Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2021, 8(11):89-96**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Fixed Point Property for Lebesgue-like Two Spaces, a Corresponding Function Space of a Köthe-Toeplitz Dual of a Cesàro Difference Sequence Space and a Degenerate Lorentz Space

Veysel Nezir

Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey
e-mail: veyselnezir@yahoo.com

Abstract In 1970, Cesàro Sequence Spaces was introduced by Shiue. In 1981, Kızmaz defined difference sequence spaces for ℓ^∞, c_0 and c. Then, in 1983, Orhan introduced Cesàro Difference Sequence Spaces. In this study, first we discuss the fixed point property for these spaces. Then, we recall some important fixed point theory oriented results on Lebesgue space $L_1[0,1]$. Firstly, we discuss Alspach's study showing the first example of a Banach space failing weak fixed point property and next we talk about earlier works of Dowling, Lennard and Turett from 2003 and 2007 where they show the existance of fixed point free contractions on some weak compact subsets of $L_1[0,1]$. Furthermore, we recall such many examples in the literature for $L_1[0,1]$. So we wonder analogs of these studies on the corresponding function space for a Köthe-Toeplitz Dual of a Cesàro Difference Sequence Space after talking about a recent study of Nezir and Mustafa where they show that the corresponding function space fails the weak fixed point property and there exist many examples of fixed point free contractive mappings on some weakly compact subsets of these spaces. Then, we consider extending their study with some examples. Next we consider another Lebesgue-like Banach space contained in $L_1[0,1]$. We show that the Banach space we consider fails the weak fixed point property for nonexpansive mappings. In fact we find a weakly compact convex subset and fixed point free contractive mappings on it.

Keywords Fixed point property, Asymptotically noexpansive mapping, Cesàro Difference Sequences, Köthe-Toeplitz Dual
2010 Mathematics Subject Classification: 46B45, 47H09, 46B10

1. Introduction and preliminaries

We say that a Banach space $(X, \|.\|)$ has the fixed point property for non-expansive mappings if every non-expansive self mappings defined on any non-empty closed, bounded and convex subset of the Banach space has a fixed point. Here we note that if C is a subset of the Banach space, then $\mathrm{T}: \mathrm{C} \to \mathrm{C}$ is said to be a nonexpansive mapping if $\| \mathrm{T}(\mathrm{x}) - \mathrm{T}(\mathrm{y}) \| \leq \| \mathrm{x} - \mathrm{y} \|, \text{for all } \mathrm{x}, \mathrm{y} \in \mathrm{C}$. Moreover, we say that a Banach space $(X, \|.\|)$ has the weak fixed point property for non-expansive mappings if every non-expansive self mappings defined on any non-empty weakly compact and convex subset of X has a fixed point. Researches showed that most classical non-reflexive Banach spaces fail the fixed point property while they were satisfying the weak fixed point property. For a long time, it was thought that Banach spaces could have the weak fixed point property for non-expansive mappings; however, Alspach [1] showed that $L_1[0,1]$, Banach space of Lebesgue integrable functions defined on $[0,1]$, fails the fixed point property for non-expansive mappings. He provided the first example of a fixed point free map on a weakly compact, convex set. It is clear that if a Banach space fails to have the weak fixed point property then it fails to have the fixed point property. Hence, Alspach's result immediately implied the failure of the fixed point property for non-expansive mappings in $L_1[0,1]$.

Alspach usedBaker's transformation for his example in $L_1[0,1]$. His construction is given as in the following. Define$C:=\{f\in L_1[0,1]\ :\ 0\leq f\leq 1,\ t\in[0,1]\}$ and consider the nonexpansive mapping

$$Tf(t)=\begin{cases}\min\{2f(2t),1\} & \text{if } 0\leq t<\frac{1}{2}\\ \max\{2f(2t-1)-1,0\} & \text{if } \frac{1}{2}\leq t<1.\end{cases}\qquad (1.1)$$

Then, set $C_{\frac{1}{2}}:=\{f\in C\ :\ \int_0^1 f dm\ =\frac{1}{2}\}$ where m is Lebesque measure. So he shows that T is a fixed point free isometry on $C_{\frac{1}{2}}$.

