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The Performance Measure Analysis on the Irreducibility in Markov Chain States Classification

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Abstract The transitions in Markov chain are assumed to occur instantaneously and the future evolution of the system depends only on its current state and not on its past history, then the system may be represented by a Markov process. Even when the system does not possess this Markov property explicitly, it is often possible to construct a corresponding implicit representation. In this study, the irreducible Markov chain where all states are positive recurrent, null recurrent and transient are investigated, in order to provide an insight into the performance measures in irreducible aperiodic Markov chains, irreducible Ergodic Markov chains and irreducible periodic Markov chain. The matrix operations and laws are use with the help of some existing equations and formulas in Markov Chain. The Equations for performance measures are derived and demonstrated with the help of illustrative examples, and the following results are obtained for illustrative example 1, the mean number of sunny days per week is 0.48125, the average number of rainy days is 5.33750, the average number of transitions from one sunny day to the next sunny day is $1/.06875 = 14.55$, the average number of days between two sunny days is 13.55, the mean number of rainy days between two sunny days is 11.09 while the mean number of cloudy days between two sunny days is 2.45. Likewise, for other two illustrative examples.

Keywords Aperiodic, ergodic, irreducibility, null recurrent, positive recurrent, cyclic classes

Introduction

The irreducibility of a Markov chain can be enunciated in terms of the reachability of the states. State j is said to be reachable or accessible from state i if there exists a path from state i to state j. We write this as $i \rightarrow j$. A discrete-time Markov chain is irreducible if every state is reachable from every other state, i.e., if there exists an integer $n$for which $P_{ij}^{n} \geq 0$, for every pair of states $i$and j. Romanovsky [1] introduced the application and simulation of a discrete Markov Chains and this was extended to the introduction of Numerical Solutions of Markov Chains by Stewart [2, 3], while the suitability of the Markov chain approach is demonstrated in the wind feed in Germany by Pesch *et al.* [4]. Uzun and Kiral [5] carried out the study to predict the direction of the gold price movement, and to determine the probabilistic transition matrix of the closing returns of gold prices, using the Markov chain model of fuzzy state, while the application of Markov chain using a data mining approach to get a prediction of the monthly rainfall data is shown by Aziza *et al.* [6]. The application of Markov chain on the spread of disease infection which shown that Hepatitis B was more infectious overtime than tuberculosis and HIV is demonstrated by Clemence [7], while the application of Markov chain to Journalism is demonstrated by Vermeer and Trilling [8], but in this study, the performance measure analysis on the irreducibility in Markov chain states classification are analysed, for Markov chains with different classes of states, and these are demonstrated with illustrative examples.

Notation

C	the set of all states in a Markov chain
$C_{1,}\ C_{2,}\ C_3, \cdots$	the subsets of states that partition C
$f_{jj}^{(n)}$	Conditional Probability that on leaving state *j* the first return to state *j* occurs *n* steps later
M	mean recurrence time
p_{ij}	Probability of moving from state i to state *j*
$p_{jj}^{(n)}$	that the Markov chain is once again in state j, $n$time steps after leaving it
v_{ij}	the average time spent by the Markov chain in state $i$at steady state between two successive visits to state *j*
$v_j(T)$	the average time spent by the chain in state *j* in a fixed period of time τ at steady state
π	the stationary probability distribution
π_j	the proportion of time that the process spends in state*j*

Materials and Methods

In the study of classifications concerning groups of states. Let C be the set of all states in a Markov chain, and let $C_{1,}\ C_{2,}\ C_3, \cdots$ be subsets of states that partition C. The subset of states C_1 is said to be closed if no one-step transition is possible from any state in C_1 to any state in C_2. This is illustrated in Figure 1 where the subset consisting of states {1, 2, 3} is closed. The subset containing states 4 through 6 is not closed. Also, the set that contains all six states is closed. More generally, any nonempty subset C_1 of Cis said to be closed if no state in C_1leads to any state outside C_1 (in any number of steps), i.e.,

$$p_{ij}^{n} = 0 \quad \text{for } i \in C_1,\ j\ not \in C_1, \quad n \geq 1. \qquad (1)$$

