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Perturbation and Asymptotic Analysis of Nonlinear Partial Differential Equations of Order Four

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Abstract This research centred on the use of perturbation and asymptotic methods in solving partial differential equations of a nonlinear elastic finite column structure that is subjected to a step load lying on a quadratic-cubic foundation. The formulation of the governing equation contains two small but mathematically independent parameters (δ and ϵ) which are used in asymptotic expansions of the variables. A two-timing regular perturbation procedure was used to analyze and solve the governing equation of displacement.

Keywords Perturbation, asymptotic analysis, nonlinear equation and analytic solution

Introduction

In view of rapid development in researches involving nonlinear system, there is high demand for asymptotic techniques to solve nonlinear problems especially nonlinear differential equations of higher order.

The characteristic problems to be solved are strictly nonlinear and no simple method exists. The perturbation approach and asymptotic series expansion are adopted to analyze the problem. This is made possible by the presence of two small but mathematically independent parameters upon which the two-timing perturbation scheme is formulated.

However, many such investigations are described by equations in which the solution cannot be obtained easily, in other words, exact solutions to such problems are impossible because of the high level of nonlinearity that are inherent in such formulations. Consequently, there is a great need to seek for analytic methods to address such uncommon problems [1].

This study aims to determine, using perturbation and asymptotic techniques, the dynamic buckling load of a viscously damped but clamped finite column on a quadratic-cubic nonlinear elastic foundation where the column is trapped by a step load [2].

The aim of perturbation theory is to determine the behavior of solution $x = x^{\epsilon}$ of $p^{\epsilon}(x) = o$ as $\epsilon \to 0$. The use of small parameter is simply for definiteness [3]. Perturbation and asymptotic analysis apply to a broad class of problems. In some cases, explicit expression for x^{ϵ} such as integral representation, and want to obtain its behavior in the limit $\epsilon \to 0$.

2. Formulation of the Problem

The usual dimensional differential equation satisfied by the deflection $W(X,T)$ of the column under consideration, as in [4] and [5] is:

$$m_0 W_{,TT} + c_0 W_{,T} + EIW_{,XXXX} + 2P(T)W_{,XX} + Wk_1 - k_2 W^2 - k_3 W^3 = -2P(T)\frac{d^2\overline{W}}{dX^2}, T > 0 \quad (2.1)$$

$$0<X<\pi \quad (2.2)$$

W (X, 0) = 0= $W_{,T}$ (X, 0) = 0, 0 <X < π

W = $W_{,X}$ = 0 at X=0, π

where m_0 is the mass per unit length, c_0 is the damping coefficient, EI is the bending stiffness, where E and I are the Young's modulus and the moment of inertia respectively. Here the nonlinear elastic foundation exerts a force per unit length given by Wk_1 - k_2W^2 – k_3W^3 on the column where k_1, k_2 and $k_3$are constants such that k_1>0, k_2>0, k_3>0. In this formulation, all nonlinearities higher than cubic are excluded, while all nonlinear derivatives of W (X,T) are also excluded. Here, $\overline{W}$ is the stress-free time independent twice-differentiable initial imperfection displacement and all aspects of axial inertia are neglected.

2.1. Non-Dimensionalization of the Problem

To reduce equation (2.1) -(2.2) to non-dimensional form, we adopt the following quantities:

$$x = (\frac{k_1}{EI})^{\frac{1}{4}}X,\quad \omega = (\frac{k_2}{k_1})^{\frac{1}{2}}W,\quad \lambda f(t) = \frac{P(T)}{2(EIk_1)^{\frac{1}{2}}},\ t = (\frac{k_1}{m_0})^{\frac{1}{2}}T,\ \epsilon\overline{\omega} = (\frac{k_3}{k_1})^{\frac{1}{2}}\overline{W},\ 2\delta = \frac{c_0}{(m_0k_1)^{\frac{1}{2}}},\quad \alpha = \frac{k_2}{\sqrt{k_1k_2}},$$

$$\beta = (\frac{k_3}{k_1})^{\frac{3}{2}} \quad (2.3)$$

Here, we shall assume the following inequalities

0< δ << 1, 0< ϵ << 1.

On substituting (2.3) in (2.1) and simplifying, the following is obtained

$$\omega_{,tt} + 2\delta\omega_{,t} + \omega_{,xxxx} + 2\lambda f(t)\omega_{,xx} + \omega - \alpha\omega^2 - \beta\omega^3 = -2\epsilon\lambda f(t)\frac{d^2\overline{\omega}}{dx^2},\ t > 0 \quad (2.4)$$

0<x<π

ω(x, 0) = 0= $\omega_{,t}$ (x,0) = 0 , 0 <x < π

ω = $\omega_{,x}$ = 0 at x= 0, π

where ω is the displacement, t is the time variable, δ is the damping coefficient, α and β are the imperfection –sensitivity parameters, ϵ is the amplitude of the imperfection, $\overline{\omega}$ is a stress-free twice-differentiable imperfection and f(t) is a time dependent loading function while λ is the non-dimensional amplitude (or magnitude) of the loading [6]. Here, a subscript following a comma indicates partial differentiation and $f(t)$ is a step load such that;

$$f(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \quad (2.5)$$

Here, it is assumed $\delta and \epsilon$ are two small but mathematically unrelated parameters that satisfy the inequalities as in (2.5). Our ultimate aim is to determine the dynamic buckling load λ_D which is obtained by using the maximization [6].

3. Perturbation Procedure

Let

$$\tau = \delta t \quad (3.1)$$

$$\hat{t} = t + \frac{1}{\delta}[\omega_1(\tau)\epsilon + \omega_2(\tau)\epsilon^2 + \omega_3(\tau)\epsilon^3 + \omega_4(\tau)\epsilon^4 + \cdots \quad (3.2)$$

where,

$\omega_i(0) = 0,\ i = 1,2,3, ...$,

Let

ω (x, t) = U (x, $\hat{t}$, τ, ϵ, δ)

From equation (3.6) we have,

$$\omega_{,t} = \left(\frac{\partial U}{\partial \hat{t}}.\frac{\partial \hat{t}}{\partial t}\right) + \left(\frac{\partial U}{\partial \hat{t}}.\frac{\partial \hat{t}}{\partial \tau}.\frac{d\tau}{dt}\right) + \left(\frac{\partial U}{\partial \tau}.\frac{d\tau}{dt}\right) \quad (3.3)$$

i.e

$$\omega_{,t} = U_{,\hat{t}} + (\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \ \dots\dots)U_{,t} + \delta U_{,\tau} \quad (3.4)$$

The following also follows:

$$\omega_{,tt} = U_{,\hat{t}\hat{t}} + (\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \dots)^2 U_{,\hat{t}\hat{t}} + \delta^2 U_{,\tau\tau} + 2(\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \cdots)U_{,\hat{t}\hat{t}} + 2\delta U_{,\hat{t}\tau} + 2\delta(\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \dots)U_{,\hat{t}\tau} + \delta(\omega_1''\epsilon + \omega_2''\epsilon^2 + \omega_3''\epsilon^3 + \dots)U_{,\hat{t}}$$

$$U_{,\hat{t}\hat{t}} + (\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \dots)^2 U_{,\hat{t}\hat{t}} + \delta^2 U_{,\tau\tau} + 2(\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \cdots)U_{,\hat{t}\hat{t}} + 2\delta U_{\hat{t},\tau} + 2\delta(\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \dots)U_{,\hat{t}\tau} + \delta(\omega_1''\epsilon + \omega_2''\epsilon^2 + \omega_3''\epsilon^3 + \dots)U_{,\hat{t}} + 2\delta\left[U_{,\hat{t}} + (\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \dots)U_{,\hat{t}} + \delta U_{,\tau}\right] + U_{,xxxx} + 2\lambda U_{,xx} + U + \alpha U^2 - \beta U^3 = -2\lambda\epsilon\frac{d^2\overline{\omega}}{dx^2}$$

Let

$$\mathrm{U}(\mathrm{x}, \epsilon, \tau) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} U_n^{(i,j)}(x,t,\tau)\epsilon^i\delta^j \tag{3.5}$$

$$= \epsilon\left(U^{(10)} + \delta U^{(11)} + \delta^2 U^{(12)} + \cdots\right) + \epsilon^2\left(U^{(20)} + \delta U^{(21)} + \delta^2 U^{(22)} + \cdots\right) + \epsilon^3\left(U^{(30)} + \delta U31 + \delta 2U32 + \ldots\right) + \ldots \tag{3.6}$$

Here, the ij in $U^{(ij)}$ are not powers but superscripts. Therefore, the following orders of equations are obtained

$$O(\epsilon):\ U_{,\hat{t}\hat{t}}^{(10)} + U_{,xxxx}^{(10)} + 2\lambda U_{,xx}^{(10)} + U^{(10)} = -2\lambda\frac{d^2\tilde{\omega}}{dx^2}, \tag{3.7}$$

$$O(\epsilon\delta):\ U_{,\hat{t}\hat{t}}^{(11)} + U_{,xxxx}^{(11)} + 2\lambda U_{,xx}^{(11)} + U^{(11)} = -2U_{,\hat{t}\tau}^{(10)} - 2U_{,\hat{t}}^{(10)} \tag{3.8}$$

$$O(\epsilon\delta^2):\ U_{,\hat{t}\hat{t}}^{(12)} + U_{,xxxx}^{(12)} + 2\lambda U_{,xx}^{(12)} + U^{(12)} = -2U_{,\hat{t}\tau}^{(11)} - 2U_{,\hat{t}}^{(11)} - U_{,\tau\tau}^{(10)} \tag{3.9}$$

$$O(\epsilon^2):\ U_{,\hat{t}\hat{t}}^{(20)} + U_{,xxxx}^{(20)} + 2\lambda U_{,xx}^{(20)} + U^{(20)} = -\alpha(U^{(10)})^2 - 2\omega_1' U_{,\hat{t}\hat{t}}^{(10)} \tag{3.10}$$

$$O(\epsilon^2\delta):\ U_{,\hat{t}\hat{t}}^{(21)} + U_{,xxxx}^{(21)} + 2\lambda U_{,xx}^{(21)} + U^{(21)} = -2\alpha U^{(10)}\mathrm{U}^{(11)} - 2U_{,\hat{t}\tau}^{(20)} - 2U_{,\hat{t}}^{(20)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(11)} - \omega_1'' U_{,\hat{t}}^{(10)} - 2\omega_1' U_{,\hat{t}}^{(10)} \tag{3.11}$$

$$O(\epsilon^2\delta^2):\ U_{,\hat{t}\hat{t}}^{(22)} + U_{,xxxx}^{(22)} + 2\lambda U_{,xx}^{(22)} + U^{(22)} = -U_{,\tau\tau}^{(20)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(12)} - 2U_{,\hat{t}\tau}^{(21)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(12)} - 2\omega_1'' U_{,\hat{t}}^{(11)} - 2U_{,\hat{t}}^{(21)} - 2\omega_1' U_{,\hat{t}}^{(11)} - \alpha\left\{\left(\mathrm{U}^{(11)}\right)^2 + \mathrm{U}^{(10)}\mathrm{U}^{(12)}\right\} \tag{3.12}$$

$$O(\epsilon^3):\ U_{,\hat{t}\hat{t}}^{(30)} + U_{,xxxx}^{(30)} + 2\lambda U_{,xx}^{(30)} + U^{(30)} = -(\omega_1')^2 U_{,\hat{t}\hat{t}}^{(10)} - 2\left(\omega_1' U_{,\hat{t}\hat{t}}^{(20)} + \omega_2' U_{,\hat{t}\hat{t}}^{(10)}\right) - 2\alpha\mathrm{U}^{(20)}\mathrm{U}^{(12)} + \beta\left(\mathrm{U}^{(10)}\right)^3 \tag{3.13}$$

$$O(\epsilon^3\delta):\ U_{,\hat{t}\hat{t}}^{(31)} + U_{,xxxx}^{(31)} + 2\lambda U_{,xx}^{(31)} + U^{(31)} = -(\omega_1')^2 U_{,\hat{t}\hat{t}}^{(10)} - 2\left(\omega_1' U_{,\hat{t}\tau}^{(21)} + \omega_2' U_{,\hat{t}\tau}^{(11)}\right) - 2U_{,\hat{t}\tau}^{(30)} + 2\left(\omega_1' U_{,\hat{t}\hat{t}}^{(20)} + \omega_2' U_{,\hat{t}\hat{t}}^{(10)}\right) - \left(\omega_1'' U_{,\hat{t}}^{(20)} + \omega_2''^{+} U_{,\hat{t}}^{(10)}\right) - 2\left\{U_{,\hat{t}}^{(30)} + \left(\omega_1' U_{,\hat{t}}^{(20)} + \omega_2'^{+} U_{,\hat{t}}^{(10)}\right)\right\} - \alpha\left(\mathrm{U}^{(10)}\mathrm{U}^{(21)} + \mathrm{U}^{(11)}\mathrm{U}^{(20)}\right) + 3\beta\left(\mathrm{U}^{(10)}\right)^2\left(\mathrm{U}^{(11)}\right) \tag{3.14}$$

$$O(\epsilon^3\delta^2):\ U_{,\hat{t}\hat{t}}^{(32)} + U_{,xxxx}^{(32)} + 2\lambda U_{,xx}^{(32)} + U^{(32)} = -(\omega_1')^2 U_{,\hat{t}\hat{t}}^{(12)} - U_{,\tau\tau}^{(30)} - 2\left(\omega_1' U_{,\hat{t}\hat{t}}^{(22)} + \omega_2' U_{,\hat{t}\hat{t}}^{(12)}\right) - 2U_{,\hat{t}\tau}^{(31)} - 2\left(\omega_1' U_{,\hat{t}\tau}^{(21)} + \omega_2' U_{,\hat{t}\tau}^{(11)}\right) - \left(\omega_1'' U_{,\hat{t}}^{(21)} + \omega_2'' U_{,\hat{t}\tau}^{(11)}\right) - 2\left(U_{,\hat{t}}^{(31)} + \omega_1 U_{,\hat{t}}^{(21)} + \omega_2' U_{,\hat{t}}^{(11)}\right) - 2U_{,\tau}^{(30)} - 2\alpha\left(\mathrm{U}^{(10)}\mathrm{U}^{(32)} + \mathrm{U}^{(11)}\mathrm{U}^{(21)} + \mathrm{U}^{(12)}\mathrm{U}^{(20)}\right) + \beta\left[\left(\mathrm{U}^{(10)}\right)^2\mathrm{U}^{(12)} + 3\mathrm{U}^{(10)}\left(\mathrm{U}^{(10)}\right)^2\right] \tag{3.15}$$

4. Application

This chapter solves the problem earlier posed in chapter three using a two-timing perturbation and asymptotic methods.

