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On Lebesgue-like Corresponding Function Space of a Köthe-Toeplitz Dual of a Generalized Cesàro Difference Sequence Space and Fixed Point Property

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Abstract In 1970, Cesàro Sequence Spaces was introduced by Shiue. In 1981, Kızmaz defined difference sequence spaces for ℓ^∞, c_0 and c. Then, in 1983, Orhan introduced Cesàro Difference Sequence Spaces. Later, Et and Tripathy et. al. generalized the space introduced by Orhan for any $m \in \mathbb{N}$. We will be interested in their generalizations. In this study, first we discuss the fixed point property for these spaces. Then, we recall that Goebel and Kuczumow showed that there exists a very large class of closed, bounded, convex subsets in ℓ^1, Banach space of absolutely summable scalar sequences, with fixed point property for nonexpansive mappings. So we consider an analogue result for the corresponding function space of a Köthe-Toeplitz Dual of a generalized Cesàro Difference Sequence Space which contains Lebesgue space $L_1[0,1]$. We show that there exists a large class of closed, bounded and convex subsets of these spaces with fixed point property for affine nonexpansive mappings.

Keywords Fixed point property, nonexpansive mapping, Cesàro Difference Sequences, Köthe-Toeplitz dual
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1. Introduction and preliminaries

We say that a Banach space $(X, \|.\|)$ has the fixed point property for non-expansive mappings if every non-expansive self mappings defined on any non-empty closed, bounded and convex subset of the Banach space has a fixed point. Here we note that if C is a subset of the Banach space, then $T: C \to C$ is said to be a nonexpansive mapping if $\| T(x) - T(y) \| \leq \| x - y \|$, for all $x, y \in C$. Researchers have been interested in checking if a nonreflexive Banach space can be renormed to have the fixed point property to see how the fixed point property is related with reflexivity. In fact, the first example of a nonreflexive Banach space which is renormable to have the fixed point property was given by Lin [13]. Lin showed this fact by setting an equivalent norm on Banach space of absolutely summable scalar sequences, ℓ^1. Because of sharing many common properties, it is natural to ask if , Banach space of scalar sequences converging to 0, c_0 can be renormed to have the fixed point property for non-expansive mappings as another well known classical non-reflexive Banach space. Maria and Hernandes Lineares [15] obtained an example for the class of nonreflexive Banach spaces which can be renormed to have the fixed point property for affine nonexpansive mappings and their space was the Banach space of Lebesgue integrable functions on [0,1], $L_1[0,1]$. It can be said that all these works are inspired by the work of Goebel and Kuczumow [10]. Goebel and Kuczumow showed that there exists very large class of non-weakly compact, closed, bounded and convex subsets of ℓ^1 respect to weak* topology of ℓ^1 with fixed point property for non-expansive mappings.

Thus, in this study, we work on Goebel and Kuczumow analogy for a Banach space containing Lebesgue space $L_1[0,1]$. The space we consider is the corresponding function space of a Köthe-Toeplitz Dual of a generalized Cesàro Difference Sequence Space which contains Lebesgue space $L_1[0,1]$. We show that there exists a very large class of closed, bounded and convex subsets of the space with the fixed point property for non-expansive mappings under affinity condition.

We recall that the Cesàro sequence spaces

$$\mathrm{ces}_p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

were introduced by Shiue [19] in 1970, where $1 \leq p < \infty$. It has been shown that $\ell^p \subset \mathrm{ces}_p$ for $1 < p \leq \infty$. Moreover, it has been shown that Cesàro sequence spaces ces_p for $1 < p < \infty$ are seperable reflexive Banach spaces. Furthermore, it was also proved by Cui-Hudzik [3], Cui-Hudzik-Li [4] and Cui-Meng-Pluciennik [5] that Cesàro sequence spaces ces_p for $1 < p < \infty$ have the fixed point property. They prove this result using different methods. One method is to calculate Garcia-Falset coefficient. It is known that if Garcia-Falset coefficient is less than 2 for a Banach space, then it has the fixed point property for nonexpansive mappings [9]. Using this fact, since they calculate this coefficient for ces_p as $2^{1/p}$ similary to what it is for ℓ^p, they point the result for the Cesàro sequence spaces. Another fact is that they see that the space has normal structure for $1 < p < \infty$. Then using the fact via Kirk [12] that reflexive Banach spaces with normal structure has the fixed point property, they easily deduce that the space has the fixed point property for $1 < p < \infty$. Their results on Cesàro sequence spaces as a survey can be seen in [2].

