Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2021, 8(12):36-40**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Using Fractional Taylor Series Expansion to Solve Some Limit Problems

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

Abstract In this paper, some limit problems are studied by using the fractional Taylor series expansions of several fractional functions. A new multiplication of fractional analytic functions is introduced, which is a natural extension of the multiplication of traditional analytic functions.

Keywords Limit problems, Fractional Taylor series expansions, Fractional functions, New multiplication, Fractional analytic functions

1. Introduction

Fractional calculus is a branch of mathematical analysis which deals with the research and applications of integrals and derivatives of arbitrary order. In recent decades, the field of fractional calculus has attracted the interest of researchers in diverse scientific fields such as mathematics, physics, chemistry, engineering, and economics [1-12]. In this article, we use a new multiplication of fractional analytic functions and the method of fractional Taylor series expansion to study the following five limit problems:

$$\lim_{x\to 0}\frac{[sin_\alpha(x^\alpha)]^{\otimes 3}}{x^{3\alpha}}, \tag{1}$$

$$\lim_{x\to 0}\frac{x^{3\alpha}\otimes E_\alpha(x^\alpha)}{\frac{2}{\Gamma(\alpha+1)}x^\alpha - sin_\alpha(2x^\alpha)}, \tag{2}$$

$$\lim_{x\to 0}\frac{sin_\alpha(x^\alpha)\otimes[1-cos_\alpha(x^\alpha)]}{x^{3\alpha}\otimes cos_\alpha(x^\alpha)}, \tag{3}$$

$$\lim_{x\to 0}\frac{\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha}-[sin_\alpha(x^\alpha)]^{\otimes 2}}{1-\frac{3}{\Gamma(2\alpha+1)}x^{2\alpha}-[cos_\alpha(x^\alpha)]^{\otimes 3}}, \tag{4}$$

$$\lim_{x\to 0}\frac{[sin_\alpha(x^\alpha)]^{\otimes 2}}{\left[E_\alpha(x^\alpha)-1-\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\otimes cos_\alpha(x^\alpha)}, \tag{5}$$

where $0<\alpha\le 1, (-1)^\alpha$ exists, and $E_\alpha, cos_\alpha, sin_\alpha$ are α-fractional exponential function, cosine function, sine function respectively. Moreover, the new multiplication $\otimes$ of fractional analytic functions is a natural generalization of ordinary multiplication of analytic functions.

2. Preliminaries and Properties

Definition 2.1([13])**:** Suppose that x, x_0 and a_n are real numbers, $x_0 \in (a,b)$, and $0<\alpha\le 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as a α-fractional power series, i.e., $f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval (x_0-r, x_0+r), then $f_\alpha(x^\alpha)$ is called α-fractional analytic at x_0, where r is the radius of convergence about x_0. If $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and is α-fractional analytic at every point in open interval (a,b), then we say that f_α is an α-fractional analytic function on $[a,b]$.

In the following,we introduce a new multiplication of fractional analytic functions.

Definition 2.2([13])**:** Assume that $0<\alpha\le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}x^{n\alpha}, \tag{6}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} x^{n\alpha}. \tag{7}$$

Then we define

$$f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} x^{n\alpha} \otimes \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} x^{n\alpha}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) x^{n\alpha}. \tag{8}$$

Definition 2.3 ([14])**:** Suppose that $0 < \alpha \le 1$, and x is a real number. The α-fractional exponential function, sine function and cosine function are defined as follows:

$$E_\alpha(x^\alpha) = 1 + \frac{1}{\Gamma(\alpha+1)} x^\alpha + \frac{1}{\Gamma(2\alpha+1)} x^{2\alpha} + \frac{1}{\Gamma(3\alpha+1)} x^{3\alpha} + \cdots = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} x^{n\alpha}, \tag{9}$$

$$sin_\alpha(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha - \frac{1}{\Gamma(3\alpha+1)} x^{3\alpha} + \frac{1}{\Gamma(5\alpha+1)} x^{5\alpha} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma((2n+1)\alpha+1)} x^{(2n+1)\alpha}, \tag{10}$$

$$cos_\alpha(x^\alpha) = 1 - \frac{1}{\Gamma(2\alpha+1)} x^{2\alpha} + \frac{1}{\Gamma(4\alpha+1)} x^{4\alpha} - \frac{1}{\Gamma(6\alpha+1)} x^{6\alpha} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(2n\alpha+1)} x^{2n\alpha}. \tag{11}$$