Let the imperfection $\overline{\omega}$ be

$$\overline{\omega} = \bar{a}_m(1 - cos2mx) \tag{4.1}$$

where $\bar{a}_m$ is a constant. Based on the clamped boundary condition which the problem assumes, it is necessary to take the displacement $U^{(ij)}$ as:

$$U^{(ij)}(t,\tau,x) = \sum_{n=1}^{\infty} U_n^{(ij)}(\hat{t}\ \tau)(1 - cos2nx) \quad (4.2)$$

Solution of equation of order $\epsilon\delta^j$, j=0,1,2

$$U_{,\hat{t}\hat{t}}^{(10)} + U_{,xxxx}^{(10)} + 2\lambda U_{,xx}^{(10)} + U^{(10)} = -2\lambda \frac{d^2\bar{\omega}}{dx^2},$$

$$\sum_{n=1}^{\infty}\left[U_{n,\hat{t}\hat{t}}^{(10)}(1 - cos2nx) + \{-16n^4 + 8\lambda n^2 + (1 - cos2nx)\}U_n^{(10)}\right] = -8\lambda m^2 \bar{a}_m cos2mx \quad (4.3)$$

i.e

$$\sum_{n=1}^{\infty}(1 - cos2nx)U_{n,\hat{t}\hat{t}}^{(10)} + \{-16n^4 + 8\lambda n^2 + (1 - cos2nx)\}U_n^{(10)} = -8\lambda m^2 \bar{a}_m cos2mx \quad (4.3)$$

Multiplying (4.3) through by cos2mx and integrating from 0 to π and for n=m, the result is,

$$U_{m,\hat{t}\hat{t}}^{(10)} + (16m^4 - 8\lambda m^2 + 1)U_m^{(10)} + U_m^{(10)} = 8\lambda m^2 \bar{a}_m \quad (4.4)$$

Let

$$16m^4 - 8\lambda m^2 + 1 = \theta^2 > 0 \ \forall m \quad (4.5)$$

Then (4.5) yields

$$U_{m,\hat{t}\hat{t}}^{(10)} + \theta^2 U_m^{(10)} + U_m^{(10)} = 8\lambda m^2 \bar{a}_m \quad (4.6)$$

With initial conditions,

$$U_m^{(10)}(0,0) = 0;\ U_{m,\hat{t}}^{(10)}(0,0) = 0$$

Therefore, the solutions of (4.7d) is

$$U_m^{(10)} = \alpha_1(\tau)cos\theta\hat{t} + \beta_1(\tau)sin\theta\hat{t} + B \quad (4.7)$$

Where

$$B = \frac{8\lambda m^2 \bar{a}_m}{\theta^2} \quad (4.8)$$

The use of initial conditions gives

$$\alpha_1(0) = -\frac{8\lambda m^2 \bar{a}_m}{\theta^2}, \quad \beta_1 = 0 \quad (4.9)$$

Thus

$$U^{(10)} = U_m^{(10)}(1 - cos2mx) \quad (4.10)$$

$$O(\epsilon\delta):\ U_{,\hat{t}\hat{t}}^{(11)} + U_{,xxxx}^{(11)} + 2\lambda U_{,xx}^{(11)} + U^{(11)} = -2U_{,\hat{t}\ \tau}^{(10)} - 2U_{,\hat{t}}^{(10)}$$

Let

$$U^{(11)} = \sum_{n=1}^{\infty} U_n^{(11)}(\hat{t},\tau)(1 - cos2nx)$$

$$\sum_{n=1}^{\infty}\left[U_{n,\hat{t}\hat{t}}^{(11)}(1 - cos2nx) + (-16n^4 + 8\lambda n^2)U_n^{(11)}cos2nx + (1 - cosnx)\ U_n^{(11)}\right]U_n^{(10)} =$$
$$-2Um,t\ \tau(10)+Um,t(10)1-cos2mx$$

(4.11)

$$U_{m,\hat{t}\hat{t}}^{(11)} + (16m^4 - 8\lambda m^2 + 1)U_m^{(11)} = -2\left(U_{m,\hat{t}\ \tau}^{(10)} + U_{m,\hat{t}}^{(10)}\right) \quad (4.12)$$

i.e.

$$U_{m,\hat{t}\hat{t}}^{(11)} + \theta^2 U_m^{(11)} = -2\left(U_{m,\hat{t}\ \tau}^{(10)} + U_{m,\hat{t}}^{(10)}\right) \quad (4.13)$$

The initial conditions are

$$U_m^{(11)}(0,0) = 0;\ U_{m,\hat{t}}^{(11)}(0,0) + U_{m,\tau}^{(10)}$$

Substituting for $U_m^{(10)}$on the right hand side of (4.13) gives

$$U_{m,\hat{t}\hat{t}}^{(11)} + \theta^2 U_m^{(11)} = -2[-\theta\alpha_1' sin\theta\hat{t} + \theta\beta_1' cos\theta\hat{t} + (-\theta\alpha_1 sin\theta\hat{t} + \theta\beta_1 cos\theta\hat{t})] = -2\theta[-(\alpha_1' +$$
$$\alpha 1sin\theta t+\beta 1'+\beta cos\theta t \quad (4.14)$$

To ensure a uniformly valid solution in $\hat{t}$, implies equating to zero the coefficients of $cos\theta\hat{t}$ and $sin\theta\hat{t}$ on the RHS of (4.11a). Therefore, the coefficient of $cos\theta\hat{t}$ gives;

$$\beta_1' + \beta = 0 \quad (4.15)$$

$$\beta_1(\tau) = Ae^{-\tau} \quad and\ \beta_1(\tau) = 0 \quad (4.16)$$

Similarly, the coefficient of$sin\theta\hat{t}$ gives,

$$\alpha_1' + \alpha_1 = 0 \quad (4.17)$$

This gives,

$$\alpha_1'(0) = -\alpha_1(0) = B \; and \; \alpha_1(\tau) = -Be^{-\tau} \qquad (4.18)$$

Therefore

$$U_m^{(10)} = \alpha_1(\tau)cos\theta\hat{t} + B \qquad (4.19)$$

The remaining equation in (4.11a) is

$$U_{m,\hat{t}\hat{t}}^{(11)} + \theta^2 U_m^{(11)} = 0 \qquad (4.20)$$

$$\therefore \; U_m^{(11)} = \alpha_2(\tau)cos\theta\hat{t} + \beta_2(\tau)sin\theta\hat{t} \qquad (4.21)$$

$$U_m^{(11)}(0,0) = 0 \; implies \quad \alpha_2(0) = 0 \qquad (4.22)$$

$$U_{m,\hat{t}}^{(11)}(0,0) + U_{m,\tau}^{(10)} = 0 \text{ Implies}$$

$$\beta_2(0)\theta + \alpha_1'(0) = 0 \text{ and } \beta_2(0) = -\frac{\alpha_1'(0)}{\theta} = \frac{-B}{\theta} \qquad (4.23)$$

Therefore

$$U_m^{(11)} = U_m^{(11)}(1 - cos2mx) \qquad (4.24)$$

$$O(\epsilon\delta^2): U_{,\hat{t}\hat{t}}^{(12)} + U_{,xxxx}^{(12)} + 2\lambda U_{,xx}^{(12)} + U^{(12)} = -2U_{,\hat{t}\,\tau}^{(11)} - 2U_{,\hat{t}}^{(11)} - U_{,\tau\tau}^{(10)} \qquad (4.25)$$

Substituting for $U_m^{(11)}$ and $U_m^{(10)}$ into (4.15), gives

$$U_{m,\hat{t}\hat{t}}^{(12)} + \theta^2 U_m^{(12)} = -2[-\theta\alpha_2'(\tau)sin\theta\hat{t} + \theta\beta_2'(\tau)cos\theta\hat{t} + (-\theta\alpha_2(\tau)sin\theta\hat{t} + \theta\beta_2(\tau)cos\theta\hat{t})] - \alpha_1''(\tau)cos\theta\hat{t}$$

$$= 2\theta\alpha_2'(\tau)sin\theta\hat{t} - 2\theta\beta_2'(\tau)cos\theta\hat{t} - \theta\alpha_2(\tau)sin\theta\hat{t} + \theta\beta_2(\tau)cos\theta\hat{t} - \alpha_1''(\tau)cos\theta\hat{t}$$

$$= \left(2\theta\alpha_2'(\tau) - 2\theta\alpha_2\right)sin\theta\hat{t} + \left(2\theta\beta_2(\tau) - 2\theta\beta_2'(\tau) - \alpha_1''(\tau)\right)cos\theta\hat{t}\Big) \qquad (4.26)$$

To remove secular terms in the solution of $U_m^{(12)}$, i.e, to ensure a uniformly valid solution in $\hat{t}$ implies equating to zero the coefficients of $cos\theta\hat{t} \; and \; sin\theta\hat{t}$ on the right hand side. These respectively give

$$cos\theta\hat{t}: -2(\theta\beta_2' + \theta\beta_2) - \alpha_1'' = 0$$

and

$$sin\theta\hat{t} : -2(-\theta\alpha_2' - \theta\alpha_2) = 0$$

Therefore

$$\beta_2' + \beta_2 = \frac{-\alpha_1''}{2\theta} \; and \; \left[\beta_2'(0) = \frac{3B}{2\theta}\right] \qquad (4.27)$$

$$\alpha_2' + \alpha_2 = 0 \qquad (4.28)$$

Therefore,

$$\alpha_2(\tau) \equiv 0, \text{ and } \beta_2(\tau) = e^{-\tau}\left[-\int_0^\tau \frac{e^s\alpha''}{2\theta}ds + \beta_2(0)\right]$$

Therefore,

$$U_m^{(11)} = \beta_2(\tau)sin\theta\hat{t} \qquad (4.29)$$

The initial conditions are

$$U_m^{(12)}(0,0) = 0, \qquad U_{m,\hat{t}}^{(12)} + U_{m,\tau}^{(11)}(0,0) = 0$$

Then,

$$U_m^{(12)}(\hat{t},\tau) = \alpha_3(\tau)\cos\theta\,\hat{t} + \beta_3(\tau)sin\theta\hat{t} \qquad (4.30)$$

Applying the initial conditions yield,

$$\alpha_3(0) = 0, \quad \beta_3(0) = 0 \qquad (4.31)$$

$$U_{m,\hat{t}}^{(12)}(0,0) = \theta\beta_3(0) = 0$$

Therefore,

$$\beta_3(0) = 0 \text{ and } U^{12} = U_m^{(12)}(1 - cos2mx)$$

Solution of equation of order$\epsilon^2\delta^j$, j=0, 1, 2

$$O(\epsilon^2): \; U_{,\hat{t}\hat{t}}^{(20)} + U_{,xxxx}^{(20)} + 2\lambda U_{,xx}^{(20)} + U^{(20)} = -\alpha(U^{(10)})^2 - 2\omega_1' U_{,\hat{t}\hat{t}}^{(10)}$$

Let

$$U^{(20)} = \sum_{n=1}^{\infty} U_n^{(20)}(\hat{t}\,\tau)(1 - \cos2nx) \qquad (4.32)$$

$$\sum_{n=1}\left[U^{(20)}_{n,\hat{t}\hat{t}}(1-\cos 2nx)+(-16n^4+8\lambda n^2)U^{(20)}_n\cos 2nx\ +(1-\cos 2nx)U^{(20)}_n\right]\ =$$
$$\alpha\left[\left(U^{(10)}_m\right)^2(1-\cos 2mx)^2\right]-2\omega_1' U^{(10)}_{m,\hat{t}\hat{t}}(1-\cos 2mx) \quad (4.33)$$

Multiplying both sides of (4.33) through by cos2mx and integrating from 0 to π and for n=m, the result gives

$$\left[-\frac{\pi}{2}U^{(20)}_{m,\hat{t}\hat{t}}+(-16m^4+8\lambda m^2)U^{(20)}_m\left(\frac{\pi}{2}\right)+\left(\frac{\pi}{2}U^{(20)}_m\right)\right]\ =-\alpha\left(U^{(10)}_m\right)^2\left[-2\left(\frac{\pi}{2}\right)-2\omega_1' U^{(10)}_{m,\hat{t}\hat{t}}\left(\frac{-\pi}{2}\right)\right]$$
$$=-\alpha\left(U^{(10)}_m\right)^2\left[-2\left(\frac{\pi}{2}\right)-2\omega_1' U^{(10)}_{m,\hat{t}\hat{t}}\left(\frac{-\pi}{2}\right)\right] \quad (4.34)$$

This gives

$$U^{(20)}_{m,\hat{t}\hat{t}}\ +\theta^2U^{(20)}_m=-\left[2\alpha\left(U^{(10)}_m\right)^2+2\omega' U^{(10)}_{m,\hat{t}\hat{t}}\right] \quad (4.35)$$

The initial conditions are;

$$U^{(20)}_m(0{,}0)=0,U^{(20)}_{m,\hat{t}}(0{,}0)+\omega'(0)U^{(10)}_{m,\hat{t}}(0{,}0)=0$$

Next multiplying equation (4.33) by cos4mx and integrating from 0 to π and for n=m, the result gives;

$$-\frac{\pi}{2}U^{(20)}_{2m,\hat{t}\hat{t}}+(-256m^4+32\lambda m^2)\left(\frac{\pi}{2}\right)U^{(20)}_{2m}-\left(\frac{\pi}{2}\right)U^{(20)}_{2m}=\ -\alpha\left(U^{(10)}_m\right)^2\left(\frac{1}{2}.\frac{\pi}{2}\right) \quad (4.36)$$

Simplifying (4.26) gives;

$$U^{(20)}_{2m,\hat{t}\hat{t}}+(256m^4-32\lambda m^2+1)U^{(20)}_{2m}\ =\frac{\alpha}{2}\left(U^{(10)}_m\right)^2 \quad (4.37)$$

Let

$$\varphi^2\ =(256m^4+32\lambda m^2+1)>0 \quad (4.38)$$

Then, the final equation is

$$U^{(20)}_{2m,\hat{t}\hat{t}}+\varphi^2U^{(20)}_m\ =\frac{\alpha}{2}\left(U^{(10)}_m\right)^2 \quad (4.39)$$

The initial conditions are;

$$U^{(20)}_{2m}(0{,}0)=0;\ U^{(20)}_{2m,\hat{t}}(0{,}0)+\omega_1' U^{(10)}_{2m,\hat{t}}(0{,}0)=0$$

On substituting for $U^{(10)}_m$on the RHS of (4.39), the simplification is

$$U^{(20)}_{2m,\hat{t}\hat{t}}+\theta^2U^{(20)}_m\ =-[\{2\alpha(\alpha_1\cos\theta\hat{t}+B)^2\}+\{2\omega_1'(-\alpha_1\cos\theta\hat{t})\}]$$
$$=-\left[2\alpha\left\{\left(\frac{\alpha_1^2}{2}+B^2\right)+2B\alpha_1\cos\theta\hat{t}+\frac{\alpha_1^2}{2}\alpha_1\cos 2\theta\hat{t}\right\}+2\omega'_1(-\alpha_1\theta^2\cos\theta\hat{t})\right] \quad (4.40)$$

To ensure a uniformly valid solution in$\hat{t}$, we equate to zero, the coefficients of $\cos\theta\hat{t}$on the right hand side of (4.18a)

That is

$$-[2B\alpha_1-2\omega'_1\theta^2\alpha_1]=0 \quad (4.41)$$

Therefore

$$\omega'_1\ =\frac{B}{\theta^2};\ \omega_1\ =\int\frac{B}{\theta^2}d\tau \quad (4.42)$$

The remaining part of equation (4.40) for $U^{(20)}_m$ is

$$U^{(20)}_{m,\hat{t}\hat{t}}+\theta^2U^{(20)}_m\ =r_0+r_1\cos 2\theta\hat{t} \quad (4.43)$$

Where

$$r_0\ =-2\alpha\left(\frac{\alpha_1^2}{2}+B^2\right),\qquad r_0(0)\ =-3\alpha B^2$$
$$r_1\ =\ -\alpha\alpha_1^2\,,r_1(0)=-\alpha B^2,r_0'(0)=2\alpha B^2\,,r_1'(0)=2\alpha B^2$$

Therefore, it follows that

$$U^{(20)}_m(\hat{t},\tau)=\alpha_4(\tau)\cos\theta\hat{t}+\beta_4(\tau)\sin\theta\hat{t}+\frac{r_0}{\theta^2}-\frac{r_1\cos 2\theta\hat{t}}{3\theta^2} \quad (4.44)$$

From the initial condition;

$$U^{(20)}_m(0{,}0)=0;$$

i.e.

$$\alpha_4(0) + \frac{r_0(0)}{\theta^2} - \frac{r_1}{3\theta^2} = 0$$

Therefore

$$\alpha_4(0) = \frac{r_1}{3\theta^2} - \frac{r_0(0)}{\theta^2} = \frac{8\alpha B^2}{3\theta^2} \quad (4.45)$$

Applying the initial condition $U^{(20)}_{m,\hat{t}}(0,0) + \omega'(0) + U^{(10)}_{m,\hat{t}}(0,0)$ yields

$$\beta_4(0) = 0$$

Simplification of (4.34) yields,

$$U^{(20)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(20)}_m = \frac{\alpha}{2}\left[\left(\frac{\alpha_1^2}{2} + B^2\right) + 2B\alpha_1 cos\theta\hat{t} + \frac{\alpha_1^2}{2}cos2\theta\hat{t}\right] \quad (4.46)$$

Therefore,

$$U^{(20)}_{2m}(\hat{t},\tau) = \alpha_5(\tau)cos\varphi\hat{t} + \beta_5(\tau)sin\varphi\hat{t} + \frac{\alpha}{2}\left[\frac{(\frac{\alpha_1^2}{2}+B^2)}{\varphi^2} + \frac{2B\alpha_1 cos\theta\,\hat{t}}{(\varphi^2-\theta^2)} + \frac{\alpha_1^2 cos2\theta\hat{t}}{2(\varphi^2-\theta^2)}\right] \quad (4.47)$$

From the initial conditions;

$$U^{(20)}_{2m}(0,0) = 0;\ U^{(20)}_{2m,\hat{t}}(0,0) + \omega_1' U^{(10)}_{2m,\hat{t}}(0,0) = 0$$

It follows that

$$\alpha_5(0) + \frac{\alpha}{2}\left[\frac{(\frac{\alpha_1^2}{2}+B^2)}{\varphi^2} + \frac{2B\alpha_1}{(\varphi^2-\theta^2)} + \frac{\alpha_1^2}{2(\varphi^2-4\theta^2)}\right] = 0$$

$$\alpha_5(0) = -\frac{\alpha}{2}\left[\frac{3B^2}{2\varphi^2} - \frac{2B^2}{(\varphi^2-\theta^2)} + \frac{B^2}{2(\varphi^2-4\theta^2)}\right] = B^2\alpha S_0 \text{ and } \beta_5(0) = 0 \quad (4.48)$$