Later, in 1981, Kızmaz [11] introduced difference sequence spaces for ℓ^∞, c and c_0 where they are the Banach spaces of bounded, convergent and null sequences $x = (x_n)_n$, respectively. As it is seen below, his definitions for these spaces were given using difference operator applied to the sequence x, $\triangle x = (x_k - x_{k+1})_k$.

$$\ell^\infty(\triangle) = \{x = (x_n)_n \subset \mathbb{R} \mid \triangle x \in \ell^\infty\},$$
$$\mathrm{c}(\triangle) = \{x = (x_n)_n \subset \mathbb{R} \mid \triangle x \in \mathrm{c}\},$$
$$\mathrm{c}_0(\triangle) = \{x = (x_n)_n \subset \mathbb{R} \mid \triangle x \in \mathrm{c}_0\}.$$

Kızmaz investigated Köthe-Toeplitz Duals and some properties of these spaces.

Furthermore, Cesàro sequence spaces X^p of non-absolute type were defined by Ng and Lee [16] in 1977 as follows:

$$X^p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| < \infty \right\},$$

where $1 \leq p < \infty$. They prove that X^p is linearly isomorphic and isometric to ℓ^p for $1 \leq p \leq \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Later, in 1983, Orhan [17] introduced Cesàro Difference Sequence Spaces by the following definitions:

$$C_p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right| < \infty \right\},$$

where $1 \leq p < \infty$ and $\triangle x_k = x_k - x_{k+1}$ for each $k \in \mathbb{N}$. He noted that their norms are given as below for any $x = (x_n)_n$:

$$\|x\|_p^* = |x_1| + \left(\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\triangle x_k\right|^p\right)^{1/p} \text{ and } \|x\|_\infty^* = |x_1| + \sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\triangle x_k\right|,$$

respectively.

Orhan showed that there exists a linear bounded operator $S: C_p \to C_p$ for $1 \le p \le \infty$ such that Köthe-Toeplitz β −Duals of these spaces are given respectively as follows:

$$S(C_p)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^q\} \text{ where } 1 < p < \infty \text{ and } q = \frac{p}{p-1},$$

$$S(C_1)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^\infty\} \text{ and}$$

$$S(C_\infty)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^1\}.$$

It might be better to use the notation $X^p(\triangle)$ instead of C_p for $1 \le p \le \infty$ since we also recalled the difference sequence spaces and used similar type of notation.

We note that Orhan also proved that $X^p \subset X^p(\triangle)$ for $1 \le p \le \infty$ strictly. Also, one can clearly see that $X^p(\triangle)$ is linearly isomorphic and isometric to ℓ^p for $1 \le p \le \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Note also that Köthe-Toeplitz Dual for $p = \infty$ case in Orhan's study and ℓ^∞ case in Kızmaz study coincides.

Furthermore, Et and Çolak [6] generalized the spaces introduced in Kızmaz's work [11] in the following way for $m \in \mathbb{N}$.

$$\ell^\infty(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \ell^\infty\},$$

$$\mathrm{c}(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \mathrm{c}\},$$

$$\mathrm{c}_0(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \mathrm{c}_0\}$$

where $\triangle x = (\triangle x_k) = (x_k - x_{k+1})_k$, $\triangle^0 x = (x_k)_k$, $\triangle^m x = (\triangle^m x_k) = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1})_k$ and $\triangle^m x_k = \sum_{i=0}^{m}(-1)^i\binom{m}{i} x_{k+i}$ for each $k \in \mathbb{N}$.