Proposition 2.4 (fractional Euler's formula) ([14])**:** *Let* $0 < \alpha \le 1$, *then*

$$E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha). \tag{12}$$

Proposition 2.5 (fractional DeMoivre's formula) ([14])**:** *If* $0 < \alpha \le 1$, *and* n *is a positive integer, then*

$$[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^{\otimes n} = cos_\alpha(nx^\alpha) + isin_\alpha(nx^\alpha). \tag{13}$$

3. Main Results

To obtain the major results in this study, we need two lemmas.

Lemma 3.1: *Let* $0 < \alpha \le 1$ *and* x *be a real number, then*

$$[sin_\alpha(x^\alpha)]^{\otimes 2} = \frac{2}{\Gamma(2\alpha+1)} x^{2\alpha} - \frac{8}{\Gamma(4\alpha+1)} x^{4\alpha} + \frac{32}{\Gamma(6\alpha+1)} x^{6\alpha} - \cdots. \tag{14}$$

Proof Using fractional DeMoivre's formula yields

$$[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^{\otimes 2} = cos_\alpha(2x^\alpha) + isin_\alpha(2x^\alpha). \tag{15}$$

Hence,

$$cos_\alpha(2x^\alpha) = [cos_\alpha(x^\alpha)]^{\otimes 2} - [sin_\alpha(x^\alpha)]^{\otimes 2} = 1 - 2[sin_\alpha(x^\alpha)]^{\otimes 2}. \tag{16}$$

Thus,

$$[sin_\alpha(x^\alpha)]^{\otimes 2}$$

$$= \frac{1}{2} - \frac{1}{2} cos_\alpha(2x^\alpha)$$

$$= \frac{1}{2} - \frac{1}{2}\left[1 - \frac{4}{\Gamma(2\alpha+1)} x^{2\alpha} + \frac{16}{\Gamma(4\alpha+1)} x^{4\alpha} - \frac{64}{\Gamma(6\alpha+1)} x^{6\alpha} + \cdots\right]$$

$$= \frac{2}{\Gamma(2\alpha+1)} x^{2\alpha} - \frac{8}{\Gamma(4\alpha+1)} x^{4\alpha} + \frac{32}{\Gamma(6\alpha+1)} x^{6\alpha} - \cdots.$$ Q.e.d.

Lemma 3.2: *If* $0 < \alpha \le 1$ *and* x *is a real number, then*

$$1 - \frac{3}{\Gamma(2\alpha+1)} x^{2\alpha} - [cos_\alpha(x^\alpha)]^{\otimes 3} = -\frac{21}{\Gamma(4\alpha+1)} x^{4\alpha} + \frac{183}{\Gamma(6\alpha+1)} x^{6\alpha} + \cdots, \tag{17}$$

and

$$[sin_\alpha(x^\alpha)]^{\otimes 3} = \frac{6}{\Gamma(3\alpha+1)} x^{3\alpha} - \frac{60}{\Gamma(5\alpha+1)} x^{5\alpha} + \cdots. \tag{18}$$

Proof Also by fractional DeMoivre's formula, we have

$$[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^{\otimes 3} = cos_\alpha(3x^\alpha) + isin_\alpha(3x^\alpha). \tag{19}$$