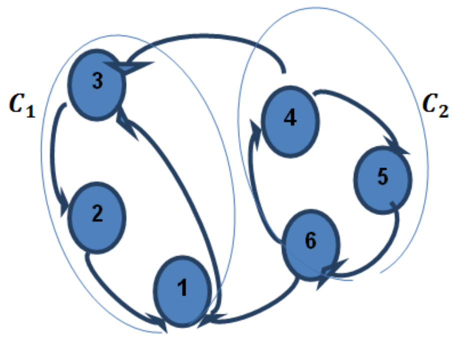

Figure 1: C_1 is a close subset of States C

If the closed subset C_1 consists of a single state, then that state is an absorbing state. While a set of states that is not closed is said to be open. It is apparent that any finite set of transient states must constitute an open set. Any individual state that is not an absorbing state constitutes by itself, an open set. If the set of all states $C$is closed and does not contain any proper subset that is closed, then the Markov chain is said to be irreducible. On the other hand, if C contains proper subsets that are closed, the chain is said to be reducible. A closed subset of states is said to be an irreducible subset if it contains no proper subset that is closed. In Figure 1, the subset consisting of states 1, 2, and 3 is the unique irreducible subset of this Markov chain. In this example, no other subset of states constitutes an irreducible subset. Any proper subset of an irreducible subset constitutes a set of states that is open. The matrix of transition probabilities of the Markov chain in Figure 1 as shown in Figure 2 has the following nonzero structure.

$$\boldsymbol{P} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & * & \mathbf{0} & * & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & * \\ * & \mathbf{0} & \mathbf{0} & * & \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{D_{11}} & \boldsymbol{U_{12}} \\ \boldsymbol{L_{21}} & \boldsymbol{D_{22}} \end{pmatrix}$$

Figure 2: Transition Probabilities Matrix for Group of States Classification I

In this matrix, the symbol $*$represents a nonzero probability corresponding to a transition of the Markov chain. The matrix has been decomposed according to the partition $\{1, 2, 3\}, \{4, 5, 6\}$ into two diagonal blocks D_{11} and D_{22}, and two off-diagonal blocks, U_{12} and L_{21}, all of size 3×3. Observe that the upper off-diagonal block U_{12} is identically equal to zero. This means that notransition is possible from any state represented by diagonal block D_{11} to any state representedby diagonal block D_{12}. On the other hand, the lower off-diagonal block L_{21} does contain nonzero elements signifying that transitions do occur from the states of D_{22} to the states of D_{11}. If state $j$is reachable from state i ($i \to j$) and state i is reachable from state j ($j \to i$) then states i and j are said to be communicating states and we write $i \leftrightarrow j$. By its very nature, this communication property is symmetric, transitive, and reflexive and thus constitutes an equivalence relationship. We have, for any states $i, j,$ and k,

$$i \leftrightarrow j \Longrightarrow j \leftrightarrow i,$$
$$i \leftrightarrow j \; and \; j \leftrightarrow k \Longrightarrow i \leftrightarrow k,$$
$$i \leftrightarrow j \; and \; j \leftrightarrow i \Longrightarrow i \leftrightarrow i$$

To prove the above communication properties: for $i \leftrightarrow j$ implies that $i \to j$and thus there exists an$n_1 > 0$ for which $P_{ij}^{n_1} > 0$. Likewise for$j \leftrightarrow k$ there exists an $n_2 > 0$for which $P_{ij}^{n_2} > 0$.

