies

$$\left|p(\rho e^{i\theta})-p(re^{i\theta})\right| \le \int_r^\rho n\left\{\frac{n|a_0|t+k^2|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|}\right\}\max_{|z|=t}|p(z)|dt$$
$$-\int_r^\rho \frac{nt^{n-1}}{k^n}\left\{1-\frac{n|a_0|t^2+k^2t|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|}\right\}m\,dt$$

$$\le \int_r^\rho n\left\{\frac{n|a_0|t+k^2|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|}\right\}\left(\frac{k+t}{k+r}\right)^n\max_{|z|=r}|p(z)|dt$$
$$-\int_r^\rho \frac{nt^{n-1}}{k^n}\left\{1-\frac{n|a_0|t^2+k^2t|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|}\right\}m\,dt,$$

by Lemma 2.3, and $m=\min_{|z|=k}|p(z)|$.

The above inequality, for $0\le r\le\rho\le k$, gives

$$M(p,\rho)\le\left[1+\frac{n}{(k+r)^n}\int_r^\rho \frac{n|a_0|t+k^2|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|}(k+t)^n\,dt\right]M(p,r)$$
$$-\frac{mn}{k^n}\int_r^\rho\left[t^{n-1}-\frac{n|a_0|t^2+k^2t|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|}t^{n-1}\right]dt$$
$$-\frac{mn}{k^n}\left[\frac{\rho^n-r^n}{n}-\frac{n|a_0|\rho^2+k^2\rho|a_1|}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\left(\frac{\rho^n-r^n}{n}\right)\right]$$

$$\le\left[1+\frac{(k+\rho)}{(k+r)^n}\left\{\frac{n|a_0|\rho+k^2|a_1|}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\right\}\left\{(k+\rho)^n-(k+r)^n\right\}\right]M(p,r)$$
$$-\frac{(\rho^n-r^n)}{k^n}\left[1-\frac{n|a_0|\rho^2+k^2\rho|a_1|}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\right]m$$

$$\leq\left[1+\frac{(k+\rho)(n|a_0|\rho+k^2|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}+\frac{(k+\rho)(n|a_0|\rho+k^2|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\left(\frac{k+\rho}{k+r}\right)^n\right]M$$

$$-\frac{(\rho^n-r^n)}{k^n}\left[\frac{k^2n|a_0|+k^2\rho|a_1|}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\right]m$$

$$\leq\left[\frac{(k-\rho)k(n|a_0|+k|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}+\left\{1-\frac{k(k-\rho)(n|a_0|-k|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\right\}\left(\frac{k+\rho}{k+r}\right)^n\right]M$$

$$-\frac{(\rho^n-r^n)}{k^{n-2}}\left[\frac{n|a_0|+\rho|a_1|}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\right]m$$

$$=\left(\frac{k+\rho}{k+r}\right)^n\left[1-\frac{k(k-\rho)(n|a_0|-k|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\left\{1-\left(\frac{k+\rho}{k+r}\right)^n\right\}\right]M(p,r)$$

$$-\frac{(\rho^n-r^n)}{k^{n-2}}\left[\frac{n|a_0|+\rho|a_1|}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\right]m$$

from which the proof the THEOREM 1.5 follows.

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