Later, different examples like his by some researchers have been given such as [8, 9] by Dowling, Lennard and Turett, [18] by Llorens-Fuster and Sims, and [24] by Sine.

For example, using the similar set to Alspach's,Sine constructed a composite isometry given by $Sf=\chi_{[0,1]}-Tf,\ \forall f\in C_{\frac{1}{2}}$.

On the other hand, Dowling, Lennard and Turett's study [8] showed that there exists a fixed point free isometry on $C_{\frac{1}{2}}$ as in the following:

Let

$$\Delta f(t):=\begin{cases}f(2t) & \text{if } 0\leq t<\frac{1}{2}\\ 1-f(2t-1) & \text{if } \frac{1}{2}\leq t<1.\end{cases}\qquad (1.2)$$

Then, they showed that $T\Delta: C_{\frac{1}{2}}\to C_{\frac{1}{2}}$ is an isometry such that it does not have any fixed point.

So all the examples we mentioned above are isometries. As we recall from the famous Banach Contraction Principle that contractions defined on complete metric spaces or closed subsets of those have fixed points. Then, researchers wondered if there exist fixed point free contractions defined on weaklycompact sets. Note that if a map $f:D\to D$satisfies the condition $\|fx-fy\|<\|x-y\|$ for all $x,y\in D$, then we call it a contraction.

The first example of a fixed point free such map on a weakly compact convex set was introduced in [2]. Later, Casini, Miglierina, and Piasecki in [4] provided the second example. There are more examples found in Sivek's Ph.D. thesis [25] written under supervision of Chris Lennard.

In this study, firstly we provide and introduction to a recent study of Nezir and Mustafa [20] where they show that the corresponding function space fails the weak fixed point property and there exist many examples of fixed point free contractive mappings on some weakly compact subsets of these spaces. Then, we give more examples.

Next we consider another Lebesgue-like Banach space contained in $L_1[0,1]$. We show that it does not have weak fixed point property and in fact there exist a weakly compact subset and invariant fixed point free contractive mappings defined on it.

Now, we recall that the Cesàro sequence spaces

$$\text{ces}_p=\left\{x=(x_n)_n\subset\mathbb{R}\,\middle|\,\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{1/p}<\infty\right\}$$

and

$$\text{ces}_\infty=\left\{x=(x_n)_n\subset\mathbb{R}\,\middle|\,\sup_n\frac{1}{n}\sum_{k=1}^{n}|x_k|<\infty\right\}$$

were introduced by Shiue [23] in 1970, where $1\leq p<\infty$.It has been shown that $\ell^p\subset\text{ces}_p$for $1<p\leq\infty$. Moreover, it has been shown that Cesàro sequence spaces $\text{ces}_p$for $1<p<\infty$ are seperable reflexive Banach spaces. Furthermore, it was also proved by Cui and Hudzik [5], Cui, Hudzik and Li [6] and Cui, Meng and Pluciennik [7] that Cesàro sequence spaces ces_p for $1<p<\infty$ have the fixed point property. They prove this result using different methods. One method is to calculate Garcia-Falset coefficient. It is known that if Garcia-Falset coefficient is less than 2 for a Banach space, then the space has the fixed point property for nonexpansive mappings [14]. Using this fact, since they calculate this coeeficient for ces_p as $2^{1/p}$ similary to what it is for ℓ^p, they point the result for the Cesàro sequence spaces. Another fact is that they see that the space has normal structure for $1<p<\infty$. Then using the fact via Kirk [15] that reflexive Banach spaces with normal structure has the fixed point property, they easily deduce that the space has the fixed point property for $1 < p < \infty$. Their results on Cesàro sequence spaces as a survey can be seen in [4].