Set$n = n_1 + n_2$

Then, from the Chapman-Kolmogorov equation, we have

$$P_{ik}^{n} = \sum_{all\, l} P_{il}^{(n_1)} P_{ik}^{(n_2)} \geq P_{ij}^{(n_1)} P_{jk}^{(n_2)} \tag{2}$$

Let the set of all states that communicate with state $i$forms a class and is denoted by $C(i)$.

If state i is recurrent, there exists an integer $n > 0$ such that

$$P_{jj}^{(n_2+n+n_1)} = P_{ji}^{n_2} P_{ii}^{n} P_{ij}^{n_1} \tag{3}$$

Since state $i$is recurrent, $\sum_{n=1}^{\infty} P_{ii}^{n} = \infty$, likewise for recurrent state j, $\sum_{n=1}^{\infty} P_{jj}^{n} = \infty$.

Since

$$\sum_{n=1}^{\infty} P_{ji}^{n_2} P_{ii}^{n} P_{ij}^{n_1} = P_{ji}^{n_2} P_{ij}^{n_1} \sum_{n=1}^{\infty} P_{ii}^{n} = \infty \tag{4}$$

Thus recurrent states can only reach other recurrent states: no transient state can be reached from are current state and the set of recurrent states must be closed. If state i is a recurrent state, then $C(i)$is an irreducible closed set and contains only recurrent states and all these states must be positive recurrent or they all must be null recurrent.

Results

Irreducible Markov Chains that are Null Recurrent or Transient

Consider the infinite (denumerable) Markov chain with probability transition matrix in random walk as given in Figure 3 below

$$P = \begin{pmatrix} 0 & 1 & 1 & \cdots & & \\ q & 0 & p & 0 & \cdots & \\ 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & q & 0 & p & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Figure 3: State transition probability matrix for a random walk on the integers

where p is a positive probability and $q = 1 - p$. Observe that every time the Markov chain reachesstate 0, it must leave it again at the next time step. State 0 is said to constitute a reflecting barrier. Since every state can reach every other state, the Markov chain is irreducible and hence all the states are positive recurrent or all the states are null recurrent or all the states are transient. When $p = q$, we obtained an irreducible, null-recurrent Markov chain and when $p > q$, we obtained an irreducible, transient Markov chain. Recall that $p$is the probability of moving from any state $i > 0$ to state $i + 1$, $q$is the probability of movingfrom any state $i > 0$ to $i - 1$. In such Markov chains, there is no stationary probability vector.The only solution to the system of equations$z = zP$ is the vector whose components are all equal to zero. Furthermore, if a limiting distribution exists, its components must all be equal to zero.

Irreducible Markov Chains that are Positive Recurrent

An example of an irreducible, positive-recurrent Markov chain is the same random walk problem but this time with $p < q$. In an irreducible, positive-recurrent Markov chain, the system of equations $z = zP$ has a unique and strictly positive solution. This solution is the stationary probabilitydistribution π, and its elements are given by

$$\pi_j = \frac{1}{M_{jj}} \qquad (5)$$

where M_{jj} is the mean recurrence time of state j (which, for a positive-recurrent state, is finite).

Equation (5) is readily verified by multiplying both sides of known matrix Equation of mean recurrence time

$$M = E + P(M - diag\{M\}) \qquad (6)$$

by the vector πandobserving that$\pi = \pi P$. We get

$$\pi M = \pi E + \pi P(M - diag\{M\}) = e^T + \pi(M - diag\{M\}) = e^T + \pi M - \pi diag\{M\} \qquad (7)$$

and thus

$$\pi diag\{M\} = e^T. \qquad (8)$$

Conversely, the states of an irreducible Markov chain which has a unique stationary probability vector, are positive recurrent. An irreducible, positive-recurrent Markov chain does not necessarily have a limiting probability distribution. This is the case when the Markov chain is periodic as the following example shows.