Also, Et [7] and Tripathy et. al. [20] generalized the space introduced by Orhan in the following way for $m \in \mathbb{N}$.

$$X^p(\triangle^m) = \left\{x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\triangle^m x_k\right|^p\right)^{1/p} < \infty\right\}$$

and

$$X^\infty(\triangle^m) = \left\{x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\triangle^m x_k\right| < \infty\right\},$$

Then, it is seen that that Köthe-Toeplitz Dual for$p = \infty$ case in Et's study [7] and ℓ^∞ case in Et and Çolak study [6] coincides such that Köthe-Toeplitz Dual was given as below for any $m \in \mathbb{N}$.

$$D_m := \{a = (a_n)_n \subset \mathbb{R} | (n^m a_n)_n \in \ell^1\}$$

$$= \left\{a = (a_k)_k \subset \mathbb{R} : \ \|a\| = \sum_{k=1}^{\infty} k^m |a_k| < \infty\right\}.$$

Note that $D_m \subset \ell^1$for any $m \in \mathbb{N}$.

One can see that corresponding function space for these duals can be given as below:

$$U_m := \left\{\begin{matrix} f: [0,1] \to \mathbb{R} \\ \text{measurable}\end{matrix} : \|f\| = \int_0^1 t^m |f(t)| dt < \infty\right\}.$$

Note that $L_1[0,1] \subset U_m$ and D_m is the space when counting measure is used for U_m.

As we have already stated, in this study, we consider Goebel and Kuczumow [10] analogy for the corresponding function space of a Köthe-Toeplitz Dual ofa Cesàro Difference Sequence Space which contains Lebesgue space $L_1[0,1]$. We show that there exists a large class of closed, bounded and convex subsets of Köthe-Toeplitz Dual for $X^\infty(\triangle)$ with fixed point property for affine nonexpansive mappings.

Now we provide some preliminaries before giving our main results.

Definition 1.1. *Let $(X, \|\cdot\|)$ be a Banach space and C is a non-empty closed, bounded, convex subset.*

1. If $T: C \to C$ is a mapping such that for all $\lambda \in [0,1]$ and for all $x, y \in C$, $T\big((1-\lambda)x + \lambda y\big) = (1-\lambda)T(x) + \lambda T(y)$ then T is said to be an affine mapping.

2. If $T: C \to C$ is a mapping such that $\parallel T(x) - T(y) \parallel \leq \parallel x - y \parallel$ $for\ all\ x, y \in C$ then T is said to be a nonexpansive mapping.
Also, if for every nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for nonexpansive mappings [fpp(ne)].

Remark 1.1. *In 1979, Goebel and Kuczumow [10] showed there exists a large class of closed, bounded and convex subsets of ℓ^1 using a key lemma they obtained. Their lemma says that if $\{x_n\}$ is a sequence in ℓ^1 converging to x in weak-star topology, then for any $y \in \ell^1$,*

$$r(y) = r(x) + \|y - x\|_1 \ where\ r(y) = \limsup_n \|x_n - y\|_1 .$$

We will call this fact ∴.
The analogue of this lemma for $L_1[0,1]$ is observed via the result in Brezis and Lieb [1]. Note that Hernández-Linares pointed this fact in his Ph.D. thesis [14], written under supervision of Maria Japon Pineda. Now we provide the lemma which is deduced by their results and will be key for our results in this section.

Lemma 1.1. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued measurable functions which are uniformly bounded in $L_1[0,1]$. Assume that f_n converges to an $f \in L_1[0,1]$ pointwise almost everywhere (a.e.). Then for any $g \in L_1[0,1]$,*

$$S(g) = S(f) + \|f - g\|_1 \ where\ S(g) = \limsup_n \|f_n - g\|_1 .$$

Since the corresponding function space of a Köthe-Toeplitz Dual of a Cesàro Difference Sequence Space which contains Lebesgue space $L_1[0,1]$ and in fact it is isometrically isomorphic to $L_1[0,1]$, for the corresponding function spaceU_1 the following lemma can be given as straight and quick result.