Later, in 1981, Kızmaz [14] introduced difference sequence spaces for ℓ^∞, c and c_0 where they are the Banach spaces of bounded, convergent and null sequences $x = (x_n)_n$, respectively. As it is seen below, his definitions for these spaces were given using difference operator applied to the sequence x, $\triangle x = (x_k - x_{k+1})_k$.

$$\ell^\infty(\triangle) = \{x = (x_n)_n \subset \mathbb{R} |\triangle x \in \ell^\infty\},$$
$$c(\triangle) = \{x = (x_n)_n \subset \mathbb{R} |\triangle x \in c\},$$
$$c_0(\triangle) = \{x = (x_n)_n \subset \mathbb{R} |\triangle x \in c_0\}.$$

Kızmaz investigated Köthe-Toeplitz Duals and some properties of these spaces.

Furthermore, Cesàro sequence spaces X^p of non-absolute type were defined by Ng and Lee [21] in 1977 as follows:

$$X^p = \left\{x = (x_n)_n \subset \mathbb{R} \middle| \left(\sum_{n=1}^{\infty} \left|\frac{1}{n}\sum_{k=1}^{n} x_k\right|^p\right)^{1/p} < \infty\right\}$$

and

$$X^\infty = \left\{x = (x_n)_n \subset \mathbb{R} \middle| \sup_n \left|\frac{1}{n}\sum_{k=1}^{n} x_k\right| < \infty\right\},$$

where $1 \le p < \infty$. They prove that X^p is linearly isomorphic and isometric to ℓ^p for $1 \le p \le \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Later, in 1983, Orhan [22] introduced Cesàro Difference Sequence Spaces by the following definitions:

$$C_p = \left\{x = (x_n)_n \subset \mathbb{R} \middle| \left(\sum_{n=1}^{\infty} \left|\frac{1}{n}\sum_{k=1}^{n} \triangle x_k\right|^p\right)^{1/p} < \infty\right\}$$

and

$$C_\infty = \left\{x = (x_n)_n \subset \mathbb{R} \middle| \sup_n \left|\frac{1}{n}\sum_{k=1}^{n} \triangle x_k\right| < \infty\right\},$$

where $1 \le p < \infty$ and $\triangle x_k = x_k - x_{k+1}$ for each $k \in \mathbb{N}$. He noted that their norms are given as below for any $x = (x_n)_n$:

$$\|x\|_p^* = |x_1| + \left(\sum_{n=1}^{\infty} \left|\frac{1}{n}\sum_{k=1}^{n} \triangle x_k\right|^p\right)^{1/p} \text{ and } \|x\|_\infty^* = |x_1| + \sup_n \left|\frac{1}{n}\sum_{k=1}^{n} \triangle x_k\right|$$

respectively.

Orhan showed that there exists a linear bounded operator $S: C_p \to C_p$ for $1 \le p \le \infty$ such that Köthe-Toeplitz β −Duals of these spaces are given respectively as follows:

$$S(C_p)^\beta = \{a = (a_n)_n \subset \mathbb{R} |(na_n)_n \in \ell^q\} \text{ where } 1 < p < \infty \text{ and } q = \frac{p}{p-1},$$
$$S(C_1)^\beta = \{a = (a_n)_n \subset \mathbb{R} |(na_n)_n \in \ell^\infty\} \text{ and}$$
$$S(C_\infty)^\beta = \{a = (a_n)_n \subset \mathbb{R} |(na_n)_n \in \ell^1\}.$$

It might be better to use the notation $X^p(\triangle)$ instead of C_p for $1 \le p \le \infty$since we also recalled the difference sequence spaces and used similar type of notation.

We note thatOrhan also proved that $X^p \subset X^p(\triangle)$for $1 \le p \le \infty$ strictly. Also, one can clearly see that $X^p(\triangle)$ is linearly isomorphic and isometric to ℓ^p for $1 \le p \le \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Note also that Köthe-Toeplitz Dual for$p = \infty$ case in Orhan's study and ℓ^∞ case in Kızmaz study coincides.