Illustrative Example 1: Consider the four-state irreducible, positive-recurrent Markov chain whose transition probability matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It may readily be verified that the vector $(0.25, 0.25, 0.25, 0.25)$ is the unique stationary distribution ofthis Markov chain, but that no matter which starting state is chosen, there is no limiting distribution.This Markov chain is periodic with period 4 which means that if it is in state 1 at time step $n$it willmove to state 2 at time step $(n + 1)$, to state 3 at time step $(n + 2)$, to state 4 at time step $(n + 3)$, andback to state 1 at time step $(n + 4)$. It will alternate forever in this fashion and will never settle into alimiting distribution. As illustrated bellow

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \cdots$$

i.e.,

$$P, P^2, \quad P^3, \quad P^4 = PP^5 = P^2, \quad P^6 = P^3, \quad P^7 = P, \cdots$$

Then $\lim_{n\to\infty} P^n$ does not exist and P does not have a limiting distribution. In this case, successive powers of P alternate. Therefore, the existence of a unique stationary distribution of a Markov chain does not necessarily mean that the Markov chain has a limiting distribution.

Irreducible, Aperiodic Markov Chains

An example of an irreducible and aperiodic Markov chain is the semi-infinite random walk problem with a Bernoulli barrier. From state 0, rather than moving to state 1 with probability 1, the Markov chain remains in state 0 with probability $q$or moves to state 1 with probability p. Thisintroduces a self-loop on state 0 and destroys the periodicity property of the original chain. The three previous characteristics of the states, transient, null recurrent, and positive recurrent, remain in effect according to whether $p > q$, $p = q$, or $p < q$, respectively. In an irreducible and aperiodic Markov chain, the limiting distribution always exists and is independent of the initial probability distribution. Moreover, exactly one of the following conditions must hold:

1. All states are transient or all states are null recurrent, in which case$\pi_j = 0$ for all j, and there exists no stationary distribution (even though the limiting distribution exists). The state space in this case must be infinite.
2. All states are positive recurrent (which, together with the aperiodicity property, makes themergodic), in which case $\pi_j > 0$, for all j, and the probabilities $\pi_j$constitute a stationarydistribution. The π_j are uniquely determined by means of

$$\pi_j = \sum_{all\ j} \pi_i P_{ij} \ \text{ and } \ \sum_j \pi_j = 1. \qquad (9)$$

In matrix terminology, this is written as

$\pi = \pi P$, and $\pi e = 1$.

When the Markov chain is irreducible and contains only a finite number of states, then these states are all positive recurrent and there exists a unique stationary distribution. If the Markov chain is also aperiodic, the aperiodicity property allows us to assert that this stationary distribution is also the unique steady-state distribution. The states of an irreducible, finite, and aperiodic Markov chain are ergodic as is the Markov chain itself.

Irreducible, Ergodic Markov Chains

In an ergodic discrete-time Markov chain all the states are positive recurrent and aperiodic. This, together with the irreducibility property, implies that in such a Markov chain the probability distribution $\pi(n)$, as a function of n, always converges to a limiting distribution π,which is independent of the initial state distribution. This limiting (steady-state) distribution is alsothe unique stationary distribution of the Markov chain. It follows from Equation of the probability that the Markov chain is in state $i$at step n given by

$$\pi_i(n) = \sum_{all\ k} P_{ki}^n \pi_k(0) \quad (10)$$

Which in matrix form

$$\pi(n) = \pi(0)P^n \quad (11)$$

where $\pi(0)$ denotes the initial state distribution and $P^{(n)} = P^n$ since we assume the chain to behomogeneous. The probability distribution $\pi(n)$is called a transient distribution, since it gives the probability of being in the various states of the Markov chain at a particular instant in time, i.e., at step n. As the Markov chain evolves onto step $(n+1)$, the distribution at time step n is discarded,hence it is only transient.Suchthat

$$\pi_j(n+1) = \sum_{all\ i} P_{ij}^n \pi_i(n) \quad (12)$$

and taking the limit as $n\rightarrow\infty$of both sides gives

$$\pi_j = \sum_{all\ i} P_{ij}^n \pi_i \quad (13)$$

Thus, the equilibrium probabilities may be uniquely obtained by solving the matrix equation

$\pi = \pi P$ with $\pi \geq 0$ and $\|\pi\|_1 = 1$.