Lemma 1.2. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued measurable functions which are uniformly bounded in U_1. Assume that f_n converges to an $f \in U_1$ pointwise almost everywhere (a.e.). Then for any $g \in U_1$,*

$$S(g) = S(f) + \|f - g\| \ where\ S(g) = \limsup_n \|f_n - g\| .$$

2. Main Result

In this section, we work on Goebel and Kuczumow analogy for a Banach space containing Lebesgue space $L_1[0,1]$. The space we consider is the corresponding function space U_m of Köthe-Toeplitz Dual ofa Cesàro Difference Sequence Space $X^\infty(\triangle^m)$ for any $m \in \mathbb{N}$, which is the corresponding function space of a Köthe-Toeplitz Dual of a generalized Cesàro Difference Sequence Space. We show that there exists a very large class of closed, bounded and convex subsets of the space with the fixed point property for non-expansive mappings under affinity condition.
Now, we consider the following class of closed, bounded and convex subsets. Note that here we will be using the ideas similar to those in the section 2 of Ph.D. thesis of Everest [8], written under supervision of Chris Lennard, where Everest firstly provides Goebel and Kuczumow's proofs in detailed.
So we demonstrate examples of these subsets and provide a theorem related with each of them.
Example 2.1. *Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1, f_2 := b\, e_2$, and $f_n := e_n$, for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n := (n+1)t^{n-m}$, $\forall n \in \mathbb{N}$. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of U_1 by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \beta_n f_n :\ \forall n \in \mathbb{N}, \quad t_n \geq 0 \ and\ \sum_{n=1}^{\infty} \beta_n = 1 \right\} .$$

Example 2.2. *Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1, f_2 := b\, e_2$, and $f_n := e_n$, for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n := \frac{n e^{nt}}{t^m (e^n - 1)}$, $\forall n \in \mathbb{N}$. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of U_1 by*

$$E^{(m)}:=\left\{\sum_{n=1}^{\infty}\beta_n f_n:\ \forall n\in\mathbb{N},\qquad \beta_n\geq 0\ \ and\ \sum_{n=1}^{\infty}\beta_n=1\right\}.$$

Example 2.3. *Fix $m\in\mathbb{N}$ and $b\in(0,1)$.Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1:=b\,e_1$,$f_2:=b\,e_2$, and $f_n:=e_n$, for all integers $n\geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n:=\frac{ne^{nt}}{t^m(e^n-1)}\chi_{[0,\frac{1}{n}]}$, $\forall n\in\mathbb{N}$, where χ is the characteristics funtion. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of U_1 by*

$$E^{(m)}:=\left\{\sum_{n=1}^{\infty}\beta_n f_n:\ \forall n\in\mathbb{N},\qquad \beta_n\geq 0\ \ and\ \sum_{n=1}^{\infty}\beta_n=1\right\}.$$

Example 2.4. *Fix $m\in\mathbb{N}$ and $b\in(0,1)$.Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1:=b\,e_1$,$f_2:=b\,e_2$, and $f_n:=e_n$, for all integers $n\geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n:=\frac{4n}{\pi t^m(1+n^2t^2)}\chi_{[0,\frac{1}{n}]}$, $\forall n\in\mathbb{N}$, where χ is the characteristics funtion. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of U_1 by*

$$E^{(m)}:=\left\{\sum_{n=1}^{\infty}\beta_n f_n:\ \forall n\in\mathbb{N},\qquad \beta_n\geq 0\ \ and\ \sum_{n=1}^{\infty}\beta_n=1\right\}.$$

Theorem 2.1. *For any $m\in\mathbb{N}$ andb $\in(0,1)$,each of the sets $E^{(m)}$ defined as in the examples above has the fixed point property for affine $\|\,,\|$-nonexpansive mappings.*

Proof. Fix $m\in\mathbb{N}$ *and* $b\in(0,1)$.Let $T:E^{(m)}\to E^{(m)}$ be an affine nonexpansive mapping. Then, since T is affine, by Lemma 1.1.2 in the Ph.D. thesis of Everest [8] written under supervision of Lennard, there exists a sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}}\in E^{(m)}$ such that $\left\|Tx^{(n)}-x^{(n)}\right\|\underset{n}{\to}0$. Without loss of generality, passing to a subsequence if necessary, there exists $x\in U_m$ such that $x^{(n)}$converges to x in weak* topology. Then, by Goebel Kuczumow analog fact, Lemma 2 given in the last part of the previous section, we can define a function $s:U_m\to[0,\infty)$ by

$$s(y)=\limsup_n\left\|x^{(n)}-y\right\|\quad,\ \forall y\in U_m$$

and so

$$s(y)=s(y)+\|x-y\|\ ,\ \forall y\in U_m\,.$$

Now define the weak* closure of the set E as it is seen below.