Furthermore, Et and Çolak [10] generalized the spaces introduced in Kızmaz's work [14] in the following way for $m \in \mathbb{N}$.

$$\ell^\infty(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} |\triangle^m x \in \ell^\infty\},$$
$$c(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} |\triangle^m x \in c\},$$
$$c_0(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} |\triangle^m x \in c_0\}$$

where $\triangle x = (\triangle x_k) = (x_k - x_{k+1})_k$, $\triangle^0 x = (x_k)_k$, $\triangle^m x = (\triangle^m x_k) = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1})_k$ and $\triangle^m x_k = \sum_{i=0}^{m}(-1)^i \binom{m}{i} x_{k+i}$ for each $k \in \mathbb{N}$.

Also, Et [11] and Tripathy et. al. [27] generalized the space introduced by Orhan in the following way for $m \in \mathbb{N}$.

$$X^p(\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle^m x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^\infty(\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle^m x_k \right| < \infty \right\},$$

Then, it is seen that that Köthe-Toeplitz Dual for$p = \infty$ case in Et's study [11] and ℓ^∞ case in Et and Çolak study [10] coincides such that Köthe-Toeplitz Dual was given as below for any $m \in \mathbb{N}$.

$$D_m := \{a = (a_n)_n \subset \mathbb{R} | (n^m a_n)_n \in \ell^1\}$$
$$= \left\{ a = (a_k)_k \subset \mathbb{R} : \ \|a\| = \sum_{k=1}^{\infty} k^m |a_k| < \infty \right\}.$$

Note that $D_m \subset \ell^1$.

One can see that corresponding function space for these duals can be given as below:

$$U_m := \left\{ \begin{matrix} f: [0,1] \to \mathbb{R} \\ \text{measurable} \end{matrix} : \|f\| = \int_0^1 t^m |f(t)| dt < \infty \right\}.$$

Note that $L_1[0,1] \subset U_m$ and D_m is the space when counting measure is used for U_m.

As we have already stated, in this study, first we introduce a recent study of Nezir and Mustafa where they show that the corresponding function space U_1 fails the weak fixed point property and there exist many examples of fixed point free contractive mappings on some weakly compact subsets of these spaces. Next, we consider another Lebesgue-like Banach space contained in $L_1[0,1]$. We show that it does not have weak fixed point property and in fact there exist a weakly compact subset and invariant fixed point free contractive mappings defined on it.

Now we provide some preliminaries before giving our main results.

Definition 1.1. *Let $(X, \|\cdot\|)$ be a Banach space and Cbe a non-empty closed, bounded, convex subset.*

1. If $T: C \to C$ is a mapping such that for all $\lambda \in [0,1]$ and for all $x, y \in K$, $T\big((1-\lambda)x + \lambda y\big) = (1-\lambda)T(x) + \lambda T(y)$ then T is said to be an affine mapping.

2. If $T: C \to C$ is a mapping such that $\| T(x) - T(y) \| \leq \| x - y \|$ $for\ all\ x, y \in C$ then T is said to be a nonexpansive mapping.

Also, if for every nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for nonexpansive mappings [fpp(ne)].

3. If $T: C \to C$ is a mapping such that there exists$\lambda \in (0,1)$and $\| T(x) - T(y) \| \leq \lambda \| x - y \|$ $for\ all\ x, y \in C$ then T is said to be a contractive mapping.

Also, if for every contractive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for contractive mappings [fpp(c)].

Definition 1.2. *Let $(X, \|\cdot\|)$ be a Banach space and Cbe a non-empty weakly compact, convex subset.*

1. If for every nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the weak fixed point property for nonexpansive mappings [w-fpp(ne)].

2. If for every contractive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the weak fixed point property for contractive mappings[w-fpp(c)].