It may be shown that, as $n \rightarrow \infty$, the rows of the n-step transition matrix P^nall becomeidentical to the vector of stationary probabilities. Letting P_{ij}^ndenote the ij^{th}element of P^n, we have

$\pi_j = \lim_{n\rightarrow\infty} P_{ij}^n$ for all i and j,

i.e., the stationary distribution is replicated on each row of P^nin the limit as $n\rightarrow\infty$. This property may be observed in the example of the Markov chain given below:

Illustrative Example 2: Consider a homogeneous, discrete-time Markov chain that describes the daily weather pattern in Calabar South South Nigeria (well known for its prolonged periods of rainy days). We simplify the situation by considering only three types of weather pattern: rainy, cloudy, and sunny. These three weather conditions describe the three states of our Markov chain: state 1 (R) represents a (mostly) rainy day; state 2 (C), a (mostly) cloudy day; and state 3 (S), a (mostly) sunny day. The weather is observed daily. On any given rainy day, the probability that it will rain the next day is estimated at 0.8; the probability that the next day will be cloudy is 0.15, while the probability that tomorrow will be sunny is only 0.05. Similarly, probabilities may be assigned when a particular day is cloudy or sunny as shown in Tables 1 -3 and Figure 1 below.

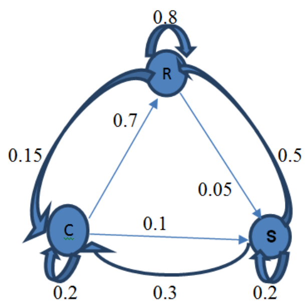

Figure 3: Transition diagram for weather pattern at Calabar, South South, Nigeria

$$P = \begin{pmatrix} 0.8 & 0.15 & 0.05 \\ 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \end{pmatrix},$$

Figure 4: Transition probability for weather pattern at Calabar, Nigeria

$$P^2 = \begin{pmatrix} 0.8 & 0.15 & 0.05 \\ 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \end{pmatrix} \times \begin{pmatrix} 0.8 & 0.15 & 0.05 \\ 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \end{pmatrix} = \begin{pmatrix} 0.770 & 0.165 & 0.065 \\ 0.750 & 0.175 & 0.075 \\ 0.710 & 0.195 & 0.095 \end{pmatrix}$$

$$\vdots$$

$$\lim_{n\to\infty} P^n = \begin{pmatrix} .76250 & 0.16875 & 0.06875 \\ .76250 & 0.16875 & 0.06875 \\ .76250 & 0.16875 & 0.06875 \end{pmatrix}$$

The matrix consists of rows that are all identical and equal to the$= (0.76250 \quad 0.16875 \quad 0.06875)$

The following are some performance measurements often deduced from the steady-state probability vector of irreducible, ergodic Markov chains.

• $v_j(T)$, the average time spent by the chain in state j in a fixed period of time τ at steady state, is equal to the product of the steady-state probability of state *j* and the duration of the observation period:

$$v_j(T) = \pi_j(T).$$

The steady-state probability $\pi_j$itself may be interpreted as the proportion of time that theprocess spends in state$j$, averaged over the long run. Returning to the weather example, themean number of sunny days per week is only 0.48125, while the average number of rainydays is 5.33750.

• $1/\pi_j$is the average number of steps between successive visits to state j. For example, the average number of transitions from one sunny day to the next sunny day in the weather example is $1/.06875 = 14.55$. Hence the average number of days between two sunny days is13.55.