It follows that

$$cos_\alpha(3x^\alpha) = [cos_\alpha(x^\alpha)]^{\otimes 3} - 3cos_\alpha(x^\alpha) \otimes [sin_\alpha(x^\alpha)]^{\otimes 2}$$

$$= [cos_\alpha(x^\alpha)]^{\otimes 3} - 3cos_\alpha(x^\alpha) \otimes [1 - [cos_\alpha(x^\alpha)]^{\otimes 2}]$$

$$= 4[cos_\alpha(x^\alpha)]^{\otimes 3} - 3cos_\alpha(x^\alpha).$$

Thus,

$$[cos_\alpha(x^\alpha)]^{\otimes 3} = \frac{1}{4}[cos_\alpha(3x^\alpha) + 3cos_\alpha(x^\alpha)]. \quad (20)$$

And hence,

$$1 - \frac{3}{\Gamma(2\alpha+1)}x^{2\alpha} - [cos_\alpha(x^\alpha)]^{\otimes 3}$$

$$= 1 - \frac{3}{\Gamma(2\alpha+1)}x^{2\alpha} - \frac{1}{4}[cos_\alpha(3x^\alpha) + 3cos_\alpha(x^\alpha)]$$

$$= 1 - \frac{3}{\Gamma(2\alpha+1)}x^{2\alpha} - \frac{1}{4}\left[1 - \frac{9}{\Gamma(2\alpha+1)}x^{2\alpha} + \frac{81}{\Gamma(4\alpha+1)}x^{4\alpha} - \frac{729}{\Gamma(6\alpha+1)}x^{6\alpha} + \cdots + 3\left[1 - \frac{1}{\Gamma(2\alpha+1)}x^{2\alpha} + \frac{1}{\Gamma(4\alpha+1)}x^{4\alpha} - \frac{1}{\Gamma(6\alpha+1)}x^{6\alpha} + \cdots\right]\right]$$

$$= 1 - \frac{3}{\Gamma(2\alpha+1)}x^{2\alpha} - \frac{1}{4}\left[4 - \frac{12}{\Gamma(2\alpha+1)}x^{2\alpha} + \frac{84}{\Gamma(4\alpha+1)}x^{4\alpha} - \frac{732}{\Gamma(6\alpha+1)}x^{6\alpha} + \cdots\right]$$

$$= -\frac{21}{\Gamma(4\alpha+1)}x^{4\alpha} + \frac{183}{\Gamma(6\alpha+1)}x^{6\alpha} + \cdots.$$

On the other hand, since

$$sin_\alpha(3x^\alpha) = 3[cos_\alpha(x^\alpha)]^{\otimes 2} \otimes sin_\alpha(x^\alpha) - [sin_\alpha(x^\alpha)]^{\otimes 3}$$

$$= 3[1 - [sin_\alpha(x^\alpha)]^{\otimes 2}] \otimes sin_\alpha(x^\alpha) - [sin_\alpha(x^\alpha)]^{\otimes 3}$$

$$= 3sin_\alpha(x^\alpha) - 4[sin_\alpha(x^\alpha)]^{\otimes 3}. \quad (21)$$

It follows that

$$[sin_\alpha(x^\alpha)]^{\otimes 3}$$

$$= \frac{1}{4}[3sin_\alpha(x^\alpha) - sin_\alpha(3x^\alpha)]$$

$$= \frac{1}{4}\left[\left[\frac{3}{\Gamma(\alpha+1)}x^\alpha - \frac{3}{\Gamma(3\alpha+1)}x^{3\alpha} + \frac{3}{\Gamma(5\alpha+1)}x^{5\alpha} - \cdots\right] - \left[\frac{3}{\Gamma(\alpha+1)}x^\alpha - \frac{27}{\Gamma(3\alpha+1)}x^{3\alpha} + \frac{243}{\Gamma(5\alpha+1)}x^{5\alpha} - \cdots\right]\right]$$

$$= \frac{6}{\Gamma(3\alpha+1)}x^{3\alpha} - \frac{60}{\Gamma(5\alpha+1)}x^{5\alpha} + \cdots.$$

Q.e.d.