2. Main Result

In this section,first we introduce a recent study of Nezir and Mustafa [20] where they show that the corresponding function space U_1 fails the weak fixed point property and there exist many examples of fixed point free contractive mappings on a weakly compact subset of these spaces. Next, we consider a degenerate Lorentz space contained in Lebesgue space $L_1[0,1]$. We consider another Lebesgue-like Banach space contained in $L_1[0,1]$. We show that it does not have weak fixed point property and in fact there exist a weakly compact subset and invariant fixed point free contractive mappings defined on it.

2.1. A Lebesgue-like space containing $L_1[0,1]$

Now firstly we consider the corresponding function space for a Köthe-Toeplitz Dual ofa Cesàro Difference Sequence Space.Nezir and Mustafa [20] show there exists a fixed point free isometry. Since it is an unpublished work, we give only some details but one can confirm that U_1 fails the weak fixed point property; that is, there exists a fixed point free nonexpansive mapping defined on a weakly compact subset of U_1since $L_1[0,1] \subset U_1$ and there exists a fixed point free nonexpansive mapping defined on a weakly compact subset of $L_1[0,1]$.Indeed, Nezir and Mustafa[20] first consider the set $D := \{f \in L_1[0,1]:\ 0 \leq f \leq 1,\ t \in [0,1]\}$ and they define $D_{\frac{1}{2}} :=$ $\{f \in D\ :\ \int_0^1 t|f(t)|dt\ = \frac{1}{2}\}$. Then they show that there exists a fixed point free isometry T on $D_{\frac{1}{2}}$.

Then, Sine's construction but defined on $D_{\frac{1}{2}}$ and using Nezir and Mustafa's map would be an isometry as well; that is, $Sf = \chi_{[0,1]} - Tf,\ \forall f \in D_{\frac{1}{2}}$ would be an isometry. Then, similarly to works of Burns, Lennard and Sivek [2], they define a map $R: D_{1/2} \to D_{1/2}$by

$$R(f) = \sum_{n=0}^{\infty} \frac{T^n(f)}{2^{n+1}}$$

and show that R is a fixed point free contraction. Using their ideas, we can also say that we can define some other maps $J: D_{1/2} \to D_{1/2}$ and by $P: D_{1/2} \to D_{1/2}$

$$J(f) = \sum_{n=0}^{\infty} q_{n+1} S^n(f) \quad \text{and} \quad P(f) = \sum_{n=0}^{\infty} q_{n+1} T^n(f)$$

where $(q_n)_n$ is any scalar sequence summing to 1; that is, $\sum_{n=0}^{\infty} q_{n+1} = 1$.

2.2. A Lebesgue-like space contained in $L_1[0,1]$

Now, we consider another Lebesgue like Banach space which is actually contained in Lebesgue space $L_1[0,1]$. It can be said that the space we consider is a degenerate Lorentz space. So firstly we recall Lorentz space.

Definition 2.1. *Let $\alpha \in (0,1)$.*

$$L_{\alpha,1}[0,1] := \left\{ f: [0,1] \to \mathbb{R}\ measurable\ \middle| \| x \|_{\alpha,1} := \int_0^1 \frac{\alpha f^*(t)}{t^{1-\alpha}} dt < \infty \right\}$$

where $f^*(t)$ is the decreasing rearrangement of $|f(t)|$; that is, $f^*(t)$ is ordered decreasing and equimeasurable with $|f(t)|$.

Note that rearrangement is a non-expansive mapping with respect to Lebesgue norms. The non-increasing rearrangement of a function was first studied by Steiner [26] and it is defined as a certain generalized inverse of the distribution function. There have been many researches on these type of functions but first most known properties were given by Hardy, Littlewood and Pólya in 1930's [13].

Another $L_1[0,1]$ analogue, Lorentz-Marcinkiewicz space, is defined as follows:

Definition 2.2. *Let $w: [0,1] \to \mathbb{R}$ be the weight function which is nonincreasing such that $\int_0^{\infty} w dm = \infty$ where m is the Lebesgue measure. Then,*

$$L_{w,1}[0,1] := \left\{ \begin{matrix} f: [0,1] \to \mathbb{R} \\ \text{measurable} \end{matrix} \middle| \| x \|^{w,1} := \int_0^1 w(t) f^*(t) dt < \infty \right\}$$

where $f^*(t)$ is the decreasing rearrangement of $|f(t)|$.