• $v_{ij}$is the average time spent by the Markov chain in state $i$at steady state between two successive visits to state$j$.It is equal to the ratio of the steady-state probabilities of states$i$and j:

$$v_{ij} = \frac{\pi_i}{\pi_j} \qquad (14)$$

The quantity $v_{ij}$is called the visit ratio, since it indicates the average number of visits to state$i$between two successive visits to state j. In our example, the mean number of rainy daysbetween two sunny days is 11.09 while the mean number of cloudy days between two sunny days is 2.45.

Irreducible, Periodic Markov Chains

We now investigate the effects that periodicity introduces when we seek limiting distributions and higher powers of the single-step transition matrix. In an irreducible discrete-time Markov chain, when the number of single-step transitions required on leaving any state to return to that same state(by any path) is a multiple of some integer$p > 1$, the Markov chain is said to be periodic of period$p$, or cyclic of index p. One of the fundamental properties of such a Markov chain, is that it ispossible by a permutation of its rows and columns to transform it to the form, called the normal form,

$$P = \begin{pmatrix} 0 & P_{12} & 0 & 0 & 0 \\ 0 & 0 & P_{23} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & P_{P-1,P} \\ P_{P1} & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (15)$$

in which the diagonal submatrices $P_{ii}$are square and the only nonzero submatrices areP_{12}, P_{23}, ..., P_{P1}. This corresponds to a partitioning of the states of the system into $p$distinctsubsets and an ordering imposed on the subsets. These subsets are referred to as the cyclic classes of the Markov chain. The imposed ordering is such that once the system is in a state of subset i itmust exit this subset in the next time step and enter a state of subset

$$(i \bmod p) + 1 \qquad (16)$$

Illustrative Example 3: Consider the Markov chain whose transition diagram is given in Figure below, in which the states have been ordered according to their cyclic classes.

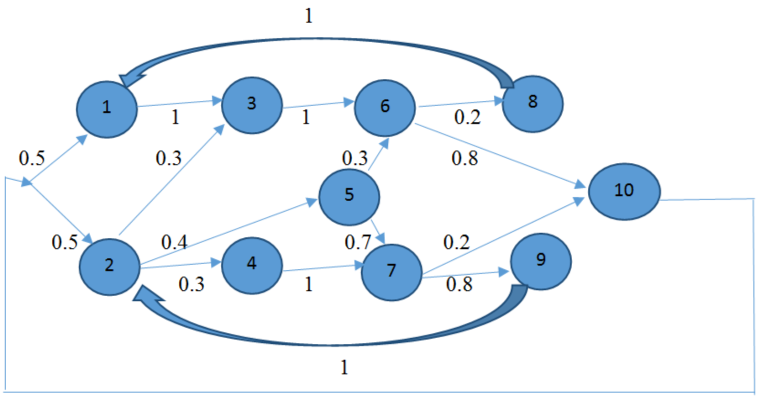

Figure 5: Cyclic Markov Chain

Its transition matrix is given by

$$P = \begin{pmatrix} & & 1 & & & & & & & \\ & & .3 & .3 & .4 & & & & & \\ & & & & & 1 & & & & \\ & & & & & & 1 & & & \\ & & & & & .3 & .7 & & & \\ & & & & & & & .2 & & .8 \\ & & & & & & & & .8 & .2 \\ 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ .5 & .5 & & & & & & & & \end{pmatrix}$$

Transition Probability of Cyclic Markov Chain

Evidently, this Markov chain has periodicity $p = 4$. There are four cyclic classes C_1 through C_4 given by

$$C_1 = \{1, 2\}, \quad C_2 = \{3, 4, 5\}, \quad C_3 = \{6, 7\}, and \quad C_4 = \{8, 9, 10\}.$$