We have the following main results in this paper.

Theorem 3.3: *Suppose that* $0 < \alpha \le 1$ *and* $(-1)^\alpha$ *exists, then*

$$\lim_{x\to 0}\frac{[sin_\alpha(x^\alpha)]^{\otimes 3}}{x^{3\alpha}} = \frac{6}{\Gamma(3\alpha+1)}, \quad (22)$$

$$\lim_{x\to 0}\frac{x^{3\alpha}\otimes E_\alpha(x^\alpha)}{\frac{2}{\Gamma(\alpha+1)}x^\alpha - sin_\alpha(2x^\alpha)} = \frac{\Gamma(3\alpha+1)}{8}, \quad (23)$$

$$\lim_{x\to 0}\frac{sin_\alpha(x^\alpha)\otimes[1-cos_\alpha(x^\alpha)]}{x^{3\alpha}\otimes cos_\alpha(x^\alpha)} = \frac{3}{\Gamma(3\alpha+1)}, \quad (24)$$

$$\lim_{x\to 0}\frac{\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha} - [sin_\alpha(x^\alpha)]^{\otimes 2}}{1 - \frac{3}{\Gamma(2\alpha+1)}x^{2\alpha} - [cos_\alpha(x^\alpha)]^{\otimes 3}} = -\frac{8}{21}, \quad (25)$$

$$\lim_{x\to 0}\frac{[sin_\alpha(x^\alpha)]^{\otimes 2}}{\left[E_\alpha(x^\alpha) - 1 - \frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\otimes cos_\alpha(x^\alpha)} = 2. \quad (26)$$

Proof $\lim_{x\to 0}\frac{[sin_\alpha(x^\alpha)]^{\otimes 3}}{x^{3\alpha}} = \lim_{x\to 0}\frac{\frac{6}{\Gamma(3\alpha+1)}x^{3\alpha} - \frac{60}{\Gamma(5\alpha+1)}x^{5\alpha} + \cdots}{x^{3\alpha}} = \frac{6}{\Gamma(3\alpha+1)}$. In addition, since

$$x^{3\alpha}\otimes E_\alpha(x^\alpha) = \Gamma(3\alpha+1)\left[\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}\otimes\left[1 + \frac{1}{\Gamma(\alpha+1)}x^\alpha + \frac{1}{\Gamma(2\alpha+1)}x^{2\alpha} + \frac{1}{\Gamma(3\alpha+1)}x^{3\alpha} + \cdots\right]\right]$$

$$= \Gamma(3\alpha+1)\left[\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha} + \frac{4}{\Gamma(4\alpha+1)}x^{4\alpha} + \frac{10}{\Gamma(5\alpha+1)}x^{5\alpha} + \cdots\right]. \quad (27)$$

And

$$\frac{2}{\Gamma(\alpha+1)}x^{\alpha}-sin_{\alpha}(2x^{\alpha})=\frac{2}{\Gamma(\alpha+1)}x^{\alpha}-\left[\frac{2}{\Gamma(\alpha+1)}x^{\alpha}-\frac{8}{\Gamma(3\alpha+1)}x^{3\alpha}+\frac{32}{\Gamma(5\alpha+1)}x^{5\alpha}-\cdots\right]$$

$$=\frac{8}{\Gamma(3\alpha+1)}x^{3\alpha}-\frac{32}{\Gamma(5\alpha+1)}x^{5\alpha}+\cdots. \quad (28)$$

It follows that

$$\lim_{x\to 0}\frac{x^{3\alpha}\otimes E_{\alpha}(x^{\alpha})}{\frac{2}{\Gamma(\alpha+1)}x^{\alpha}-sin_{\alpha}(2x^{\alpha})}=\lim_{x\to 0}\frac{\Gamma(3\alpha+1)\left[\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}+\frac{4}{\Gamma(4\alpha+1)}x^{4\alpha}+\frac{10}{\Gamma(5\alpha+1)}x^{5\alpha}+\cdots\right]}{\frac{8}{\Gamma(3\alpha+1)}x^{3\alpha}-\frac{32}{\Gamma(5\alpha+1)}x^{5\alpha}+\cdots}=\frac{\Gamma(3\alpha+1)}{8}.$$