But in this study we will be working on the Banach space defined below. Note that this space is originated from the space introduced in previous two definitions but here we do not use decreasing rearrengements instead we use the absolute value. One may consider subspaces of decreasing functions in Lorentz spaces or Lorentz-Marcinkiewicz spaces. So our space would generalize this type of subspaces.Standard references for Lorentz-Marcinkiewicz spaces are[16,17,18].

Definition 2.3. Let

$$\mathcal{M} := \left\{ \begin{matrix} f:[0,1] \to \mathbb{R} \\ \text{measurable} \end{matrix} : \|f\|^{\sim} = \int_0^1 \frac{|f(t)|}{t} dt < \infty \right\}.$$

Then, it is easy to see that $\mathcal{M}$ is a Banach space contained in Lebesgue space $L_1[0,1]$.

The following theorem shows that $\mathcal{M}$does not have weak fixed point property and in fact there exist aweakly compact subset and invariant fixed point free contractive mappings defined on it.

Theorem 2.1.*There exists a weakly compact subsets of $\mathcal{M}$ such that there exist invariant fixed point free nonexpansive mappings and invariant fixed point free contractive mappings defined on that set.*

Proof. In $\mathcal{M}$, consider the subset

$K:= \left\{ f \in \mathcal{M} \ : \ 0 \le f \le 1,\ t \in [0,1],\ \int_0^1 \frac{f(t)}{t} dt \le 1 \right\}$, then also consider the mapping

$$\psi f(t) = tT\left(\frac{f(t)}{t}\right) \qquad (2.1)$$

where T is Alspach's mapping with the formula (1.1).

Then, for any $f, g \in K$, we have

$$\begin{aligned} \|\psi(f) - \psi(g)\|^{\sim} &= \int_0^1 \frac{\left| tT\left(\frac{f(t)}{t}\right) - tT\left(\frac{g(t)}{t}\right)\right|}{t} dt \\ &= \int_0^1 \left| T\left(\frac{f(t)}{t}\right) - T\left(\frac{g(t)}{t}\right)\right| dt \\ &= \left\| \mathrm{T}\left(\frac{f}{t}\right) - \mathrm{T}\left(\frac{g}{t}\right)\right\|_1 \\ &= \left\| \frac{f}{t} - \frac{g}{t} \right\|_1 \\ &= \int_0^1 \left| \frac{f(t)}{t} - \frac{g(t)}{t} \right| dt \\ &= \int_0^1 \frac{|f(t) - g(t)|}{t} dt \\ &= \|f - g\|^{\sim} \end{aligned}$$

Next define $K_{\frac{1}{2}} := \left\{ f \in K \ : \ \int_0^1 \frac{f(t)}{t} dt = \frac{1}{2} \right\}$. Then, ψ is a fixed point free nonexpansive mapping on $K_{\frac{1}{2}}$ by the proof of Alspach's theorem in [1]. Also, similarly to Sine's result [24], the mapping defined by $\Omega f := \chi_{[0,1]} - \psi f,\ \forall f \in K_{\frac{1}{2}}$ is another fixed point free nonexpansive mapping on $K_{\frac{1}{2}}$

Now, define $R^{\sim}: K_{\frac{1}{2}} \to K_{\frac{1}{2}}$ by

$$R^{\sim}(f) = \sum_{n=0}^{\infty} \frac{\psi^n(f)}{2^{n+1}},$$

then using the strategy in the proof of the main theorem in [2], it is seen that $R^{\sim}$ is a fixed point free contractive invariant mapping on $K_{\frac{1}{2}}$ and so $\mathcal{M}$fails w-fpp for contractive mappings (so does for nonexpansive mappings).