Therefore

$$U^{(20)} = U^{(20)}_m(1-cos2mx) + U^{(20)}_{2m}(1-cos4mx) \quad (4.49)$$

From (3.21),

$$O(\epsilon^2\delta):\ U^{(21)}_{,\hat{t}\hat{t}} + U^{(21)}_{,xxxx} + 2\lambda U^{(21)}_{,xx} + U^{(21)}$$
$$= -2\alpha U^{(10)}U^{(11)} - 2U^{(20)}_{,\hat{t}\,\tau} - 2U^{(20)}_{,\hat{t}} - 2\omega_1' U^{(11)}_{,\hat{t}\hat{t}} - \omega_1'' U^{(10)}_{,\hat{t}} - 2\omega_1' U^{(10)}_{,\hat{t}}$$

i.e,

$$U^{(21)}_{,\hat{t}\hat{t}} + U^{(21)}_{,xxxx} + 2\lambda U^{(21)}_{,xx} + U^{(21)} = -2\alpha U^{(10)}_m(1-cos2mx)U^{(11)}_m(1-cos2mx) - 2U^{(20)}_{m,\hat{t}\,\tau}(1-cos2mx) - 2U^{(20)}_{m,\hat{t}}(1-cos2mx) - 2\omega_1' U^{(11)}_{m,\hat{t}\hat{t}}(1-cos2mx) - \omega_1'' U^{(10)}_{m,\hat{t}}(1-cos2mx) - 2\omega_1' U^{(10)}_{m,\hat{t}}(1-cos2mx) \quad (4.50)$$

Let

$$U^{(21)} = \sum_{n=1}^{\infty} U^{(21)}_n(\hat{t}\,\tau)(1-\cos 2nx)$$

Substituting into (4.50), gives

$$\sum_{n=1}\left[U^{(21)}_{n,\hat{t}\hat{t}}(1-\cos 2nx) + (-16n^4 + 8\lambda n^2)U^{(21)}_n\, cos2nx + (1-cos2nx)U^{(21)}_n\right] = -2\alpha U^{(10)}_m U^{(11)}_m\left[\frac{3}{2} - 2cos2mx + \frac{1}{2}cos4mx\right] - 2U^{(20)}_{m,\hat{t}\tau}(1-cos2mx) - 2U^{(20)}_{m,\hat{t}}(1-cos2mx) - 2\omega_1' U^{(11)}_{m,\hat{t}\hat{t}}(1-cos2mx) - \omega_1'' U^{(10)}_{m,\hat{t}}(1-cos2mx) - 2\omega_1' U^{(10)}_{m,\hat{t}}(1-cos2mx) \quad (4.51)$$

Multiplying both sides of (4.41) through by cos2mx and integrating from 0 to π and for n = m gives

$$\left[-\frac{\pi}{2}U^{(21)}_{m,\hat{t}\hat{t}} + (-16m^4 + 8\lambda m^2)U^{(21)}_m\left(\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}U^{(21)}_m\right)\right] = \left[\begin{array}{c} -2\alpha U^{(10)}_m U^{(11)}_m\left(-\frac{\pi}{2}\right) - 2U^{(20)}_{m,\hat{t}\tau}\left(-\frac{\pi}{2}\right) - 2U^{(20)}_{m,\hat{t}} \\ \left(-\frac{\pi}{2}\right) - 2\omega_1' U^{(11)}_{m,\hat{t}\hat{t}}\left(-\frac{\pi}{2}\right) - \omega_1'' U^{(10)}_{m,\hat{t}}\left(-\frac{\pi}{2}\right) - 2\omega_1' U^{(10)}_{m,\hat{t}}\left(-\frac{\pi}{2}\right)\end{array}\right] \quad (4.52)$$

Further simplification yields

$$U^{(21)}_{m,\hat{t}\hat{t}} + (16m^4 - 8\lambda m^2 + 1)U^{(21)}_m = -2\alpha U^{(10)}_m U^{(11)}_m - 2U^{(20)}_{m,\hat{t}\tau} - 2U^{(20)}_{m,\hat{t}} - 2\omega_1' U^{(11)}_{m,\hat{t}\hat{t}} - \omega_1'' U^{(10)}_{m,\hat{t}} - 2\omega_1' U^{(10)}_{m,\hat{t}} \quad (4.53)$$

The above finally yields

$$U_{m,\hat{t}\hat{t}}^{(21)} + \theta^2 U_m^{(21)} = -2\alpha U_m^{(10)} U_m^{(11)} - 2U_{m,\hat{t}\tau}^{(20)} - 2U_{m,\hat{t}}^{(20)} - 2\omega_1' U_{m,\hat{t}\hat{t}}^{(11)} - \omega_1'' U_{m,\hat{t}}^{(10)} - 2\omega_1' U_{m,\hat{t}}^{(10)} \quad (4.54)$$

The initial conditions are

$$U_m^{(21)}(0,0) = 0;\ U_{m,\hat{t}}^{(21)}(0,0) + \omega_1'(0) U_{m,\hat{t}}^{(11)}(x,0,0) + U_{m,\tau}^{(20)}(0,0) = 0$$

Next, multiplying (4.52) by cos4mx and integrating from 0 to π for n =m, gives

$$\left[-\frac{\pi}{2} U_{2m,\hat{t}\hat{t}}^{(21)} + (-256m^4 + 32\lambda m^2) U_{2m}^{(21)} \left(\frac{\pi}{2}\right) - \frac{\pi}{2} U_{2m}^{(21)})\right] = -2\alpha U_m^{(10)} U_m^{(11)} \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right) - 2\left(U_{2m,\hat{t}\tau}^{(20)} + U_{m,\hat{t}}^{(20)}\right) \quad (4.55)$$

$$U_{2m,\hat{t}\hat{t}}^{(21)} + \varphi^2 U_{2m}^{(21)} = \alpha U_m^{(10)} U_m^{(11)} + 2\left(U_{2m,\hat{t}\tau}^{(20)} + U_{2m,\hat{t}}^{(20)}\right) \quad (4.56)$$

The initial conditions are

$$U_{2m}^{(21)}(0,0) = 0;\ U_{2m,\hat{t}}^{(21)}(0,0) = 0$$

Substituting for $U_m^{(10)}, U_m^{(11)} and\ U_m^{(20)}$ in (4.52) yields

$$U_{m,\hat{t}\hat{t}}^{(21)} + \theta^2 U_m^{(21)} = -2\alpha U_m^{(10)} U_m^{(11)} - 2U_{m,\hat{t}\tau}^{(20)} - 2U_{m,\hat{t}}^{(20)} - 2\omega_1' U_{m,\hat{t}\hat{t}}^{(11)} - \omega_1'' U_{m,\hat{t}}^{(10)} - 2\omega_1' U_{m,\hat{t}}^{(10)} \quad (4.57)$$

i.e

$$U_{m,\hat{t}\hat{t}}^{(21)} + \theta^2 U_m^{(21)} = -2\alpha\left(\frac{\alpha_1\beta_2}{2} sin\theta\hat{t} + B\beta_2 sin\theta\hat{t}\right) - 2\left(-\theta\alpha_4' sin\theta\hat{t} + \theta\beta_4' cos\theta\hat{t} + \frac{2\theta r_1' sin2\theta\hat{t}}{3\theta^2}\right) - 2\left(-\theta\alpha_4 sin\theta\hat{t} + \theta\beta_4 cos\theta\hat{t} + \frac{2\theta r_1 sin2\theta\hat{t}}{3\theta^2}\right) - 2\omega_1'(-\theta^2)\beta_2 sin\theta\hat{t} - \omega_1''(-\theta\alpha_1 sin\theta\hat{t}) - 2\omega_1'(-\alpha_1\theta sin\theta\hat{t}) \quad (4.58)$$

Where,

$$\alpha_4(0) = \frac{8\alpha B^2}{3\theta^2},\quad \alpha_4'(0) = \frac{-13\alpha B^2}{3\theta^2} + \frac{4B^2}{\theta} \quad (4.59)$$

The remaining equation in (4.58) is

$$U_{m,\hat{t}\hat{t}}^{(21)} + \theta^2 U_m^{(21)} = r_2 + r_3 cos2\theta\hat{t} + r_4 sin2\theta\hat{t} \quad (4.60)$$

With the initial conditions;

$$U_m^{(21)}(0,0) = 0;\ U_{m,\hat{t}}^{(21)}(0,0) + \omega_1'(0) U_{m,\hat{t}}^{(11)} + U_{m,\hat{\tau}}^{(20)}(0,0) = 0$$

The solution of (4.60) is

$$U_m^{(21)}(\hat{t},\tau) = \alpha_6 cos\theta\hat{t} + \beta_6 sin\theta\hat{t} + \frac{r_2}{\theta^2} - \left(\frac{r_3 cos2\theta\hat{t} + r_4 sin2\theta\hat{t}}{3\theta^2}\right) \quad (4.61)$$

With the initial conditions,

$$\alpha_6(0) + \frac{r_2}{\theta^2} - \frac{r_3}{3\theta^2} = 0$$

Therefore

$$\alpha_6(0) = \frac{r_3 - 3r_2}{3\theta^2} = 0,\qquad \beta_6(0) = 0 \quad (4.62)$$

$$U_{2m,\hat{t}\hat{t}}^{(21)} + \varphi^2 U_{2m}^{(21)} = \alpha\left[\frac{\alpha_1\beta_2}{2} sin2\theta\hat{t} + B\beta_2 sin\theta\hat{t}\right] + 2\left(U_{2m,\hat{t}\tau}^{(20)} + U_{2m,\hat{t}}^{(20)}\right) = \alpha\left[\frac{\alpha_1\beta_2}{2} sin2\theta\hat{t} + B\beta_2 sin\theta\hat{t}\right] + 2\left[-\varphi\alpha_5' sin\varphi\hat{t} + \beta_5'\varphi cos\varphi\hat{t} + \frac{\alpha}{2}\left\{\frac{-2\theta\alpha_1' Bsin\theta\,\hat{t}}{\varphi^2-\theta^2} - \frac{2\theta(\alpha_1^2)' sin2\theta\hat{t}}{2(\varphi^2-4\theta^2)}\right\} + \{\{-\varphi\alpha_5 sin\varphi\hat{t} + \beta_5 cos\varphi\hat{t} + \frac{\alpha}{2}\left\{\frac{-2\theta\alpha_1 Bsin\theta\,\hat{t}}{\varphi^2-\theta^2} - \right.\right.$$

2θα12sin2θt2(φ2−4θ2)

(4.38a)

To ensure a uniformly valid solution in $\hat{t}$, we equate to zero the coefficient of $cos\varphi\hat{t}\ and\ sin\varphi\hat{t}$

$$2\varphi\beta_5' + 2\varphi\beta_5 = 0 \Rightarrow \beta_5' + \beta_5 = 0 \Rightarrow \beta_5'(0) = -\beta_5(0)$$

$$-2\varphi\alpha_5' - 2\varphi\alpha_5 = 0 \Rightarrow \alpha_5' + \alpha_5 = 0 \Rightarrow \alpha_5'(0) = -\alpha_5(0)$$

$$\beta_5 = \beta_5(0)e^{-\tau} = 0,\ \alpha_5 = \alpha_5(0)e^{-\tau} = 0$$

The remaining equation of (4.37a) is

$$U_{2m,\hat{t}\hat{t}}^{(21)} + \varphi^2 U_{2m}^{(21)} = \left[\alpha B\beta_2 + \frac{\alpha}{2}\left(\frac{-2\theta B}{\varphi^2-\theta^2}\right)(\alpha_1' + \alpha_1)\right] sin\theta\hat{t} + \left[\frac{\alpha\alpha_1\beta_2}{2} + \frac{\alpha}{2}\left\{\frac{-\theta(\alpha_1^2)' + \alpha_1^2}{2(\varphi^2-4\theta^2)}\right\}\right] sin2\theta\hat{t}$$
$$= r_5 sin\theta\hat{t} + r_6 sin2\theta\hat{t} \tag{4.63}$$

Where,

$r_5 = \left[\alpha B\beta_2 - \frac{-2\theta B}{(\varphi^2-\theta^2)}(\alpha_1' + \alpha)\right] = \alpha B\beta_2$, since $\alpha_1' + \alpha = 0$,

$\alpha_1' = B;\quad r_5(0) = \frac{-\alpha B^2}{\theta},\ r_6 = \left[\frac{\alpha\alpha_1\beta_2}{2} + \frac{\alpha}{2}\left\{\frac{-\theta(\alpha_1^2)'+\alpha_1^2}{2(\varphi^2-4\theta^2)}\right\}\right]$

Therefore

$$r_6(0) = \left[\frac{\alpha\alpha_1(0)\beta_2(0)}{2} + \frac{\alpha}{2}\left\{\frac{-\theta(\alpha_1^2(0))'+\alpha_1^2(0)}{2(\varphi^2-4\theta^2)}\right\}\right] = \frac{B^2\alpha}{2\theta} + \frac{B^2\theta\alpha}{4(\varphi^2-4\theta^2)} = B^2 S_1 \tag{4.64}$$

Therefore the result is

$$U_{2m}^{(21)} = \alpha_7(\tau)cos\varphi\hat{t} + \beta_7(\tau)sin\varphi\hat{t} + \frac{r_5 cos\theta\,\hat{t}}{\varphi^2-\theta^2} + \frac{r_6 cos2\theta\hat{t}}{\varphi^2-4\theta^2} \tag{4.65}$$

The initial conditions are

$$U_{2m}^{(21)}(0,0) = 0;\ U_{2m,\hat{t}}^{(21)}(0,0) + U_{2m,\hat{\tau}}^{(20)}(0,0) = 0;$$

This implies

$$-\varphi\alpha_7(0)sin\varphi\hat{t} + \varphi\beta_7(0)cos\varphi\hat{t} + \frac{\theta r_5(0)cos\theta\,\hat{t}}{\varphi^2-\theta^2} + \frac{2\theta r_6(0)cos\theta\,\hat{t}}{\varphi^2-4\theta^2} + \alpha_5'(0)cos\varphi\hat{t} + \frac{\alpha}{2}\left[\frac{\alpha_1'\alpha_1}{\varphi^2} + \frac{2B\theta\alpha_1'cos\theta\,\hat{t}}{\varphi^2-\theta^2} + \right.$$

2 α1'α1cos2θt2φ2−4θ2=0

(4.66)

Therefore

$$\alpha_7(0) = 0 \tag{4.67}$$

Similarly, the following is obtained

$$\varphi\beta_7(0) + \frac{\theta r_5(0)}{\varphi^2-\theta^2} + \frac{2\theta r_6(0)}{\varphi^2-4\theta^2} + \alpha_5'(0) + \frac{\alpha}{2}\left[\frac{\alpha_1'(0)\alpha_1(0)}{\varphi^2} + \frac{2B\theta\alpha_1'(0)}{\varphi^2-\theta^2} + \frac{\alpha_1'(0)\alpha_1(0)}{(\varphi^2-4\theta^2)}\right] = 0 \tag{4.42a}$$

$$\beta_7(0) = -\frac{1}{\varphi}\left[\frac{\theta r_5(0)}{\varphi^2-\theta^2} + \frac{2\theta r_6(0)}{\varphi^2-4\theta^2} + \alpha_5'(0) + \frac{\alpha}{2}\left(\frac{\alpha_1'(0)\alpha_1(0)}{\varphi^2} + \frac{2B\theta\alpha_1'(0)}{\varphi^2-\theta^2} + \frac{\alpha_1'(0)\alpha_1(0)}{(\varphi^2-4\theta^2)}\right)\right] \tag{4.68}$$

i.e

$$\beta_7(0) = B^2\left(\frac{\alpha S_0}{\varphi} + \frac{\alpha}{2\varphi^3} + \frac{\alpha}{2\alpha(\varphi^2-4\theta^2)} - \frac{\alpha}{\alpha(\varphi^2-\theta^2)} - \frac{2\theta\alpha S_1}{\varphi(\varphi^2-4\theta^2)}\right) \tag{4.69}$$

So far, it follows that

$$U^{(21)} = U_m^{(21)}(1-cos2mx) + U_{2m}^{(21)}(1-cos4mx)$$

$$O(\epsilon^2\delta^2):\ U_{,\hat{t}\hat{t}}^{(22)} + U_{,xxxx}^{(22)} + 2\lambda U_{,xx}^{(22)} = -U_{,\tau\tau}^{(20)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(12)} - 2U_{,\hat{t}\,\tau}^{(21)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(12)} - 2\omega_1'' U_{,\hat{t}}^{(11)} -$$
$$2U_{,\hat{t}}^{(21)} - 2\omega_1' U_{,\hat{t}}^{(11)} - \alpha\left\{\left(U^{(11)}\right)^2 + U^{(10)}\right\} \tag{4.70}$$