On leaving any state of class $C_i, i = 1, 2, 3, 4$, the Markov chain can only go to states of class $C(i \, mod \, p) + 1$ in the next time step and therefore it can only return to a starting state after four, or somemultiple of four, steps.Our interest in periodic Markov chains such as this is in determining its behavior at time step n in the limit as $n \to \infty$. Specifically, we wish to investigate the behavior of P_n as $n \to \infty$, the existence or nonexistence of a limiting probability distribution and the existence or nonexistence ofa stationary distribution. We begin by examining the case when the Markov chain possesses four cyclic classes, i.e., the case of Equation (15) with $p = 4$. We have

$$P = \begin{pmatrix} 0 & P_{12} & 0 & 0 \\ 0 & 0 & P_{23} & 0 \\ 0 & 0 & 0 & P_{34} \\ P_{41} & 0 & 0 & P_{44} \end{pmatrix} \tag{16}$$

and taking successive powers, we obtain

$$P^2 = \begin{pmatrix} 0 & 0 & P_{12}P_{23} & 0 \\ 0 & 0 & 0 & P_{23}P_{34} \\ P_{34}P_{41} & 0 & 0 & 0 \\ 0 & P_{41}P_{12} & 0 & 0 \end{pmatrix} \tag{17}$$

$$P^3 = \begin{pmatrix} 0 & 0 & 0 & P_{12}P_{23}P_{34} \\ P_{23}P_{34}P_{41} & 0 & 0 & 0 \\ 0 & P_{34}P_{41}P_{12} & 0 & 0 \\ 0 & 0 & P_{41}P_{12}P_{23} & 0 \end{pmatrix} \tag{18}$$

$$P^4 = \begin{pmatrix} P_{12}P_{23}P_{34}P_{41} & 0 & 0 & 0 \\ 0 & P_{23}P_{34}P_{41}P_{12} & 0 & 0 \\ 0 & 0 & P_{34}P_{41}P_{12}P_{23} & 0 \\ 0 & 0 & 0 & P_{41}P_{12}P_{23}P_{34} \end{pmatrix}. \tag{19}$$

Illustrative Example 4: Consider the Markov chain of Example 3. We find that P^4=

$$P = \begin{pmatrix} 0.60 & 0.40 & & & & & & & & \\ 0.31 & 0.69 & & & & & & & & \\ & & 0.72 & 0.12 & 0.16 & & & & & \\ & & 0.37 & 0.27 & 0.36 & & & & & \\ & & 0.475 & 0.225 & 0.30 & & & & & \\ & & & & & 0.768 & 0.232 & & & \\ & & & & & 0.478 & 0.522 & & & \\ & & & & & & & 0.20 & 0.00 & 0.80 \\ & & & & & & & 0.084 & 0.464 & 0.452 \\ & & & & & & & 0,142 & 0.232 & 0.626 \end{pmatrix}$$

$$\lim_{n\to\infty} P^4 =$$

$$\begin{pmatrix} 0.4366 & 0.5634 & & & & & & & & \\ 0.4366 & 0.5634 & & & & & & & & \\ & & 0.6056 & 0.1690 & 0.2254 & & & & & \\ & & 0.6056 & 0.1690 & 0.2254 & & & & & \\ & & 0.6056 & 0.1690 & 0.2254 & & & & & \\ & & & & & 0.6732 & 0.3268 & & & \\ & & & & & 0.6732 & 0.3268 & & & \\ & & & & & & & 0.1347 & 0.2614 & 0.6039 \\ & & & & & & & 0.1347 & 0.2614 & 0.6039 \\ & & & & & & & 0.1347 & 0.2614 & 0.6039 \end{pmatrix}$$