Now, we will provide another invariant fixed point free nonexpansive mapping and a different invariant fixed point free contractive mapping defined on the set $K_{\frac{1}{2}}$ using ideas in [8].

Claim: Using Dowling, Lennard and Turett's mapping given in the formula (1.2), define a composite mapping by

$$\Delta^{\sim} f(t) := \begin{cases} \frac{f(2t)}{2t} & \text{if } 0 \le t \le \frac{1}{2} \\ 1 - \frac{f(2t-1)}{2t-1} & \text{if } \frac{1}{2} < t < 1 \end{cases} \text{ and } \Delta^{*} f(t) := t\Delta^{\sim} f(t).$$

One can see that $\Delta^{*} f(t) = t\Delta\left(\frac{f}{t}\right)(t)$ is an invariant mapping defined on $K_{\frac{1}{2}}$.

Then, $\psi\Delta^{*}: K_{\frac{1}{2}} \to K_{\frac{1}{2}}$ is a fixed point free nonexpansive mapping where ψ is given as in the formula (2.1). Moreover, there exists a fixed point free contractive mapping $\varphi^{\sim}: K_{\frac{1}{2}} \to K_{\frac{1}{2}}$.

Proof of the claim: For any $f, g \in K_{\frac{1}{2}}$, we have

$$\begin{aligned}
\|\psi\Delta^{*}(f) - \psi\Delta^{*}(g)\|^{\sim} &= \int_0^1 \frac{\left|tT\left(\frac{\Delta^{*}(f)(t)}{t}\right) - tT\left(\frac{\Delta^{*}(g)(t)}{t}\right)\right|}{t} dt \\
&= \int_0^1 \left|T\left(\frac{t\Delta^{\sim}(f)(t)}{t}\right) - T\left(\frac{t\Delta^{\sim}(g)(t)}{t}\right)\right| dt \\
&= \| \mathrm{T}(\Delta^{\sim}(f)) - \mathrm{T}(\Delta^{\sim}(g)) \|_1 \\
&= \| \Delta^{\sim}(f) - \Delta^{\sim}(g) \|_1 \\
&= \int_0^1 \left|\Delta\left(\frac{f}{t}\right)(t) - \Delta\left(\frac{g}{t}\right)(t)\right| dt \\
&= \left\|\Delta\left(\frac{f}{t}\right) - \Delta\left(\frac{g}{t}\right)\right\|_1 \\
&= \left\|\frac{f}{t} - \frac{g}{t}\right\|_1 \\
&= \int_0^1 \left|\frac{f(t)}{t} - \frac{g(t)}{t}\right| dt \\
&= \int_0^1 \frac{|f(t) - g(t)|}{t} dt \\
&= \|f - g\|^{\sim}
\end{aligned}$$

Now define $\varphi^{\sim}: K_{\frac{1}{2}} \to K_{\frac{1}{2}}$ by

$$\varphi^{\sim}(f) := \sum_{n=0}^{\infty} \frac{(\psi\Delta^{*})^n(f)}{2^{n+1}},$$

then using the strategy in the proof of the main theorem in [2], it is seen that $\varphi^{\sim}$ is a fixed point free contractive invariant mapping on $K_{\frac{1}{2}}$.

References

[1]. Alspach, D. E. (1981). A fixed point free nonexpansive map. *Proceedings of the American Mathematical Society*, *82*(3):423-424.

[2]. Burns, J., Lennard, C., & Sivek, J. (2014). A contractive fixed point free mapping on a weakly compact convex set. *Studia Mathematica*, *3*(223):275-283.

[3]. Casini, E., Miglierina, E., & Piasecki, Ł. (2017). Separable Lindenstrauss spaces whose duals lack the weak* fixed point property for nonexpansive mappings. *Studia Mathematica*, *238*:1-16.

[4]. Chen, S., Cui, Y., Hudzik, H., & Sims, B. (2001). Geometric properties related to fixed point theory in some Banach function lattices. In *Handbook of metric fixed point theory*. Springer, Dordrecht, 339-389.