On the other hand, since

$$sin_{\alpha}(x^{\alpha})\otimes[1-cos_{\alpha}(x^{\alpha})]$$
$$=sin_{\alpha}(x^{\alpha})-sin_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(x^{\alpha})$$
$$=sin_{\alpha}(x^{\alpha})-\frac{1}{2}sin_{\alpha}(2x^{\alpha})$$
$$=\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}+\frac{1}{\Gamma(5\alpha+1)}x^{5\alpha}-\cdots\right]$$
$$-\frac{1}{2}\left[\frac{2}{\Gamma(\alpha+1)}x^{\alpha}-\frac{8}{\Gamma(3\alpha+1)}x^{3\alpha}+\frac{32}{\Gamma(5\alpha+1)}x^{5\alpha}-\cdots\right]$$
$$=\frac{3}{\Gamma(3\alpha+1)}x^{3\alpha}-\frac{15}{\Gamma(5\alpha+1)}x^{5\alpha}+\cdots.$$

And

$$x^{3\alpha}\otimes cos_{\alpha}(x^{\alpha})$$
$$=\Gamma(3\alpha+1)\left[\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}\otimes\left[1-\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+\frac{1}{\Gamma(4\alpha+1)}x^{4\alpha}-\frac{1}{\Gamma(6\alpha+1)}x^{6\alpha}+\cdots\right]\right]$$
$$=\Gamma(3\alpha+1)\left[\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}-\frac{10}{\Gamma(5\alpha+1)}x^{5\alpha}+\frac{35}{\Gamma(7\alpha+1)}x^{7\alpha}-\cdots\right].$$

It follows that

$$\lim_{x\to 0}\frac{sin_{\alpha}(x^{\alpha})\otimes[1-cos_{\alpha}(x^{\alpha})]}{x^{3\alpha}\otimes cos_{\alpha}(x^{\alpha})}=\lim_{x\to 0}\frac{\frac{3}{\Gamma(3\alpha+1)}x^{3\alpha}-\frac{15}{\Gamma(5\alpha+1)}x^{5\alpha}+\cdots}{\Gamma(3\alpha+1)\left[\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}-\frac{10}{\Gamma(5\alpha+1)\,by}x^{5\alpha}+\frac{35}{\Gamma(7\alpha+1)}x^{7\alpha}-\cdots\right]}=\frac{3}{\Gamma(3\alpha+1)}.$$

Moreover, by Lemma 3.1 and 3.2, we obtain

$$\lim_{x\to 0}\frac{\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha}-[sin_{\alpha}(x^{\alpha})]^{\otimes 2}}{1-\frac{3}{\Gamma(2\alpha+1)}x^{2\alpha}-[cos_{\alpha}(x^{\alpha})]^{\otimes 3}}$$
$$=\lim_{x\to 0}\frac{\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha}-\left[\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha}-\frac{8}{\Gamma(4\alpha+1)}x^{4\alpha}+\frac{32}{\Gamma(6\alpha+1)}x^{6\alpha}-\cdots\right]}{-\frac{21}{\Gamma(4\alpha+1)}x^{4\alpha}+\frac{183}{\Gamma(6\alpha+1)}x^{6\alpha}+\cdots}$$
$$=\lim_{x\to 0}\frac{\frac{8}{\Gamma(4\alpha+1)}x^{4\alpha}-\frac{32}{\Gamma(6\alpha+1)}x^{6\alpha}+\cdots}{-\frac{21}{\Gamma(4\alpha+1)}x^{4\alpha}+\frac{183}{\Gamma(6\alpha+1)}x^{6\alpha}+\cdots}$$
$$=-\frac{8}{21}.$$