Substituting on the right hand side gives

$$U_{,\hat{t}\hat{t}}^{(22)} + U_{,xxxx}^{(22)} + 2\lambda U_{,xx}^{(22)}$$
$$= -\left[U_{m,\tau\tau}^{(20)}(1-\cos2mx) + U_{2m,\tau\tau}^{(20)}(1-\cos4mx) + 2\omega_1' U_{m,\hat{t}\hat{t}}^{(12)}(1-cos2mx)\right.$$
$$+ 2\left\{U_{m,\hat{t}\,\tau}^{(21)}(1-cos2mx) + U_{2m,\hat{t}\,\tau}^{(21)}(1-cos2mx)\right\} + 2\omega_1'' U_{m,\hat{t}}^{(11)}(1-cos2mx)$$
$$+ 2\left\{U_{m,\hat{t}}^{(21)}(1-cos2mx) + U_{2m,\hat{t}}^{(21)}(1-cos4mx)\right\} + +\ 2\omega_1' U_{m,\hat{t}\hat{t}}^{(11)}(1-cos2mx)$$
$$\left. + \alpha\left(U_m^{(11)}\right)^2\left\{\frac{3}{2} - 2\cos2mx + \frac{1}{2}\cos4mx\right\} + 2\left\{U_m^{(10)}U_m^{(12)}\right\}\left\{\frac{3}{2} - 2\cos2mx + \frac{1}{2}\cos4mx\right\}\right]$$

Let

$$U^{(22)} = \sum_{n=1}^{\infty} U_n^{(22)}(\hat{t}\,\tau)(1-cos2nx)$$

The left hand side of (4.60) simplifies to

$$\sum_{n=1}\left[U_{n,\hat{t}\hat{t}}^{(22)}(1-\cos2nx) + (-16n^4 + 8\lambda n^2)U_n^{(22)} + U_n^{(22)}(1-cos2nx)\right]$$

Multiplying (4.60) through by cos2mx and integrating from 0 to π and for n=m, gives;

$$-\frac{\pi}{2}U^{(22)}_{m,\hat{t}\hat{t}} + (-16m^4 + 8\lambda m^2)U^{(22)}_m\left(\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}U^{(22)}_m\right) = -\left[\left(-\frac{\pi}{2}\right)U^{(20)}_{m,\tau\tau} + 2\omega_1' U^{(12)}_{m,\hat{t}\hat{t}}\left(-\frac{\pi}{2}\right) + \right.$$

2Um, tτ 12−π2+ 2ω1''Um, t11−π2+2Um, t21−π2++ 2ω1'Um, t11−π2+αUm 112−2.−π2+αUm 10Um12−2.−π2 (4.71)

Further simplification of (4.71) gives

$$-\frac{\pi}{2}U^{(22)}_{m,\hat{t}\hat{t}} + (16m^4 - 8\lambda m^2 + 1)U^{(22)}_m = -\frac{\pi}{2}\left[-U^{(20)}_{m,\tau\tau} - 2\omega_1' U^{(12)}_{m,\hat{t}\hat{t}} - 2U^{(21)}_{m,\hat{t}\tau} - 2\omega_1'' U^{(11)}_{m,\hat{t}} - 2U^{(21)}_{m,\hat{t}} - \right.$$

2ω1'Um, t11+2αUm112+2αUm10Um12 (4.72)

Further simplification yields

$$U^{(22)}_{m,\hat{t}\hat{t}} + \theta^2 U^{(22)}_m = -\left[U^{(20)}_{m,\tau\tau} + 2\omega_1' U^{(12)}_{m,\hat{t}\hat{t}} + 2U^{(21)}_{m,\hat{t}\tau} + 2\omega_1'' U^{(11)}_{m,\hat{t}} + 2U^{(21)}_{m,\hat{t}} + 2\omega_1' U^{(11)}_{m,\hat{t}} - \right.$$

2Um112+Um10Um12 (4.73)

The initial conditions are

$$U^{(22)}_m(0,0) = 0;\;\; U^{(22)}_{m,\hat{t}}(0,0) + \omega_1'(0)U^{(12)}_{m,\hat{t}} + U^{(21)}_{m,\tau}(0,0) = 0$$

Next from equation (4.72) for n=2m, let

$$U^{(22)} = \sum_{n=1}^{\infty} U^{(22)}_n(1 - cos4mx)$$

Multiplying (4.72) through by cos4mx and integrating from 0 to π and for n = 2m, gives

$$-\frac{\pi}{2}U^{(22)}_{2m,\hat{t}\hat{t}} + \frac{\pi}{2}(-256m^4 + 32\lambda m^2)U^{(22)}_{2m} - \frac{\pi}{2}U^{(22)}_{2m} = \frac{\alpha}{2}\left\{\left(U^{(11)}_m\right)^2 + \left(U^{(10)}_m U^{(12)}_m\right)\left(\frac{\pi}{2}\right)\right\} \quad (4.37a)$$

This further gives

$$U^{(22)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(22)}_{2m} = -\frac{\alpha}{2}\left\{\left(U^{(11)}_m\right)^2 + \left(U^{(10)}_m U^{(12)}_m\right)\right\} \quad (4.74)$$

The initial conditions are

$$U^{(22)}_m(0,0) = 0;\;\; U^{(22)}_{2m,\hat{t}}(0,0) + \omega_1'(0)U^{(12)}_{2m,\hat{t}} + U^{(21)}_{m,\tau}(0,0) = 0$$

On substituting for terms in (4.74) and simplifying, the result is

$$U^{(22)}_{2m,\hat{t}\hat{t}} + \theta^2 U^{(22)}_m =$$

$$-\left[\left\{\alpha_4'' cos\theta\hat{t} + \frac{r_0''}{\theta^2} - \frac{r_1'' cos2\theta\hat{t}}{3\theta^2}\right\} + 2\omega_1'\{-\theta^2\alpha_3 sin\theta\hat{t} + \beta_3\theta^2 cos\theta\hat{t}\} + 2\left\{-\theta\alpha_6' sin\theta\hat{t} + \theta\beta_6' cos\theta\hat{t} - \right.\right.$$

2θr3'sin2θt+2θr4'cos2θt3θ2+2ω1'θβ2cosθt+2−α6θsinθt+β6θcosθt+2θr3sinθt−2θr4cosθt −3θ2+2ω1'θβ2cosθt+2αβ221−cos2θt+α1β22sin2θt+Bβ2sin2θt (4.75)

To ensure a uniformly valid solution in $\hat{t}$; equate to zero the coefficients of $cos\theta\hat{t}$ *and* $sin\theta\hat{t}$ of (4.75) and this yields respectively

$$-\alpha_4'' + 2\omega_1'\theta\beta_3 - 2\theta\beta_6' - 2\omega_1''\theta\beta_2 - 2\beta_6\theta - 2\omega_1'\theta\beta_2 = 0$$

and

$$2\omega_1'\theta^2\alpha_3 + 2\theta\alpha_6' + 2\alpha_6\theta - 2\alpha B\beta_2 = 0$$

Simplification gives,

$$\beta_6' + \beta_6 = \frac{1}{2\theta}[\alpha_4'' - 2\omega_1'\theta^2\beta_3 + 2\omega_1''\beta_2 + 2\omega_1'\theta\beta_2] = \rho_2(\tau)$$

This gives

$$\beta_6(\tau) = e^{-\tau}[\textstyle\int e^{\tau}\rho_2(\tau)d\tau + \beta_6(0)] = e^{-\tau}[\int e^{\tau}\rho_2(\tau)d\tau]$$

Similarly, simplification yields

$$\alpha_6' + \alpha_6 = \frac{1}{2\theta}[-2\omega_1'\theta^2\alpha_3 + 2\alpha B\beta_2] = \rho_3(\tau)$$

Therefore

$$\alpha_6 = e^{-\tau}[\int e^{\tau}\rho_3(\tau)d\tau + \alpha_6(0)]$$

The remaining part of equation (4.75) is;

$$U_{m,\hat{t}\hat{t}}^{(22)} + \theta^2 U_m^{(22)} = r_7 + r_8 cos2\theta\hat{t} + r_9 sin2\theta\hat{t} \quad (4.76)$$

$$r_7 = -\left[\frac{r_0''}{\theta^2} - \frac{2r_4'}{3\theta} + \alpha\beta_2\right], r_8 = \left[\frac{r_1''}{3\theta^2} - \frac{4r_4}{3\theta} - \alpha\beta_2\right], \quad r_9 = -\left[\frac{2r_3'}{3\theta^2} - \frac{4r_3}{3\theta} - \alpha\alpha_1\beta_2\right]$$

It is to be recalled that

$$r_0 = -2\alpha(\frac{\alpha_1^2}{2} + B^2)$$

It therefore follows that

$$U_m^{(22)} = \alpha_8 cos\theta + \beta_8 sin\theta + \frac{r_7}{3\theta^2} + \frac{r_8 cos2\theta}{\theta^2} + \frac{r_9 sin2\theta\hat{t}}{\theta^2} \quad (4.77)$$

The initial conditions are

$$U_m^{(22)}(0,0) = 0\,; \quad U_m^{(22)}(0,0) + \omega_1'(0)U_{m,\hat{t}}^{(12)}(0,0) + U_{m,\tau}^{(21)}(0,0) = 0$$

It follows that,

$$\alpha_8(0) = \left(\frac{r_8(0)}{3\theta^2} - \frac{r_7(0)}{\theta^2}\right) = \frac{4\alpha B}{3\theta^3} - \frac{17\alpha B^2}{3\theta^4}$$

$$\theta\beta_8(0) + \frac{2\theta r_9(0)}{\theta^2} = 0$$

Therefore

$$\beta_8(0) = \frac{-2\theta r_9(0)}{\theta^2} = \frac{-3\alpha B^2}{\theta^3}$$

Solution for equations of order $\epsilon^3\delta^j$ for j=0, 1, 2

$$O(\epsilon^3)\ :\ U_{,\hat{t}\hat{t}}^{(30)} + U_{,xxxx}^{(30)} + 2\lambda U_{,xx}^{(30)} + U^{(30)}$$
$$= -(\omega_1')^2 U_{,\hat{t}\hat{t}}^{(10)} - 2\left(\omega_1' U_{,\hat{t}\hat{t}}^{(20)} + \omega_2' U_{,\hat{t}\hat{t}}^{(20)}\right) - 2\alpha \mathrm{U}^{(20)}\mathrm{U}^{(10)} + \beta\left(\mathrm{U}^{(10)}\right)^3$$

$$U_{,\hat{t}\hat{t}}^{(30)} + U_{,xxxx}^{(30)} + 2\lambda U_{,xx}^{(30)} + U^{(30)} = -(\omega_1')^2 U_{m,\hat{t}\hat{t}}^{(10)}(1-\cos 2mx) - 2\Big[\omega_1' U_{m,\hat{t}\hat{t}}^{(20)} + (1-cos2mx) + \omega_1' U_{2m,\hat{t}\hat{t}}^{(20)}(1-cos2mx)\Big] - 2\omega_2' U_{,\hat{t}\hat{t}}^{(10)}(1-cos2mx) - 2\alpha\Big[U_m^{(10)}(1-cos2mx)\Big\{U_m^{(20)}(1-cos2mx) + U_{2m}^{(20)}(1-cos4mx)\Big\}\Big] + \beta\left(\mathrm{U}_m{}^{(10)}\right)^3(1-cos2mx)^3 \quad (4.78)$$

The following simplifications are necessary as they appear in equation (4.78);

$$\mathrm{U}_m{}^{(10)}\mathrm{U}_m{}^{(20)}(1-cos2mx)(1-cos2mx) = \mathrm{U}_m{}^{(10)}\mathrm{U}_m{}^{(20)}(1-cos2mx)^2$$
$$= \mathrm{U}_m{}^{(10)}\mathrm{U}_m{}^{(20)}[1-cos2mx+cos^2 2mx]$$
$$= \mathrm{U}_m{}^{(10)}\mathrm{U}_m{}^{(20)}\left[1-2cos2mx+\left(\frac{(1+cos4mx)}{2}\right)\right]$$
$$= \mathrm{U}_m{}^{(10)}\mathrm{U}_m{}^{(20)}\left(\frac{3}{2}-2cos2mx+\frac{1}{2}cos4mx\right)$$

$$\beta\left(\mathrm{U}_m{}^{(10)}\right)^3(1-cos2mx)^3 = \beta\left(\mathrm{U}_m{}^{(10)}\right)^3[1-3cos2mx+3cos^2 2mx - cos2^3 mx]$$
$$= \beta\left(\mathrm{U}_m{}^{(10)}\right)^3\left[1-3cos2mx+\frac{3}{2}(1+cos4mx)-\frac{1}{4}(3cos2mx+cos6mx)\right]$$
$$= \beta\left(\mathrm{U}_m{}^{(10)}\right)^3\left[\frac{5}{2}-\frac{15}{4}cos2mx+\frac{3}{2}cos4mx-\frac{1}{4}cos6mx\right]$$

Therefore; substituting all these on the right hand side of (4.78) yields

$$U_{,\hat{t}\hat{t}}^{(30)} + U_{,xxxx}^{(30)} + 2\lambda U_{,xx}^{(30)} + U^{(30)} = -(\omega_1')^2 U_{m,\hat{t}\hat{t}}^{(10)}(1-\cos 2mx)$$
$$-2\Big[\omega_1' U_{m,\hat{t}\hat{t}}^{(20)}(1-cos2mx) + \omega_1' U_{2m,\hat{t}\hat{t}}^{(20)}(1-cos4mx) + \omega_2' U_m^{(10)}(1-cos2mx)\Big]$$
$$-2\alpha\Big[U_m^{(10)}U_m^{(20)}\Big\{\frac{3}{2}-2cos2mx+\frac{1}{2}cos4mx\Big\} + U_m^{(10)}U_m^{(20)}\Big\{1-\frac{1}{2}cos2mx-cos4mx+\frac{1}{2}cos6mx\Big\}\Big] + \beta\left(\mathrm{U}_m{}^{(10)}\right)^3\left[\frac{5}{2}-\frac{15}{4}cos2mx+\frac{3}{2}cos4mx+\frac{1}{4}cos6mx\right] \quad (4.79)$$

Let

$$U^{(30)} = \sum_{n=1}^{\infty} U_n^{(30)}(1 - cos2nx)$$

Left hand side of (4.79) becomes

$$\sum_{n=1}^{\infty}\left[U_{n,\hat{t}\hat{t}}^{(30)}(1 - cos2nx) + (-16n^4 + 8\lambda n^2 + 1)U_n^{(30)}cos2nx\right]$$

Multiplying (4.79) through by cos2mx and integrating from 0 to π and for n =m, the result is

$$-\frac{\pi}{2}U_{m,\hat{t}\hat{t}}^{(30)} + (-16m^4 + 8\lambda m^2 + 1)U_m^{(30)}\left(-\frac{\pi}{2}\right)$$
$$= -\left[(\omega_1')^2U_{m,\hat{t}\hat{t}}^{(10)}\left(-\frac{\pi}{2}\right) + 2\omega' U_{m,\hat{t}\hat{t}}^{(20)}\left(-\frac{\pi}{2}\right) + 2\omega_2' U_{m,\hat{t}\hat{t}}^{(10)}\left(-\frac{\pi}{2}\right) + 2\alpha U_m^{(10)}U_m^{(20)}\left(-2.-\frac{\pi}{2}\right)\right.$$
$$\left. - \alpha U_m^{(10)}U_{2m}^{(20)}\left(-\frac{\pi}{2}\right) - \frac{15}{4}\left(\mathrm{U}_m{}^{(10)}\right)^3\left(-\frac{\pi}{2}\right)\right] \tag{4.79}$$

Further simplification of (4.79) yields

$$-\frac{\pi}{2}\left[U_{m,\hat{t}\hat{t}}^{(30)} + (16m^4 - 8\lambda m^2 + 1)U_m^{(30)}\right] = -\frac{\pi}{2}\left[-(\omega_1')^2U_{m,\hat{t}\hat{t}}^{(10)} - 2\omega'{}_1U_{m,\hat{t}\hat{t}}^{(20)} - 2\omega_2'U_{m,\hat{t}\hat{t}}^{(10)} - 2\alpha\left[2U_m^{(10)}U_m^{(20)} + U_m^{(10)}U_{2m}^{(20)}\right] - \frac{15}{4}\beta\left(\mathrm{U}_m{}^{(10)}\right)^3\right] \tag{4.80}$$

Yet, further simplification yields

$$U_{m,\hat{t}\hat{t}}^{(30)} + \theta^2U_m^{(30)} = -(\omega_1')^2U_{m,\hat{t}\hat{t}}^{(10)} - 2\omega'{}_1U_{m,\hat{t}\hat{t}}^{(20)} - 2\omega_2'U_{m,\hat{t}\hat{t}}^{(10)} - 2\alpha\left[2U_m^{(10)}U_m^{(20)} + U_m^{(10)}U_{2m}^{(20)}\right] - \frac{15}{4}\beta\left(\mathrm{U}_m{}^{(10)}\right)^3 \tag{4.81}$$