[5]. Cui, Y. (1999). Some geometric properties related to fixed point theory in Cesàro spaces. *Collectanea Mathematica*, 277-288.

[6]. Cui, Y., Hudzik, H., & Li, Y. (2000). On the Garcfa-Falset Coefficient in Some Banach Sequence Spaces. In *Function Spaces*. CRC Press, 163-170.

[7]. Cui, Y., Meng, C., & Płuciennik, R. (2000). Banach—Saks Property and Property (β) in Cesàro Sequence Spaces. *Southeast Asian Bulletin of Mathematics*, *24*(2):201-210.

[8]. Dowling, P. N., Lennard, C. J., & Turett, B. (2003). Some more examples of subsets of c_0 and $L_1[0, 1]$ failing the fixed point property. *Contemporary Mathematics*, *328*:171-176.

[9]. Dowling, P. N., Lennard, C. J., & Turett, B. (2007). New fixed point free nonexpansive maps on weakly compact, convex subsets of $L_1[0, 1]$. *Studia Math*, *180*(3):271-284.

[10]. Et, M., & Çolak, R. (1995). On some generalized difference sequence spaces. *Soochow journal of mathematics*, *21*(4):377-386.

[11]. Et, M. (1996). On some generalized Cesàro difference sequence spaces. *İstanbul University Science Faculty The Journal Of Mathematics, Physics and Astronomy*, *55*:221-229.

[12]. Falset, J. G. (1997). The fixed point property in Banach spaces with the NUS-property. *Journal of Mathematical Analysis and Applications*, *215*(2):532-542.

[13]. Hardy, G. H., Littlewood, J. E., & Polya, G. (1952). Inequalities cambridge univ. *Press, Cambridge*.

[14]. Kızmaz, H. (1981). On certain sequence spaces. *Canadian mathematical bulletin*, *24*(2):169-176.

[15]. Kirk, W. A. (1965). A fixed point theorem for mappings which do not increase distances. *The American mathematical monthly*, *72*(9):1004-1006.

[16]. Lindenstraus, J., & Tzafriri, L. (1977). Classical Banach spaces I: Sequence spaces Springer-Verlag. *Berlin and New York*.

[17]. Lindenstrauss, J., & Tzafriri, L. (2013). *Classical Banach spaces II: function spaces* (Vol. 97). Springer Science & Business Media.

[18]. Llorens-Fuster, E., & Sims, B. (1998). The fixed point property in c0. *Canadian Mathematical Bulletin*, *41*(4):413-422.

[19]. Lorentz, G. G. (1950). Some new functional spaces. *Annals of Mathematics*, 37-55.

[20]. Nezir, V., Mustafa, N. A Köthe-Toeplitz dual of a generalized Cesàro difference sequence space, a degenerate Lorentz space, their corresponding function spaces and fpp, in preparation.

[21]. NgPeng-Nung, N. N., & LeePeng-Yee, L. Y. (1978). Cesàro sequence spaces of nonabsolute type. *Commentationes mathematicae*, *20*(2):429–433.

[22]. Orhan, C. (1983). Casaro Differance Sequence Spaces and Related Matrix Transformations. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, *32*:55-63.

[23]. Shiue, J. S. (1970). On the Cesaro sequence spaces. *Tamkang J. Math*, *1*(1):19-25.

[24]. Sine, R. (2020). Remarks on the example of Alspach. His example and a slight. *Nonlinear Anal. and Appl.*, Marcel Dekker, 237-241.

[25]. Sivek, J. (2014). *Differentiability, summability, and fixed points in Banach spaces* (Doctoral dissertation, University of Pittsburgh).

[26]. Steiner, J. (1881). Gesammelte Werke I, II, Reimer.

[27]. Tripathy, B. C., Esi, A., & Tripathy, B. (2005). On new types of generalized difference Cesaro sequence spaces. *Soochow Journal of Mathematics*, *31*(3):333-340.