Finally, since

$$\left[E_{\alpha}(x^{\alpha})-1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]\otimes cos_{\alpha}(x^{\alpha})$$
$$=\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}+\cdots\right]\otimes\left[1-\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+\frac{1}{\Gamma(4\alpha+1)}x^{4\alpha}-\frac{1}{\Gamma(6\alpha+1)}x^{6\alpha}+\cdots\right]$$
$$=\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}-\frac{5}{\Gamma(4\alpha+1)}x^{4\alpha}-\cdots.$$

It follows that

$$\lim_{x\to 0}\frac{[sin_{\alpha}(x^{\alpha})]^{\otimes 2}}{\left[E_{\alpha}(x^{\alpha})-1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]\otimes cos_{\alpha}(x^{\alpha})}=\lim_{x\to 0}\frac{\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha}-\frac{8}{\Gamma(4\alpha+1)}x^{4\alpha}+\frac{32}{\Gamma(6\alpha+1)}x^{6\alpha}+\cdots}{\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}-\frac{5}{\Gamma(4\alpha+1)}x^{4\alpha}-\cdots}=2.$$

Q.e.d.

4. Conclusion

From the above discussion, we know that the fractional Taylor series expansion is an important method to evaluate the limit problems in this paper. Furthermore, the new multiplication of fractional analytic functions we used is a natural generalization of multiplication of classical analytic functions. In the future, we will make use of the new multiplication to study the engineering mathematics problems and fractional differential equations.

References

[1]. Mainardi, F. (1996). Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos, Solitons & Fractals*, *7*(9), 1461-1477.

[2]. Oldham, K.B. and Zoski, C. G. (1983). Analogue instrumentation for processing polorographic data, *Journal of Electroanalytical Chemistry and Interfacial Electrochemistry*, *157*(1), 27-51.

[3]. Wang, H. (2019). Research on application of fractional calculus in signal real-time analysis and processing in stock financial market, *Chaos, Solitons & Fractals*, *128*, 92-97.

[4]. Sun, H., Zhang, Y., Baleanu, D., Chen, W., & Chen, Y. (2018). A new collection of real world applications of fractional calculus in science and engineering, *Communications in Nonlinear Science and Numerical Simulation*, *64*, 213-231.

[5]. Meng, R., Yin, D., Zhou, C., & Wu, H. (2016). Fractional description of time-dependent mechanical property evolution in materials with strain softening behavior, *Applied Mathematical Modelling*, *40*(1), 398-406.

[6]. Sun, H., Zhang, Y., Wei, S., Zhu, J., & Chen, W. (2018). A space fractional constitutive equation model for non-Newtonian fluid flow, *Communications in Nonlinear Science and Numerical Simulation*, *62*, 409-417.

[7]. Hilfer, R. (2000). Applications of fractional calculus in physics, World Scientific.

[8]. Ortigueira, M. D. (2011). Fractional calculus for scientists and engineers, Springer.

[9]. West, B. J. (2016). Fractional calculus view of complexity: Tomorrow's science, CRC Press.

[10]. Yu, C. H. (2021). A new approach to study fractional integral problems, *International Journal of Mathematics and Physical Sciences Research*, *9*(1), 7-14.

[11]. Yu, C. H. (2021). Study on fractional Newton's law of cooling, *International Journal of Mechanical and Industrial Technology*, *9*(1), 1-6.

[12]. Yu, C. H. (2021). A new insight into fractional logistic equation, *International Journal of Engineering Research and Reviews*, *9*(2), 13-17.

[13]. Yu, C. H. (2021). Study of fractional analytic functions and local fractional calculus, *International Journal of Scientific Research in Science, Engineering and Technology*, *8*(5), 39-46.

[14]. Yu, C. H. (2021). Formulas involving some fractional trigonometric functions based on local fractional calculus, *Journal of Research in Applied Mathematics*, *7*(10), 59-67.