The initial conditions are

$$U_m^{(30)}(0,0) = 0;$$
$$U_{m,\hat{t}}^{(30)}(0,0) + \omega'(0)U_{m,\hat{t}}^{(20)}(0,0) + \omega_2'(0)U_\tau^{(10)}(0,0) = 0$$

Multiplying (4.81) through by cos4mx and integrating from 0 to π and for n=2m, the result gives

$$-\frac{\pi}{2}\left[U_{2m,\hat{t}\hat{t}}^{(30)} + (256m^4 - 32\lambda m^2 + 1)U_{2m}^{(30)}\right]$$
$$= 2\left[\omega_1'U_{2m,\hat{t}\hat{t}}^{(20)}\left(-\frac{\pi}{2}\right)\right] + 2\alpha\left[U_m^{(10)}U_m^{(20)}.\frac{1}{2}\left(-\frac{\pi}{2}\right) + U_m^{(10)}U_{2m}^{(20)}\left(-\frac{\pi}{2}\right)\right] - \beta\left(\mathrm{U}_m{}^{(10)}\right)^3.\frac{3}{2}\left(-\frac{\pi}{2}\right)$$
$$U_{2m,\hat{t}\hat{t}}^{(30)} + \varphi^2U_{2m}^{(30)}$$
$$= -2\left[-\omega_1'U_{2m,\hat{t}\hat{t}}^{(20)}\right] + 2\alpha\left[U_m^{(10)}U_m^{(20)} - U_m^{(10)}U_{2m}^{(20)}\right]$$
$$-\frac{3}{2}\beta\left(\mathrm{U}_m{}^{(10)}\right)^3$$

The initial conditions are

$$U_{2m}^{(30)}(0,0) = 0;\ U_{2m}^{(30)}(0,0) + \omega_1'(0)U_{2m,\hat{t}}^{(20)}(0,0) = 0$$

Multiplying (4.81) through by cos6mx and integrating from 0 to π and for n=3m and get,

$$U_{3m,\hat{t}\hat{t}}^{(30)} + (1296m^4 - 72\lambda m^2 + 1)U_{3m}^{(30)} = \alpha U_m^{(10)}U_{2m}^{(20)} - \frac{1}{4}\beta\left(\mathrm{U}_m{}^{(10)}\right)^3 \tag{4.82}$$

Let

$\Omega^2 = 1296m^4 - 72\lambda m^2 + 1 > 0$ for all m

Therefore

$$U_{3m,\hat{t}\hat{t}}^{(30)} + \Omega^2U_{3m}^{(30)} = \alpha U_m^{(10)}U_{2m}^{(20)} - \frac{1}{4}\beta\left(\mathrm{U}_m{}^{(10)}\right)^3 \tag{4.83}$$

The initial conditions for (4.83) are;

$$U_{3m}^{(30)}(0,0) = 0;\ U_{3m,\hat{t}}^{(30)}(0,0) = 0$$

$$U_{m,\hat{t}\hat{t}}^{(30)} + \theta^2 U_m^{(30)} = -(\omega_1')^2(-\theta^2\alpha_1 cos\theta\hat{t}) - 2\omega'_1\left(-\alpha_4\theta^2 cos\theta\hat{t} + \frac{4r_1 cos2\theta\hat{t}}{3}\right) - 2\omega_2'(-\theta^2\alpha_1 cos\theta\hat{t}) -$$

$$2\alpha\left[\left(\frac{\alpha_1\alpha_4}{4} + \frac{Br_0}{\theta^2}\right) + \left(\frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B\alpha_4\right)cos\theta\hat{t} + \left(\frac{\alpha_1\alpha_4}{4} - \frac{Br_1}{3\theta^2}\right)cos2\theta\hat{t} - \frac{\alpha_1 r_1}{6\theta^2}cos3\theta\hat{t}\right] - 2\alpha\left[\frac{\alpha\alpha_1^2 B}{2(\varphi^2-\theta^2)} +\right.$$

αα12α122+B22φ2+αα138φ2−4θ2cosθt+α1α52cosφ+θt+αα12Bcos2θt2φ2−θ2+α1β52sinφ+θt+α1 α52cosφ−θt+α1β52sinφ−θt+αα13cos3θt8φ2−4θ2−15β4B3+3α12B2+3α134+α1B2cosθt+3α122B

$$cos2\theta\hat{t} + \frac{\alpha_1^3}{4}cos3\theta\hat{t} \qquad (4.84)$$

To ensure a uniformly valid solution in$\hat{t}$, equate to zero the coefficients of $cos\theta\hat{t}$ and this yields

$$(\omega_1')^2\theta^2 + 2\omega_1'\alpha_4\theta^2 + 2\omega_2'\theta^2\alpha_1 - 2\alpha\left(\frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B\alpha_4\right) - \left\{\frac{\alpha^2\alpha_1\left(\frac{\alpha_1^2}{2} + B^2\right)}{\varphi^2} + \frac{\alpha_1^3\alpha_1}{4(\varphi^2 - 4\theta^2)}\right\}$$
$$- \frac{45\beta}{4}\left(\frac{\alpha_1^3}{4} + \alpha_1 B^2\right) = 0$$

Therefore;

$$\omega_2' = -\frac{1}{2\theta^2\alpha_1}\left[(\omega_1')^2\theta^2 + 2\omega_1'\alpha_4\theta^2 - 2\alpha\left(\frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B\alpha_4\right) - \left\{\frac{\alpha^2\alpha_1\left(\frac{\alpha_1^2}{2} + B^2\right)}{\varphi^2} + \frac{\alpha_1^3\alpha_1}{4(\varphi^2 - 4\theta^2)}\right\}\right.$$
$$\left. - \frac{45\beta}{4}\left(\frac{\alpha_1^3}{4} + \alpha_1 B^2\right)\right] \qquad (4.85)$$

The remaining equation in (4.84) is

$U_{m,\hat{t}\hat{t}}^{(30)} + \theta^2 U_m^{(30)} = r_{10} + r_{11}cos2\theta\hat{t} + r_{12}cos3\theta\hat{t} + r_{13}cos(\varphi + \theta)\hat{t} + r_{14}sin(\varphi + \theta)\hat{t} + r_{15}cos(\varphi -$ *θt+r16sinφ−θt* M (4.85)

Solving (4.85) gives

$$U_m^{(30)}(\hat{t},\tau) = \alpha_9(\tau)cos\theta\hat{t} + \beta_9(\tau)sin\theta\hat{t} + \frac{r_{10}}{\theta^2}$$

$-\frac{r_{11}cos\theta\,\hat{t}}{3\theta^2} - \frac{r_{12}cos3\theta\hat{t}}{8\theta^2} - \frac{1}{\varphi(2\theta+\varphi)}[r_{13}cos(\varphi + \theta)\hat{t} + r_{14}sin(\varphi + \theta)\hat{t}] + \frac{1}{\varphi(2\theta-\varphi)}[\,r_{15}cos(\varphi - \theta)\hat{t} +$ *r16sinφ−θt* (4.86)

The initial conditions are

$$U_m^{(30)}(0,0) = 0$$
$$U_{m,\hat{t}}^{(30)}(0,0) + \omega_1'(0)U_{m,\hat{t}}^{(20)}(0,0) + \omega_2'(0)U_{m,\tau}^{(10)}(0,0) = 0$$

Where,

$$\alpha_9(0) = \left[-\frac{r_{10}}{\theta^2} + \frac{r_{11}}{3\theta^2} + \frac{r_{12}}{8\theta^2} + \frac{r_{13}}{\varphi(2\theta + \varphi)} - \frac{r_{15}}{\varphi(2\theta - \varphi)}\right] at\ \tau = 0$$

and

$$\beta_9(0) = \frac{1}{\theta}\left[\frac{r_{14}(\varphi + \theta)}{\varphi(2\theta + \varphi)} - \frac{r_{16}(\varphi - \theta)}{\varphi(2\theta - \varphi)}\right] at\ \tau = 0$$

Substituting in (4.84) gives

$$U^{(30)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(30)}_{2m}$$
$$= 2\omega_1^{'}\left[-\varphi^2\alpha_5 cos\varphi\hat{t} - \varphi^2\beta_5 sin\varphi\hat{t} + \frac{\alpha}{2}\left\{\frac{-2\theta^2 B\alpha_1 cos\theta\hat{t}}{\varphi^2-\theta^2} - \frac{2\alpha_1^2\theta^2 cos2\theta\hat{t}}{\varphi^2-4\theta^2}\right\}\right]$$
$$+ 2\alpha\left[\left(\frac{\alpha_1\alpha_4}{4} + \frac{Br_0}{\theta^2}\right) + \left(\frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B\alpha_4\right)cos\theta\hat{t} + \left(\frac{\alpha_1\alpha_4}{4} + \frac{Br_1}{3\theta^2}\right)cos2\theta\hat{t} - \frac{\alpha_1 r_1}{6\theta^2}cos3\theta\hat{t}\right]$$
$$- 2\alpha\left[\frac{\alpha\alpha_1^2 B}{2(\varphi^2-\theta^2)} + \left\{\frac{\alpha\alpha_1\left(\frac{\alpha_1^2}{2}+B^2\right)}{2\varphi^2} + \frac{\alpha_1^3\alpha}{8(\varphi^2-4\theta^2)}\right\}cos\theta\hat{t} + \frac{\alpha_1\alpha_5}{2}cos(\varphi+\theta)\hat{t}\right.$$
$$+ \frac{\alpha\alpha_1^2 Bcos2\theta\hat{t}}{2(\varphi^2-\theta^2)} + \frac{\alpha_1\beta_5}{2}sin(\varphi+\theta)\hat{t} + \frac{\alpha_1\alpha_5}{2}cos(\varphi-\theta)\hat{t} + \frac{\alpha_1\beta_5}{2}sin(\varphi-\theta)\hat{t}$$
$$\left. + \frac{\alpha_1^3\alpha cos3\theta\hat{t}}{8(\varphi^2-4\theta^2)}\right]$$
$$- \frac{3\beta}{2}\left[\left(B^3 + \frac{3\alpha_1^2 B}{2}\right) + 3\left(\frac{\alpha_1^3}{4} + \alpha_1 B^2\right)cos\theta\hat{t} + \frac{3\alpha_1^2 B}{2}cos2\theta\hat{t} + \frac{\alpha_1^3}{4}cos3\theta\hat{t}\right]$$

(4.87)

To ensure a uniformly valid solution in $\hat{t}$, needs equating to zero the coefficients of $cos\varphi\hat{t}$ and $sin\varphi\hat{t}$. A further simplification of (4.87) gives

$$U^{(30)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(30)}_{2m} = r_{17} + r_{18}cos\theta\hat{t} + r_{19}cos2\theta\hat{t} + r_{20}cos3\theta\hat{t} \quad (4.88)$$

The solution of (4.88) is

$$U^{(30)}_{2m} = \alpha_{10}cos\varphi\hat{t} + \beta_{10}sin\varphi\hat{t} + \frac{r_{17}}{\varphi^2} + \frac{r_{18}cos\theta\,\hat{t}}{(\varphi^2-\theta^2)} + \frac{r_{19}cos2\theta\hat{t}}{(\varphi^2-4\theta^2)} + \frac{r_{20}cos3\theta\hat{t}}{(\varphi^2-9\theta^2)} \quad (4.89)$$

The initial conditions for (4.89) are

$$U^{(30)}_{2m}(0,0) = 0;\ U^{(30)}_{2m,\hat{t}}(0,0) + \omega_1^{'}(0)U^{(20)}_{2m,\hat{t}}(0,0) = 0$$

Therefore,

$$\alpha_{10}(0) = -\left[\frac{r_{18}}{(\varphi^2-\theta^2)} + \frac{r_{19}}{(\varphi^2-4\theta^2)} + \frac{r_{20}}{(\varphi^2-9\theta^2)}\right]\ at\ \tau = 0 \quad (4.90)$$

$$\beta_{10}(0) = 0 \quad (4.91)$$

Therefore,

$$U^{(30)}_{3m,\hat{t}\hat{t}} + \Omega^2 U^{(30)}_{3m} = \alpha\left[\frac{\alpha\alpha_1^2 B}{2(\varphi^2-\theta^2)} + \left\{\frac{\alpha\alpha_1\left(\frac{\alpha_1^2}{2}+B^2\right)}{2\varphi^2} + \frac{\alpha_1^3\alpha}{8(\varphi^2-4\theta^2)}\right\}cos\theta\hat{t} + \frac{\alpha_1\alpha_5}{2}\cos(\varphi+\theta)\,\hat{t} + \frac{\alpha\alpha_1^2 Bcos2\theta\hat{t}}{2(\varphi^2-\theta^2)} + \right.$$

*α1β52*sin*φ+θt+α1α52*cos*φ−θt+α1β52*sin*φ−θt+α13αcos3θt8φ2−4θ2−β4B3+3α12B2+3α134+α1B2cosθt+3α122Bcos2θt+α134cos3θt*

(4.92)

Therefore;

$$U^{(30)}_{3m}(\hat{t},\tau) =$$
$$\alpha_{11}(\tau)cos\Omega\hat{t} + \beta_{11}(\tau)sin\Omega\hat{t} + \frac{r_{22}cos\theta\hat{t}}{\Omega^2-\theta^2} + \frac{r_{23}cos2\theta\hat{t}}{\Omega^2-4\theta^2} + \frac{r_{24}cos3\theta\hat{t}}{\Omega^2-9\theta^2} + \left\{\frac{r_{25}\cos(\varphi+\theta)\hat{t}+r_{26}\sin(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2}\right\} +$$
$$\left\{\frac{r_{27}\cos(\varphi-\theta)\hat{t}+r_{28}\sin(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2}\right\} \quad (4.93)$$

$$\therefore \alpha_{11}(0) = -\left[\frac{r_{22}}{\Omega^2-\theta^2} + \frac{r_{23}}{\Omega^2-4\theta^2} + \frac{r_{24}}{\Omega^2-9\theta^2} + \frac{r_{25}}{\Omega^2-(\varphi+\theta)^2} + \frac{r_{27}}{\Omega^2-(\varphi+\theta)^2}\right]\Bigg|\ \tau = 0 \quad (4.94)$$

$$\beta_{11}(0) = \frac{-1}{\Omega}\left[\frac{r_{26}(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} + \frac{r_{28}(\varphi-\theta)}{\Omega^2-(\varphi+\theta)^2}\right]\Bigg|\ \tau = 0 \quad (4.95)$$

So far, it follows that

$$U^{(30)} = U_m^{(30)}(1-cos2mx) + U_{2m}^{(30)}(1-cos4mx) + U_{3m}^{(30)}(1-cos6mx) \quad (4.96)$$

From,

$$O(\epsilon^3\delta) : U^{(31)}_{,\hat{t}\hat{t}} + U^{(31)}_{,xxxx} + 2\lambda U^{(31)}_{,xx} + U^{(31)} = -(\omega_1')^2 U^{(11)}_{,\hat{t}\hat{t}} - 2\left(\omega_1' U^{(21)}_{,\hat{t}\,\tau} + \omega_2 U^{(11)}_{,\hat{t}\,\tau}\right) - 2U^{(30)}_{,\hat{t}\,\tau} + 2\left(\omega_1' U^{(20)}_{,\hat{t}\hat{t}} + \omega_2' U^{(10)}_{,\hat{t}\hat{t}}\right) - \left(\omega_1'' U^{(20)}_{,\hat{t}} + \omega_2''{}^{+} U^{(10)}_{,\hat{t}}\right) - 2\left\{U^{(30)}_{\hat{t}} + \left(\omega_1' U^{(20)}_{,\hat{t}} + \omega_2'{}^{+} U^{(10)}_{,\hat{t}}\right)\right\} - \alpha\left(\mathrm{U}^{(10)}\mathrm{U}^{(21)} + \mathrm{U}^{(11)}\mathrm{U}^{(20)}\right) + 3\beta\left(\mathrm{U}^{(10)}\right)^2\left(\mathrm{U}^{(11)}\right) \quad (4.97)$$