Discussion

In illustrative Example 2**,** The matrix consists of rows that are all identical and equal to the $= (0.76250 \quad 0.16875 \quad 0.06875)$, and the mean number of sunny days per week is 0.48125, while the average number of rainydays is 5.33750. Also, the average number of transitions from one sunny day to the next sunny day is $1/.06875 = 14.55$. Hence the average number of days between two sunny days is 13.55 and the mean number of rainy days between two sunny days is 11.09 while the mean number of cloudy days between two sunny days is 2.45. In Illustrative Example 3, these successive powers show that, beginning in one of the states of C_1, a state of C_2 may bereached in one step, at least two steps are needed to reach a state of C_3, and a minimum of threesteps to reach a state of C_4. After four steps, the Markov chain is back in a state of C_1. Thus, afterfour steps, the transition matrix is block diagonal, and each block is a stochastic matrix. It followsthen that at any time step $4n, n = 1, 2, \ldots$, the transition matrix has this block structure and eachblock represents the transition probability matrix of an irreducible, recurrent and aperiodic Markovchain; aperiodic since a single-step in the new chain corresponds to four steps in the original chain. This means that we may now apply the previously discussed theory on irreducible, recurrent and aperiodic Markov chains to each of the blocks. In particular, each has a limit as $n \to \infty$. It follows that $\lim_{n\to\infty} P^{4n}$ exists. In illustrative Example 4, each diagonal block is treated as the transition probability matrix of a finite, aperiodic, irreducible Markov chain whose limiting distribution is then equal to its stationary distribution. This stationary distribution may be computed for each of the four blocks separately and, in the limit as $n \to \infty$, each block row must become equal to this distribution. Now observe that concatenating the fourstationary distributions together yields a vector $z$for which $z = zP$. We have

$z = (0.4366, 0.5634, 0.6056, 0.1690, 0.2254, 0.6732, 0.3268, 0.1367, 0.2614, 0.6039)$,

which when normalized becomes the stationary distribution of the original Markov chain:

$$(0.1092, 0.1408, 0.1514, 0.0423, 0.0563, 0.1683, 0.0817, 0.0337, 0.0654, 0.1510).$$

Notice that this assumes (correctly) that the Markov chain spends equal amounts of time in all periodic classes. Since an irreducible, positive-recurrent Markov chain has a unique and strictly positive stationary distribution, it follows that this computed solution is the only possible solution.

Conclusion

The irreducible Markov chain where all states are positive recurrent, null recurrent and transient are investigated, in order to provide an insight into the performance measures in irreducible, aperiodic Markov chains, irreducible, Ergodic Markov chains and irreducible, periodic Markov chain. The matrix operations and laws are use with the help of some existing equations and formulas in Markov Chain. The Equations for performance measures are derived and demonstrated with the help of illustrative examples.

References

[1]. Romanovsky, V.I. (1970). *Discrete Markov Chains*, Wolters-Noord off, Groningen, Netherlands. pp. 23 – 44.

[2]. Stewart, W. J. (1994). *Introduction to the Numerical Solution of Markov Chains*. Princeton UniversityPress, Princeton, N.J. pp. 14 – 38.

[3]. Stewart, W. J. (2009). *Probability, Markov Chain, Queues and Simulation,* Princeton University Press, United Kingdom. pp. 1- 42.

[4]. Pesch, T., Schroder, S., Allelein, H. and Hake, J. (2015). A new Markov chain related statistical approach for modelling synthetic wind power time series, *New Journal of Physics, Dentsche Physikalishe,* 35(2): 64 – 85.

[5]. Uzun, B. and Kiral, E. (2017). Application of Markov chain-fuzzy states to gold price, *Science Direct. Elsevier,* 120 (1): 365 – 371.

[6]. Azizah, A., Welastica, R., Nur, F., Ruchjana, B. and Abdullah, A.(2019). An application of Markov chain for predicting rainfall data at West Java using data mining approach, *Earth and Environmental Science*, 303(1): 203 – 216.

[7]. Clemence. T. (2019). Markov chain modelling of HIV, Tuberculosis, and Hepatitis B transmission in Ghana, *Hindawi, Interdisciplinary Perspective on Infectious Disease*, 27(1): 204 – 214.

[8]. Vermeer, S. And Trilling, D. (2020): Toward a better understanding of a new user journeys: A Markov chain approach. *Journalism Journal*, 21(1):879 – 894.