Substituting on the right hand side of (4.87) gives

$$\begin{aligned}
\mathrm{U}^{(31)}_{,\hat{t}\hat{t}} + \mathrm{U}^{(31)}_{,xxxx} + 2\lambda \mathrm{U}^{(31)}_{,xx} + \mathrm{U}^{(31)}
&= -\Big[(\omega_1')^2 U^{(11)}_{m,\hat{t}}(1-cos2mx) \\
&+ 2\left\{\omega_1'\left(U^{(21)}_{m,\hat{t}\tau}(1-cos2mx) + U^{(21)}_{2m,\hat{t}\tau}(1-cos4mx)\right) + \omega_2 U^{(11)}_{2m,\hat{t}\tau}(1-cos2mx)\right\} \\
&+ 2\left\{U^{(30)}_{m,\hat{t}\tau}(1-cos2mx) + U^{(30)}_{2m,\hat{t}\tau}(1-cos4mx) + U^{(30)}_{3m,\hat{t}\tau}(1-cos6mx)\right\} \\
&- 2\left\{\omega_1'\left(U^{(20)}_{m,\hat{t}\hat{t}}(1-cos2mx) + U^{(20)}_{2m,\hat{t}\hat{t}}(1-cos4mx)\right) + \omega_2' U^{(10)}_{m,\hat{t}\hat{t}}(1-cos2mx)\right\} \\
&+ \left\{\omega_1'' U^{(20)}_{m,\hat{t}}(1-cos2mx) + U^{(20)}_{2m,\hat{t}}(1-cos4mx)\right. \\
&\left.+ \omega_2'' U^{(10)}_{m,\hat{t}}(1-cos2mx)\right\} + 2\left\{\mathrm{U}^{(30)}_{m,\hat{t}}(1-\cos2mx) + \mathrm{U}^{(30)}_{2m,\hat{t}}(1-\cos4mx)\right. \\
&+ \mathrm{U}^{(30)}_{3m,\hat{t}}(1-\cos6mx) \\
&\left.+ \omega_1'\left(\mathrm{U}^{(20)}_{m,\hat{t}}(1-\cos2mx) + \mathrm{U}^{(20)}_{2m\hat{t}}(1-\cos4mx) + \omega_2'\mathrm{U}^{(10)}_{m,\hat{t}}(1-\cos2mx)\right)\right\} \\
&+ \alpha\left\{\mathrm{U}^{(10)}_{m}(1-\cos2mx)\left(\mathrm{U}^{(21)}_{m}(1-\cos2mx) + \mathrm{U}^{(21)}_{2m}(1-\cos4mx)\right)\right. \\
&\left.+ \mathrm{U}^{(11)}_{m}(1-\cos2mx)\left(\mathrm{U}^{(20)}_{m}(1-\cos2mx) + \mathrm{U}^{(20)}_{2m}(1-\cos4mx)\right)\right\} \\
&- 3\beta\left\{\left(\mathrm{U}^{(10)}_{m}\right)^2 \mathrm{U}^{(11)}_{m}(1-\cos2mx)^3\right\}\Big]
\end{aligned} \quad (4.98)$$

Therefore, simplifying (4.98) yields

$$\begin{aligned}
U^{(31)}_{,\hat{t}\hat{t}} + U^{(31)}_{,xxxx} + 2\lambda U^{(31)} + U^{(31)}
&= -\Big[(\omega_1')^2 U^{(21)}_{m,\hat{t}\hat{t}}(1-cos2mx) \\
&+ 2\left\{\omega_1'\left(U^{(21)}_{m,\hat{t}\tau}(1-cos2mx) + U^{(21)}_{2m,\hat{t}\tau}(1-cos4mx)\right)\omega_2' U^{(11)}_{m,\hat{t}\tau}(1-cos2mx)\right\} \\
&+ 2\left\{U^{(30)}_{m,\hat{t}\tau}(1-cos2mx) + U^{(30)}_{2m,\hat{t}\tau}(1-cos4mx) + U^{(30)}_{3m,\hat{t}\tau}(1-cos6mx)\right\} \\
&- 2\left\{\omega_1' U^{(20)}_{m,\hat{t}\hat{t}}(1-cos2mx) + U^{(20)}_{2m,\hat{t}\hat{t}}(1-cos4mx) + \omega_2' U^{(10)}_{m,\hat{t}\hat{t}}(1-cos2mx)\right\} \\
&+ \left\{\omega'{}'_1 U^{(20)}_{m,\hat{t}}(1-cos2mx) + U^{(20)}_{2m,\hat{t}}(1-cos4mx)\right. \\
&\left.+ \omega_2'' U^{(10)}_{m,\hat{t}}(1-cos2mx)\right\} + 2\left\{U^{(30)}_{m,\hat{t}}(1-cos2mx) + U^{(30)}_{2m,\hat{t}}(1-cos4mx)\right. \\
&+ U^{(30)}_{3m,\hat{t}}(1-cos6mx) + \omega_1'\left(U^{(20)}_{m,\hat{t}}(1-cos2mx) + U^{(20)}_{2m,\hat{t}}(1-cos4mx)\right) \\
&\left.+ \omega_2'\left(U^{(10)}_{m,\hat{t}}(1-cos2mx)\right)\right\} \\
&+ \alpha\left\{U^{(10)}_m U^{(21)}_m\left(\frac{3}{2} - 2cos2mx + \frac{1}{2}cos4mx\right)\right. \\
&\left.+ U^{(10)}_m U^{(21)}_{2m}\left(1 - \frac{1}{2}cos2mx - cos4mx + \frac{1}{2}cos6mx\right)\right\} \\
&+ \alpha\left\{U^{(11)}_m U^{(20)}_m\left(\frac{3}{2} - 2cos2mx + \frac{1}{2}cos4mx\right)\right. \\
&\left.+ U^{(11)}_m U^{(20)}_{2m}\left(1 - \frac{1}{2}cos2mx - cos4mx + \frac{1}{2}cos6mx\right)\right\} \\
&- 3\beta\left(U^{(10)}_m\right)^2 U^{(11)}_m\left(\frac{5}{2} - \frac{15}{4}cos2mx - \frac{3}{2}cos4mx - \frac{1}{4}cos6mx\right)\Big]
\end{aligned} \quad (4.99)$$

Let

$$U^{(31)}\sum_{n=1}^{\infty}U^{(31)}(1-cos2nx)$$

The left hand side of (4.89) yields

$$\sum_{n=1}^{\infty}\left[U_{n,\hat{t}\hat{t}}^{(31)}(1-cos2nx)+(-16n^4+8\lambda n^2+1)U_n^{(31)}cos2nx\right]$$

Multiplying (4.89) through cos2mx and integrating from 0 to π and from n=m, gives

$$-\frac{\pi}{2}\left[U_{m,\hat{t}\hat{t}}^{(31)}+(16m^4-8\lambda m^2+1)U_m^{(31)}\right]=$$
$$-\left[(\omega_1')^2U_{m,\hat{t}\hat{t}}^{(11)}\left(-\frac{\pi}{2}\right)+\ 2\left\{\omega_1'U_{m,\hat{t}\tau}^{(21)}\left(-\frac{\pi}{2}\right)+\omega_2'U_{m,\hat{t}\tau}^{(11)}\left(-\frac{\pi}{2}\right)\right\}+2\left\{U_{m,\hat{t}\,\tau}^{(30)}\left(-\frac{\pi}{2}\right)\right\}-\right.$$
$$2\left\{\begin{matrix}\omega_1'U_{m,\hat{t}\hat{t}}^{(20)}\\+\omega_2'U_{m,\hat{t}\hat{t}}^{(10)}\left(-\frac{\pi}{2}\right)\end{matrix}\right\}+\left\{\omega_1'U_{m,\hat{t}\tau}^{(20)}\left(-\frac{\pi}{2}\right)+\omega_2''U_{m,\hat{t}}^{(10)}\left(-\frac{\pi}{2}\right)\right\}+2\left\{U_{m,\hat{t}}^{(30)}\left(-\frac{\pi}{2}\right)+\omega_1'U_{m,\hat{t}\tau}^{(20)}\left(-\frac{\pi}{2}\right)+\right.$$
$$\left.\omega_2'U_{m,\hat{t}}^{(10)}\left(-\frac{\pi}{2}\right)\right\}+$$
$$\alpha\left\{-2U_m^{(10)}U_m^{(21)}\left(-\frac{\pi}{2}\right)-U_m^{(10)}U_m^{(21)}\left(-\frac{\pi}{2}\right)-2U_m^{(11)}U_m^{(21)}\left(-\frac{\pi}{2}\right)-2U_m^{(11)}U_m^{(20)}\left(-\frac{\pi}{2}\right)-\right.$$
$$\left.\left.U_m^{(11)}U_{2m}^{(20)}\left(-\frac{\pi}{2}\right)\right\}+3\beta\left(U_m^{(10)}\right)^2U_m^{(11)}\left(-\frac{15}{4}\right)\right]$$

(4.100)

A further simplification of the above yields

$$U_{m,\hat{t}\hat{t}}^{(31)}+\theta^2U_m^{(31)}=(\omega_1')^2U_{m,\hat{t}\hat{t}}^{(11)}-2\left\{\omega_1'U_{m,\hat{t}\tau}^{(21)}+\omega_2'U_{m,\hat{t}\tau}^{(11)}\right\}-2\left\{U_{m,\hat{t}\tau}^{(30)}\right\}+2\left\{\omega_1'U_{m,\hat{t}\hat{t}}^{(20)}+\omega_2'U_{m,\hat{t}\hat{t}}^{(10)}\right\}$$
$$-\left\{\omega_1'U_{m,\hat{t}}^{(20)}+\omega_2''U_{m,\hat{t}}^{(10)}\right\}-2\left\{U_{m,\hat{t}}^{(30)}+\omega_1'U_{m,\hat{t}}^{(20)}+\omega_2'U_{m,\hat{t}}^{(10)}\right\}$$
$$+\alpha\left\{2U_m^{(10)}U_m^{(21)}+U_m^{(10)}U_m^{(21)}+2U_m^{(11)}U_m^{(21)}+U_m^{(11)}U_{2m}^{(20)}\right\}-\frac{45}{4}\beta\left(U_m^{(10)}\right)^2U_m^{(11)}$$

(4.101)

The initial conditions for (4.100) are

$$U_m^{(31)}(0,0)=0;$$
$$U_{m,\hat{t}}^{(31)}(0,0)+\omega_1'(0)U_{m,\hat{t}}^{(21)}(0,0)+\omega_2'(0)U_{m,\hat{t}}^{(11)}(0,0)+U_{m,\tau}^{(30)}(0,0)=0$$

Multiplying (4.89) through cos4mx and integrating from 0 to π and for n=2m gives

$$-\frac{\pi}{2}\left[U_{m,\hat{t}\hat{t}}^{(31)}+(256m^4-32\lambda m^2+1)U_{2m}^{(30)}\right]$$
$$=-\left[2\omega_1'U_{2m,\hat{t}\tau}^{(21)}\left(-\frac{\pi}{2}\right)+2U_{2m,\hat{t}\tau}^{(30)}\left(-\frac{\pi}{2}\right)-2\omega_1'U_{2m,\hat{t}\hat{t}}^{(20)}\left(-\frac{\pi}{2}\right)+\omega_1''U_{2m,\hat{t}}^{(20)}\left(-\frac{\pi}{2}\right)\right.$$
$$+2U_{2m,\hat{t}}^{(30)}\left(-\frac{\pi}{2}\right)+2\omega_1'U_{2m,\hat{t}}^{(20)}\left(-\frac{\pi}{2}\right)$$
$$+\alpha\left\{\frac{1}{2}U_m^{(10)}U_m^{(21)}\left(-\frac{\pi}{2}\right)-U_m^{(10)}U_{2m}^{(21)}\left(-\frac{\pi}{2}\right)+\frac{1}{2}U_m^{(11)}U_m^{(20)}\left(-\frac{\pi}{2}\right)\right.$$
$$\left.\left.-U_m^{(11)}U_{2m}^{(20)}\left(-\frac{\pi}{2}\right)\right\}-3\beta\left(U_m^{(10)}\right)^2U_m^{(11)}\left(\frac{3}{2}\right)\right]$$

(4.102)

This yield

$$U_{2m,\hat{t}\hat{t}}^{(31)}+\varphi^2U_{2m}^{(30)}$$
$$=-\left[2\omega_1'U_{2m,\hat{t}\tau}^{(21)}+2U_{2m,\hat{t}\tau}^{(30)}-2\omega_1'U_{2m,\hat{t}\hat{t}}^{(20)}+\omega_1''U_{2m,\hat{t}}^{(20)}+2U_{2m,\hat{t}}^{(30)}+2\omega_1'U_{2m,\hat{t}}^{(20)}\right.$$
$$\left.+\ \alpha\left\{\frac{1}{2}U_m^{(10)}U_m^{(21)}-U_m^{(10)}U_{2m}^{(21)}+\frac{1}{2}U_m^{(11)}U_m^{(20)}-U_m^{(11)}U_{2m}^{(20)}\right\}-\frac{9}{4}\beta\left(U_m^{(10)}\right)^2U_m^{(11)}\right]$$

(4.103)

The initial conditions are

$$U_{2m}^{(31)}(0{,}0) = 0;\ U_{2m}^{(31)}(0{,}0) + \omega_1'(0)U_{2m,\hat{t}}^{(20)}(0{,}0) = 0$$

Multiplying through cos6mx and integrating from 0 to π and for n=3m, the result is

$$-\frac{\pi}{2}\left[U_{m,\hat{t}\hat{t}}^{(31)} + (1296m^4 - 72\lambda m^2 + 1)U_{2m}^{(31)}\right]$$
$$= -\left[2U_{3m,\hat{t}\tau}^{(30)}\left(-\frac{\pi}{2}\right) + 2U_{3m,\hat{t}}^{(30)}\left(-\frac{\pi}{2}\right) + \alpha\left\{\frac{1}{2}U_m^{(10)}U_{2m}^{(21)}\left(-\frac{\pi}{2}\right) + \frac{1}{2}U_m^{(11)}U_{2m}^{(20)}\left(-\frac{\pi}{2}\right)\right\} - 3\beta\left(-\frac{1}{4}\right)\left(U_m^{(10)}\right)^2U_m^{(11)}\right] \tag{4.104}$$

A further simplification of above yields

$$U_{3m,\hat{t}\hat{t}}^{(31)} + \Omega^2U_{3m}^{(30)} = -\left[2U_{3m,\hat{t}\tau}^{(30)} + 2U_{3m,\hat{t}}^{(30)} + \alpha\left\{\frac{1}{2}U_m^{(10)}U_{2m}^{(21)} + \frac{1}{2}U_m^{(11)}U_{2m}^{(20)}\right\} + \frac{3}{4}\beta\left(U_m^{(10)}\right)^2U_m^{(11)}\right] \tag{4.105}$$

The initial conditions for (4.105) are

$$U_{3m}^{(31)}(0{,}0) = 0;\ \ U_{3m,\hat{t}}^{(31)}(0{,}0) = 0$$

Simplifying (4.105) yields;

$$\begin{aligned}
&U_{m,\hat{t}\hat{t}}^{(31)} + \theta^2U_m^{(31)}\\
&= (\omega_1')^2\theta^2\beta_2\sin\theta\hat{t} - 2\left[\omega_1'(-\theta\alpha_6'\sin\theta\hat{t} + \theta\beta_6'\cos\theta\hat{t}) - \left(\frac{-2\theta r_3'\sin2\theta\hat{t} + 2\theta r_4'\cos2\theta\hat{t}}{3\theta^2}\right)\right]\\
&- 2\omega_2'\beta_2'\theta\cos\theta\hat{t}\\
&- 2\Bigg[-\alpha_9'\theta\sin\theta\hat{t} + \beta_9'\theta\cos\theta\hat{t} + \frac{2r_{11}'\sin2\theta\hat{t}}{3\theta} + \frac{3r_{12}'\sin3\theta\hat{t}}{8\theta}\\
&- \frac{1}{\varphi(2\theta+\varphi)}\{-r_{13}'(\varphi+\theta)\sin(\varphi+\theta)\hat{t} + r_{14}'(\varphi+\theta)\cos(\varphi+\theta)\hat{t}\}\\
&+ \frac{1}{\varphi(2\theta-\varphi)}\{-r_{15}'(\varphi-\theta)\sin(\varphi-\theta)\hat{t} + r_{16}'(\varphi-\theta)\cos(\varphi-\theta)\hat{t}\}\Bigg]\\
&+ 2\omega_1'\left\{-\theta^2\alpha_4\cos\theta\hat{t} + \frac{4r_1\cos2\theta\hat{t}}{3}\right\} + 2\omega_2'(-\alpha_1\theta^2\cos\theta\hat{t}) - \omega_1'\left\{-\theta\alpha_4\sin\theta\hat{t} + \frac{2r_1\sin2\theta\hat{t}}{3\theta}\right\}\\
&- \omega_2''(-\alpha_1\theta\sin\theta\hat{t})\\
&- 2\Bigg\{-\alpha_9\sin\theta\hat{t} + \beta_9\cos\theta\hat{t} + \frac{2r_{11}\sin2\theta\hat{t}}{3\theta} + \frac{3r_{12}\sin3\theta\hat{t}}{8\theta}\\
&- \frac{1}{\varphi(2\theta+\varphi)}\{-(\varphi+\theta)r_{13}\sin(\varphi+\theta)\hat{t} + r_{14}(\varphi+\theta)\cos(\varphi+\theta)\hat{t}\}\\
&+ \frac{1}{\varphi(2\theta-\varphi)}\{-(\varphi+\theta)r_{15}\sin(\varphi-\theta)\hat{t} + (\varphi-\theta)r_{16}\cos(\varphi-\theta)\hat{t}\}\Bigg\}\\
&- 2\omega_1'\left\{-\theta\alpha_4\sin\theta\hat{t} + \frac{2r_1\sin2\theta\hat{t}}{3\theta}\right\} - 2\omega_2'(-\theta\alpha_1\sin\theta\hat{t})\\
&+ 2\alpha\Bigg\{\left(\frac{\alpha_1\alpha_6}{2} - \frac{Br_2}{\theta^2}\right) + \left(\frac{\alpha_1r_2}{2} - \frac{Br_3}{6\theta^2} + B\alpha_6\right)\cos\theta\hat{t} + \left(\frac{B\beta_6 - \alpha_1\alpha_4}{6\theta^2}\right)\sin\theta\hat{t}\\
&+ \left(\frac{\alpha_1\alpha_6}{2} - \frac{Br_3}{3\theta^2}\right)\cos2\theta\hat{t} + \left(\frac{\alpha_1\beta_6}{2} - \frac{Br_4}{3\theta^2}\right)\sin\theta\hat{t} - \frac{\alpha_1r_3}{6\theta^2}\cos3\theta\hat{t} - \frac{\alpha_1r_4}{6\theta^2}\sin3\theta\hat{t}\Bigg\}\\
&+ \alpha\Bigg\{\left(\frac{\alpha_1r_6}{2(\varphi^2-4\theta^2)} + \frac{Br_5}{\varphi^2-\theta^2}\right)\sin\theta\hat{t} + \left(\frac{\alpha_1r_5}{2(\varphi^2-4\theta^2)} - \frac{Br_6}{\varphi^2-4\theta^2}\right)\sin2\theta\hat{t}\\
&+ \frac{\alpha_1r_6\sin3\theta\hat{t}}{2(\varphi^2-4\theta^2)} + B\alpha_7\cos\varphi\hat{t} + B\beta_7\sin\varphi\hat{t} + \frac{\alpha_1\alpha_7}{2}\cos(\varphi+\theta)\hat{t} + \frac{\alpha_1\beta_7}{2}\sin(\varphi+\theta)\hat{t}\\
&+ \frac{\alpha_1\alpha_7}{2}\cos(\varphi-\theta)\hat{t} + \frac{\alpha_1\beta_7}{2}\sin(\varphi-\theta)\hat{t}\Bigg\}\\
&+ 2\alpha\left\{\frac{\beta_2\alpha_4}{2}\sin2\theta\hat{t} + \frac{\beta_2r_0}{\theta^2}\sin\theta\hat{t} - \frac{\beta_2r_1}{6\theta^2}(\sin3\theta\hat{t} - \sin\theta\hat{t})\right\}
\end{aligned} \tag{4.106}$$

To ensure a uniformly valid solution in $\hat{t}$, demands equating to zero the coefficients of $\sin\theta\,\hat{t}$ and $\cos\theta\,\hat{t}$ in (4.76) as further expanded.

The coefficient of $\sin\theta\,\hat{t}$ leads to

$$\alpha_9' + \alpha_9 = h_1(\tau) \tag{4.107}$$

$$h_1(\tau) = -\frac{1}{2\theta}\left[2\omega_1'\theta\alpha_6' + \omega_2''\alpha_1\theta + \omega_1'\theta\alpha_4 + 2\omega_1'\theta\alpha_4 + 2\omega_2'\theta\alpha_1 + 2\alpha\left(\frac{B\beta_6-\alpha_1 r_4}{6\theta^2}\right)\right] \tag{4.78a}$$

Therefore

$$\alpha_9 = e^{-\tau}\left[\int e^s h_1(s)ds + \alpha_9(0)\right] \tag{4.108}$$

Equating the coefficient of $\cos\theta\,\hat{t}$ yields

$$\beta_9' + \beta_9 = h_2(\tau) \tag{4.109}$$

$$h_2(\tau) = -\frac{1}{2\theta}\left[2\omega_1'\theta\beta_6' + 2\omega_2'\beta_2'\theta + 2\omega_1'\theta^2\alpha_4 + 2\omega_2'\alpha_1\theta^2 - 2\alpha\left(\frac{\alpha_1 r_2}{\theta^2} - \frac{\alpha_1 r_3}{6\theta^2} + B\alpha_6\right)\right] \tag{4.78d}$$

Therefore

$$\beta_9 = e^{-\tau}\left[\int e^s h_2(s)ds + \beta_9(0)\right] \tag{4.110}$$

The remaining equation in (4.96)

$$U^{(31)}_{m,\hat{t}\hat{t}} + \theta^2 U^{(31)}_m = r_{29} + r_{30}\sin 2\theta\hat{t} + r_{31}\cos 2\theta\hat{t} + r_{32}\cos 3\theta\hat{t} + r_{33}\cos 3\theta\hat{t} + r_{34}\cos\varphi\hat{t} + r_{35}\sin\varphi\hat{t} + r_{36}\cos(\varphi+\theta)\,\hat{t} + r_{37}\sin(\varphi+\theta)\,\hat{t} + r_{38}\cos(\varphi-\theta)\,\hat{t} + r_{39}\sin(\varphi-\theta)\,\hat{t} \tag{4.111}$$

The initial conditions are

$$U^{(31)}_m(0,0) = 0;\; U^{(31)}_{m,\hat{t}}(0,0) + \omega_1'(0)U^{(21)}_{m,\hat{t}}(0,0) + \omega_2'(0)U^{(11)}_{m,\hat{t}}(0,0) + U^{(30)}_{m,\tau}(0,0) = 0$$

Therefore the following is obtained

$$U^{(31)}_m = \alpha_{12}\cos\theta\hat{t} + \beta_{12}\sin\theta\hat{t} + \frac{r_{29}}{\theta^2} + \frac{r_{30}\sin 2\theta\hat{t} + r_{31}\cos 2\theta\hat{t}}{\theta^2 - 4\theta^2} + \frac{r_{32}\cos 3\theta\hat{t} + r_{33}\sin 3\theta\hat{t}}{\theta^2 - 9\theta^2} + \frac{r_{34}\cos\varphi\hat{t} + r_{35}\sin\varphi\hat{t}}{\theta^2 - \varphi^2} + \frac{r_{36}\cos(\varphi+\theta)\hat{t} + r_{37}\sin(\varphi+\theta)\hat{t}}{\varphi(2\theta-\varphi)} + \frac{r_{38}\cos(\varphi-\theta)\hat{t} + r_{39}\sin(\varphi-\theta)\hat{t}}{\varphi(2\theta-\varphi)} \tag{4.112}$$

$$\alpha_{12}(0) = -\left[\frac{r_{29}}{\theta^2} + \frac{r_{31}}{\theta^2-4\theta^2} + \frac{r_{32}}{\theta^2-9\theta^2} + \frac{r_{34}}{\theta^2-\varphi^2} - \frac{r_{36}}{\varphi(2\theta-\varphi)} + \frac{r_{38}}{\varphi(2\theta-\varphi)} - \frac{2\alpha B^3}{3\theta^4} + \frac{1}{2\theta^2}\left(\frac{B^2}{\theta^4} + \frac{16\alpha B^3}{3\theta^2} - \frac{10\alpha^2 B^3}{\theta^2} + \frac{3\alpha^2 B^3}{2\varphi^2} + \frac{\alpha^2 B^3}{4(\varphi^2-4\theta^2)} + \frac{225\beta B^3}{16}\right) + \left(\alpha_9' + \frac{r_{10}'}{\theta^2} - \frac{r_{11}'}{3\theta^2} - \frac{r_{12}'}{8\theta^2} - \frac{r_{13}'}{\varphi(2\theta+\varphi)} + \frac{r_{15}'}{\varphi(2\theta-\varphi)}\right)\right]\Bigg|\,\tau = 0 \tag{4.113}$$

$$\beta_{12}(0) = \frac{-1}{\theta}\left[\frac{2\theta r_{30}}{\theta^2-4\theta^2} + \frac{3\theta r_{33}}{\theta^2-9\theta^2} + \frac{\varphi r_{35}}{\theta^2-\varphi^2} + \frac{(\varphi+\theta)r_{37}}{\varphi(2\theta-\varphi)} + \frac{(\varphi-\theta)r_{39}}{\varphi(2\theta-\varphi)}\right] at\ \tau = 0 \tag{4.114}$$

Substituting in (4.111) gives,

$$U^{(31)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(31)}_{2m}$$
$$= -\Bigg[2\omega_1'\left\{-\varphi\alpha_7'\sin\varphi\hat{t} + \varphi\beta_7'\cos\varphi\hat{t} + \frac{\theta r_5'\cos\theta\hat{t}}{\varphi^2-\theta^2} + \frac{2\theta r_6'\cos 2\theta\hat{t}}{\varphi^2-4\theta^2}\right\}$$
$$+ 2\left\{-\varphi\alpha_{10}'\sin\varphi\hat{t} + \varphi\beta_{10}'\cos\varphi\hat{t} - \frac{\theta r_{18}'\sin\theta\hat{t}}{\varphi^2-\theta^2} - \frac{2\theta r_{19}'\sin 2\theta\hat{t}}{\varphi^2-4\theta^2} - \frac{3\theta r_{20}'\sin 3\theta\hat{t}}{\varphi^2-9\theta^2}\right\}$$
$$- \alpha\omega_1'\left\{\frac{2B\alpha_1\theta^2\cos\theta\hat{t}}{\varphi^2-\theta^2} - \frac{2\theta\alpha_1^2\cos 2\theta\hat{t}}{\varphi^2-4\theta^2}\right\} + \alpha(\omega_1'' + 2\omega_1')\left\{\frac{B\alpha_1\theta\sin\theta\hat{t}}{\varphi^2-\theta^2} - \frac{\theta\alpha_1^2\sin 2\theta\hat{t}}{\varphi^2-4\theta^2}\right\}$$
$$+ 2\left\{-\varphi\alpha_{10}\sin\varphi\hat{t} + \varphi\beta_{10}\cos\varphi\hat{t} - \frac{\theta r_{18}\sin\theta\hat{t}}{\varphi^2-\theta^2} - \frac{2\theta r_{19}\sin 2\theta\hat{t}}{\varphi^2-4\theta^2} - \frac{3\theta r_{20}\sin 3\theta\hat{t}}{\varphi^2-9\theta^2}\right\}$$
$$+ \frac{\alpha}{2}\left\{\left(\frac{\alpha_1\alpha_6}{2} + \frac{r_2 B}{\theta^2}\right) + \left(\frac{\alpha_1 r_2}{\theta^2} - \frac{\alpha_1 r_3}{6\theta^2} + B\alpha_6\right)\cos\theta\hat{t} + \left(B\beta_6 - \frac{\alpha_1 r_4}{6\theta^2}\right)\sin\theta\hat{t}\right.$$
$$\left. + \left(\frac{\alpha_1\beta_6}{2} - \frac{r_4 B}{3\theta^2}\right)\sin 2\theta\hat{t} - \frac{\alpha_1 r_3}{6\theta^2}\cos 3\theta\hat{t} - \frac{\alpha_1 r_4}{6\theta^2}\sin 3\theta\hat{t}\right\}$$
$$- \alpha\left\{\left(\frac{\alpha_1 r_6}{2(\varphi^2-4\theta^2)} + \frac{Br_5}{\varphi^2-\theta^2}\right)\sin\theta\hat{t} + \frac{\alpha_1\alpha_7}{2}\cos(\varphi+\theta)\hat{t} + \frac{\alpha_1\beta_7}{2}\sin(\varphi+\theta)\hat{t}\right.$$
$$+ \frac{\alpha_1\alpha_7}{2}\cos(\varphi-\theta)\hat{t} + \frac{\alpha_1\beta_7}{2}\sin(\varphi-\theta)\hat{t} + \left(\frac{\alpha_1 r_5}{2(\varphi^2-\theta^2)} + \frac{Br_6}{\varphi^2-4\theta^2}\right)\sin 2\theta\hat{t}$$
$$\left. + \frac{\alpha_1 r_6}{2(\varphi^2-4\theta^2)}\sin 3\theta\hat{t} + B\alpha_7\cos\varphi\hat{t} + B\beta_7\sin\varphi\hat{t}\right\}$$
$$+ \frac{\alpha}{2}\left\{\frac{\beta_2\alpha_4\sin 2\theta\hat{t}}{2} + \frac{\beta_2 r_0\sin\theta\hat{t}}{\theta^2} - \frac{\beta_2 r_1}{6\theta^2}(\sin 3\theta\hat{t} - \sin\theta\hat{t})\right\}$$
$$- \alpha\left\{\frac{\alpha\beta_2}{2}\left(\frac{\frac{\alpha_1^2}{2} + B^2}{\varphi^2}\right) - \frac{\alpha\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)}\sin\theta\hat{t} + \frac{\alpha\alpha_1 B\beta_2}{2(\varphi^2-\theta^2)}\sin 2\theta\hat{t} + \frac{\alpha\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)}\sin 3\theta\hat{t}\right\}$$
$$+ \frac{9\beta}{4}\left[\left\{\beta_2\left(B^2 + \frac{\alpha_1^2}{2}\right) - \frac{\beta_2\alpha_1^2}{4}\right\}\sin\varphi\hat{t} + \beta_2 B\alpha_1\sin 2\theta\hat{t} + \frac{\beta_2\alpha_1^2}{4}\sin 3\theta\hat{t}\right]\Bigg] \tag{4.115}$$

To ensure uniformly valid solution in $\hat{t}$ needs equating the coefficients of $\cos\varphi\hat{t}$ and $\sin\varphi\hat{t}$ to zero

Equating the coefficient of $\cos\varphi\hat{t}$ yields

$$-2\omega_1'\varphi\beta_7' - 2\varphi\beta_{10}' - 2\beta_{10}\varphi + B\alpha_1\alpha_7 = 0$$

Therefore

$$\beta_{10}' + \beta_{10} = \frac{1}{2\varphi}[-2\omega_1'\varphi\beta_7' + B\alpha_1\alpha_7] \tag{4.116}$$
$$\beta_{10}' + \beta_{10} = h_3(\tau) \tag{4.117}$$

Where,

$$h_3(\tau) = \frac{1}{2\varphi}[-2\omega_1'\varphi\beta_7' + B\alpha_1\alpha_7] \tag{4.118}$$

It therefore follows that

$$\beta_{10} = e^{-\tau}[\textstyle\int h_3(s)e^s ds + \beta_{10}(0)] \tag{4.119}$$

The coefficient of $\sin\varphi\hat{t}$ leads to

$$2\omega_1'\varphi\alpha_7' + 2\varphi\alpha_{10}' + 2\alpha_{10}\varphi + B\beta_7\alpha = 0$$
$$\alpha_{10}' + \alpha_{10} = h_4(\tau) \tag{4.120}$$
$$h_4(\tau) = -\frac{1}{2\varphi}[2\omega_1'\varphi\alpha_7' + B\beta_7\alpha] \tag{4.121}$$

Therefore

$$\alpha_{10} = e^{-\tau}[\textstyle\int h_4(s)e^s ds + \alpha_{10}(0)] \tag{4.122}$$

The remaining equation in (4.115) is

$$U^{(31)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(31)}_{2m} = r_{40}\cos\theta\hat{t} + r_{41}\sin\theta\hat{t} + r_{42}\cos 2\theta\hat{t} + r_{43}\sin 2\theta\hat{t} + r_{44}\cos 3\theta\hat{t} + r_{45}\sin 3\theta\hat{t} + r_{46}\cos(\varphi+\theta)\hat{t} + r_{47}\sin(\varphi+\theta)\hat{t} + r_{48}\cos(\varphi-\theta)\hat{t} + r_{49}\sin(\varphi-\theta)\hat{t} \tag{4.123}$$

The initial conditions are

$$U_{2m}^{(31)}(0,0)=0;\ U_{2m,\hat{t}}^{(31)}(0,0)+\omega_1'(0)U_{2m,\hat{t}}^{(21)}(0,0)+U_{2m,\tau}^{(30)}(0,0)=0$$

Therefore;

$$U_{2m}^{(31)}=\alpha_{13}\cos\varphi\hat{t}+\beta_{13}\sin\varphi\hat{t}+\frac{r_{42}\cos\theta\hat{t}+r_{43}\sin\theta\hat{t}}{\varphi^2-\theta^2}+\frac{r_{44}\cos2\theta\hat{t}+r_{45}\sin2\theta\hat{t}}{\varphi^2-4\theta^2}+\frac{r_{46}\cos3\theta\hat{t}+r_{47}\sin3\theta\hat{t}}{\varphi^2-9\theta^2}-\frac{r_{48}\cos(\varphi+\theta)\hat{t}+r_{49}\sin(\varphi+\theta)\hat{t}}{\theta(2\varphi+\theta)}+\frac{r_{50}\cos(\varphi-\theta)\hat{t}+r_{39}\sin(\varphi-\theta)\hat{t}}{\theta(2\varphi-\theta)}\tag{4.124}$$

Where, from the first initial condition

$$\alpha_{13}(0)=-\left[\frac{r_{42}}{\varphi^2-\theta^2}+\frac{r_{44}}{\varphi^2-4\theta^2}+\frac{r_{46}}{\varphi^2-9\theta^2}-\frac{r_{48}}{\theta(2\varphi+\theta)}+\frac{r_{50}}{\theta(2\varphi-\theta)}\right]\ at\ \tau=0\tag{4.125}$$

And from the second initial condition, it follows that

$$\left[\beta_{13}(0)\varphi+\frac{\theta\, r_{43}}{\varphi^2-\theta^2}+\frac{2\theta r_{45}}{\varphi^2-4\theta^2}+\frac{3\theta r_{47}}{\varphi^2-9\theta^2}-\frac{(\theta+\varphi)r_{49}}{\theta(2\varphi+\theta)}+\frac{(\theta-\varphi)r_{51}}{\theta(2\varphi-\theta)}+\alpha_{10}'(0)+\frac{r_{17}'}{\varphi^2}+\frac{r_{18}'}{\varphi^2-\theta^2}+\frac{r_{19}'}{\varphi^2-4\theta^2}+\frac{r_{20}'}{\varphi^2-9\theta^2}\right]=0$$

Therefore,

$$\beta_{13}(0)=-\frac{1}{\varphi}\left[\frac{\theta\, r_{43}}{\varphi^2-\theta^2}+\frac{2\theta r_{45}}{\varphi^2-4\theta^2}+\frac{3\theta r_{47}}{\varphi^2-9\theta^2}-\frac{(\theta+\varphi)r_{49}}{\theta(2\varphi+\theta)}+\frac{(\theta-\varphi)r_{51}}{\theta(2\varphi-\theta)}+\alpha_{10}'(0)+\frac{r_{17}'}{\varphi^2}+\frac{r_{18}'}{\varphi^2-\theta^2}+\frac{r_{19}'}{\varphi^2-4\theta^2}+\frac{r_{20}'}{\varphi^2-9\theta^2}\right]\tag{4.126}$$

Substituting in (4.111);

$$U_{3m,\hat{t}\hat{t}}^{(31)}+\Omega^2U_{3m}^{(31)}=-\left[2U_{3m,\hat{t}\tau}^{(30)}+2U_{3m,\hat{t}}^{(30)}+\alpha\left\{\frac{1}{2}U_m^{(10)}U_{2m}^{(21)}+\frac{1}{2}U_m^{(11)}U_{2m}^{(20)}\right\}+\frac{3}{4}\beta\left(U_m^{(10)}\right)^2U_m^{(11)}\right]$$

Simplification yields

$$\begin{aligned}U_{3m,\hat{t}\hat{t}}^{(31)}+\Omega^2U_{3m}^{(31)}=&-\Bigg[2\Bigg\{-\Omega\alpha_{11}'\sin\Omega\hat{t}+\Omega\beta_{11}'\cos\Omega\hat{t}-\frac{\theta r_{22}'\sin\theta\hat{t}}{\Omega^2-\theta^2}-\frac{2\theta r_{23}'\sin2\theta\hat{t}}{\Omega^2-4\theta^2}-\frac{3\theta r_{24}'\sin3\theta\hat{t}}{\Omega^2-9\theta^2}\\&-\frac{(\varphi+\theta)r_{25}'\sin(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2}+\frac{(\varphi+\theta)r_{26}'\cos(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2}-\frac{(\varphi-\theta)r_{27}'\sin(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2}\\&+\frac{(\varphi-\theta)r_{28}'\cos(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2}\Bigg\}\\&+2\Bigg\{-\Omega\alpha_{11}\sin\Omega\hat{t}+\Omega\beta_{11}\cos\Omega\hat{t}-\frac{\theta r_{22}\sin\theta\hat{t}}{\Omega^2-\theta^2}-\frac{2\theta r_{23}\sin2\theta\hat{t}}{\Omega^2-4\theta^2}-\frac{3\theta r_{24}\sin3\theta\hat{t}}{\Omega^2-9\theta^2}\\&-\frac{(\varphi+\theta)r_{25}\sin(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2}+\frac{(\varphi+\theta)r_{26}\cos(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2}-\frac{(\varphi-\theta)r_{27}\sin(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2}\\&+\frac{(\varphi-\theta)r_{28}\cos(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2}\Bigg\}\\&+\frac{\alpha}{2}\Bigg\{\left(\frac{\alpha_1r_6}{2(\varphi^2-4\theta^2)}+\frac{Br_5}{\varphi^2-\theta^2}\right)\sin\theta\hat{t}+\frac{\alpha_1\alpha_7}{2}\cos(\varphi+\theta)\hat{t}+\frac{\alpha_1\beta_7}{2}\sin(\varphi+\theta)\hat{t}\\&+\frac{\alpha_1\alpha_7}{2}\cos(\varphi-\theta)\hat{t}+\frac{\alpha_1\beta_7}{2}\sin(\varphi-\theta)\hat{t}+\left(\frac{\alpha_1r_5}{2(\varphi^2-\theta^2)}+\frac{Br_6}{\varphi^2-4\theta^2}\right)\sin2\theta\hat{t}\\&+\frac{\alpha_1r_6}{2(\varphi^2-4\theta^2)}\sin3\theta\hat{t}+B\alpha_7\sin\varphi\hat{t}+B\beta_7\sin\varphi\hat{t}\Bigg\}\\&+\frac{\alpha}{2}\Bigg\{\frac{\beta_2\alpha}{2}\left(\frac{B^2+\frac{\alpha_1^2}{2}}{\varphi^2}\right)-\frac{\alpha\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)}\sin\theta\hat{t}+\frac{\alpha\alpha_1B\beta_2}{2(\varphi^2-\theta^2)}\sin2\theta\hat{t}+\frac{\alpha\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)}\sin3\theta\hat{t}\\&-\frac{\beta_2\beta_5}{2}\cos(\varphi+\theta)\hat{t}\Bigg\}+\frac{3\beta}{4}\left\{\left(\beta_2\left(B^2+\frac{\alpha_1^2}{2}\right)-\frac{\beta_2\alpha_1^2}{4}\right)\sin\theta\hat{t}+\beta_2B\alpha_1\sin2\theta\hat{t}+\frac{\beta_2\alpha_1^2}{4}\sin3\theta\hat{t}\right\}\Bigg]\end{aligned}\tag{4.127}$$

To ensure uniformly valid solution in $\hat{t}$, needs equating the coefficients of $\cos\Omega\hat{t}$ and $\sin\Omega\hat{t}$ to zero. The coefficients of $\cos\Omega\hat{t}$ yields

$$-2\Omega\beta_{11}' - 2\Omega\beta_{11} - \frac{\alpha B\beta_7}{2} = 0 \tag{4.128}$$

Therefore

$$\beta_{11}' + \beta_{11} = -\frac{\alpha B\alpha_7}{2\Omega} = h_5(\tau) \tag{4.129}$$

where

$$h_5(\tau) = -\frac{\alpha B\alpha_7}{2\Omega} \tag{4.130}$$

Therefore

$$\beta_{11} = e^{-\tau}[\int h_5(\tau)e^s ds + \beta_{11}(0)] \tag{4.131}$$

The coefficients $\sin\Omega\hat{t}$ of yields

$$-2\Omega\alpha_{11}' - 2\Omega\alpha_{11} - \frac{\alpha B\beta_7}{2} = 0 \tag{4.132}$$

where

$$h_6(\tau) = \frac{\alpha B\beta_7}{4\Omega} \tag{4.133}$$

Therefore

$$\alpha_{11} = e^{-\tau}[\int h_6(\tau)e^s ds + \alpha_{11}(0)] \tag{4.134}$$

The remaining equation (4.117) is:

$$U_{3m,\hat{t}\hat{t}}^{(31)} + \Omega^2 U_{2m}^{(31)} = r_{50}\sin\theta\hat{t} + r_{51}\sin 2\theta\hat{t} + r_{52}\sin 3\theta\hat{t} + r_{53}\cos(\varphi+\theta)\hat{t} + r_{54}\sin(\varphi+\theta)\hat{t} + r_{55}\cos(\varphi-\theta)\hat{t} + r_{56}\sin(\varphi-\theta)\hat{t} \tag{4.135}$$

The initial conditions are

$$U_{3m}^{(31)}(0,0) = 0;\ U_{3m,\hat{t}}^{(31)}(0,0) + U_{3m,\tau}^{(30)}(0,0) = 0$$

$$r_{50} = \frac{2\theta r_{22}^1}{\Omega^2-\theta^2} + \frac{2\theta r_{22}}{\Omega^2-\theta^2} - \frac{\alpha\alpha_1 r_6}{4(\varphi^2-4\theta^2)} - \frac{B\alpha r_5}{2(\varphi^2-4\theta^2)} - \frac{\alpha^2\beta_2}{4}\left(\frac{\frac{\alpha_1^2}{2}+B^2}{\varphi^2}\right) + \frac{\alpha^2\alpha_1\beta_2}{16(\varphi^2-4\theta^2)} - \frac{3\alpha\beta\beta_2}{8}\left(B^2+\frac{\alpha_1^2}{2}\right) + \frac{3\alpha\beta\alpha_1^2}{32}$$

$$r_{50}(0) = B^3\left(\frac{2\theta S_{17}}{(\Omega^2-\theta^2)} + \frac{2\theta S_{11}}{(\Omega^2-\theta^2)} + \frac{\alpha S_1}{4(\varphi^2-4\theta^2)} + \frac{\alpha^2}{2\theta(\varphi^2-4\theta^2)} + \frac{3\alpha^2}{8\theta\varphi^2} + \frac{\alpha^2}{16B(\varphi^2-4\theta^2)} + \frac{9\alpha\beta}{16\theta} + \frac{3\alpha\beta}{32B}\right)$$

$$r_{51} = \frac{4\theta r_{23}^1}{\Omega^2-4\theta^2} + \frac{4\theta r_{23}}{\Omega^2-4\theta^2} - \frac{\alpha\alpha_1 r_5}{4(\varphi^2-\theta^2)} - \frac{B\alpha r_6}{2(\varphi^2-4\theta^2)} - \frac{\alpha^2\alpha_1 B\beta_2}{4(\varphi^2-4\theta^2)} - \frac{\alpha\alpha_1\beta_2 B}{2}$$

$$r_{51}(0) = B^3\left(\frac{4\theta S_{18}}{(\Omega^2-4\theta^2)} + \frac{4\theta S_{12}}{(\Omega^2-4\theta^2)} + \frac{\alpha^2}{2\theta(\varphi^2-\theta^2)} - \frac{\alpha S_1}{2(\varphi^2-4\theta^2)} - \frac{\alpha}{2\theta}\right)$$

$$r_{52} = \frac{6\theta r_{24}^1}{\Omega^2-9\theta^2} + \frac{6\theta r_{24}}{\varphi^2-9\theta^2} - \frac{\alpha\alpha_1 r_6}{4(\varphi^2-4\theta^2)} - \frac{\alpha^2\beta_2\alpha_1^2}{16(\varphi^2-4\theta^2)} + \frac{\alpha\alpha_1^2\beta_2}{8}$$

$$r_{52}(0) = B^3\left(\frac{6\theta S_{19}}{(\Omega^2-9\theta^2)} + \frac{6\theta S_{13}}{(\Omega^2-9\theta^2)} + \frac{\alpha S_1}{4(\varphi^2-4\theta^2)} - \frac{\alpha^2}{16\theta(\varphi^2-4\theta^2)} + \frac{\alpha}{8\theta}\right)$$

$$r_{53} = -\frac{2r_{26}'(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} - \frac{2r_{26}(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} - \frac{\alpha\alpha_1\alpha_7}{4},\ r_{53}(0) = 0$$

$$r_{54} = \frac{2r_{25}'(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} + \frac{2r_{25}(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} - \frac{\alpha\alpha_1\beta_7}{4},\ r_{54}(0) = B^3\left(\frac{6\alpha S_0(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} + \frac{\alpha S_{43}}{4}\right)$$

$$r_{55} = \frac{-2r_{28}'(\varphi-\theta)}{\Omega^2-(\varphi-\theta)^2} - \frac{2r_{28}(\varphi-\theta)}{\Omega^2-(\varphi-\theta)^2} - \frac{\alpha\alpha_1\alpha_7}{4},\ r_{55}(0) = \frac{-4\alpha S_0 B^3}{\Omega^2-(\varphi-\theta)^2}$$

$$r_{56} = \frac{2r_{27}'(\varphi-\theta)}{\Omega^2-(\varphi-\theta)^2} + \frac{2r_{27}(\varphi-\theta)}{\Omega^2-(\varphi-\theta)^2} + \frac{\alpha\alpha_1\beta_7}{4},\ r_{56}(0) = B^3\left(\frac{6\alpha S_0(\varphi-\theta)}{\Omega^2-(\varphi-\theta)^2} + \frac{\alpha S_{43}}{4}\right)$$

Therefore;

$$U_{3m}^{(31)} = \alpha_{14}\cos\Omega\hat{t} + \beta_{14}\sin\Omega\hat{t} + \frac{r_{50}\sin\theta\hat{t}}{\Omega^2-\theta^2} + \frac{r_{51}\sin 2\theta\hat{t}}{\Omega^2-4\theta^2} + \frac{r_{52}\sin 3\theta\hat{t}}{\Omega^2-9\theta^2} + \left(\frac{r_{53}\cos(\varphi+\theta)\hat{t}+r_{54}\sin(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2}\right) + \left(\frac{r_{55}\cos(\varphi-\theta)\hat{t}+r_{56}\sin(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2}\right)$$

(4.136)

$$\alpha_{14}(0) = -\left[\frac{r_{53}}{\Omega^2-(\varphi+\theta)^2} + \frac{r_{55}}{\Omega^2-(\varphi-\theta)^2}\right]\Big|\tau = 0 \quad (4.137a)$$

$$\Omega\beta_{14}(0) = -\frac{\theta r_{50}}{\Omega^2-\theta^2} - \frac{2\theta r_{51}}{\Omega^2-4\theta^2} - \frac{3\theta r_{52}}{\Omega^2-9\theta^2} - \frac{(\varphi+\theta)r_{54}}{\Omega^2-(\varphi+\theta)^2} - \frac{(\varphi-\theta)r_{56}}{\Omega^2-(\varphi+\theta)^2} - \alpha_{11}' - \frac{r_{22}'}{\Omega^2-\theta^2} - \frac{r_{23}'}{\Omega^2-4\theta^2} - \frac{r_{24}'}{\Omega^2-9\theta^2} - \frac{r_{25}'}{\Omega^2-(\varphi+\theta)^2} - \frac{r_{27}'}{\Omega^2-(\varphi-\theta)^2}$$

Therefore;

$$\beta_{14}(0) = \frac{-1}{\Omega}\left[\frac{(\theta r_{50}+r_{22}')}{\varphi^2-\theta^2} + \frac{(2\theta r_{51}+r_{23}')}{\varphi^2-4\theta^2} + \frac{(3\theta r_{52}+r_{24}')}{\varphi^2-9\theta^2} - \frac{((\theta+\varphi)r_{54}+r_{24}')}{\Omega^2-(\varphi+\theta)^2} + \frac{(\theta-\varphi)r_{56}+r_{27}'}{\Omega^2-(\varphi-\theta)^2} + \alpha_{11}'(0)\right]$$

(4.138b)

So far, it follows that

$$U^{(31)} = U_m^{(31)}(1-\cos 2mx) + U_{2m}^{(31)}(1-\cos 4mx) + U_{3m}^{(31)}(1-\cos 6mx)$$

(4.139)

The summary of the solution so far is,

$$U(x,t,\tau) = \epsilon\left(U^{(10)} + \delta U^{(11)} + \delta^2 U^{(12)} + \cdots\right) + \epsilon^2\left(U^{(20)} + \delta U^{(21)} + \delta^2 U^{(22)} + \cdots\right) + \epsilon^3\left(U^{(30)} + \delta U31+\delta 2U32+\ldots+\ldots\right.$$

(4.140)

Conclusion

The perturbation and asymptotic techniques was used to analyze nonlinear partial differential equations, which in most cases are usually approached by numerical analysis through Finite Element methods. The perturbation and asymptotic techniques applied in this work made it possible to change ordinary differential equations to partial differential equations. The asymptotic method was possible by the presence of two small but independent parameters (δ *and* ϵ). Perturbation and asymptotic methods were adopted to solve elastic model structures and can also be applicable to other structural forms such as cylindrical shells, plates.

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