$$W:=\overline{E^{(m)}}^{w^*}=\left\{\sum_{n=1}^{\infty}\beta_n f_n:\ each\ \beta_n\geq 0\ \ and\ \sum_{n=1}^{\infty}\beta_n\leq 1\right\}$$

Case 1:$x\in E^{(m)}$.

Then, $\forall n\in\mathbb{N}$, we have $s(Tx)=s(x)+\|Tx-x\|$ and

$$\begin{aligned}s(Tx)&=\limsup_n\left\|Tx-x^{(n)}\right\|\\&\leq\limsup_n\left\|Tx-T\left(x^{(n)}\right)\right\|+\limsup_n\left\|x^{(n)}-T\left(x^{(n)}\right)\right\|\\&\leq\limsup_n\left\|x-x^{(n)}\right\|+0\\&=s(x).\end{aligned}\tag{2.1}$$

Therefore, $s(Tx)=s(x)+\|Tx-x\|\leq s(x)$ and so $\|Tx-x\|=0$. Thus, $Tx=x$.

Case 2: $x\in W\backslash E^{(m)}$.

Then, x is of the form $\sum_{n=1}^{\infty}\gamma_n f_n$ such that $\sum_{n=1}^{\infty}\gamma_n<1$ *and* $\gamma_n\geq 0,\ \forall n\in\mathbb{N}$.

Define $\delta:=1-\sum_{n=1}^{\infty}\gamma_n$and for $\alpha\in\left[\frac{-\gamma_1}{\delta},\frac{\gamma_2}{\delta}+1\right]$ define

$$\mathrm{h}_\alpha:=(\gamma_1+\alpha\delta)f_1+(\gamma_2+(1-\alpha)\delta)f_2+\sum_{n=3}^{\infty}\gamma_n f_n.$$

Then, $\|\mathrm{h}_\alpha-x\|=\|\alpha b\delta e_1+(1-\alpha)\delta e_2\|\leq \mathrm{b}|\alpha|\delta+b|1-\alpha|\delta=\mathrm{b}\delta$.

So $\|\mathrm{h}_\alpha-x\|$ is minimized for $\alpha\in[0,1]$ and its minimum value would be less than or equal to $b\delta$.

Now fix $y\in E^{(m)}$ of the form $\sum_{n=1}^{\infty}\beta_n f_n$ such that $\sum_{n=1}^{\infty}\beta_n=1$ with $\beta_n\geq 0,\ \forall n\in\mathbb{N}$.

Then,

$$\begin{aligned}\|y - x\| &= \left\|\sum_{k=1}^{\infty} \beta_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k\right\| = \left\|\sum_{k=1}^{\infty} \beta_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k\right\| \\ &= \left\|\sum_{k=1}^{\infty} (\beta_k - \gamma_k) f_k\right\| \\ &= \int_0^1 t \left|\sum_{k=1}^{\infty} (\beta_k - \gamma_k) f_k\right| dm = \int_0^1 \left|\sum_{k=1}^{\infty} (\beta_k - \gamma_k) t f_k\right| dm \\ &\geq \left|\int_0^1 \sum_{k=1}^{\infty} (\beta_k - \gamma_k) t f_k dm\right| \\ &= \left|\sum_{k=1}^{\infty} (\beta_k - \gamma_k)\right| \\ &= \left|1 - \sum_{k=1}^{\infty} \gamma_k\right| \\ &= \delta\end{aligned}$$

Hence,

$$\|y - x\| \geq b\delta \geq \|h_\alpha - x\|.$$

Now, define $\Lambda := \{h_\alpha \;:\; \alpha \in [0,1]\}$. Clearly, Λ is the contiunous image of a compact set and so it is a compact subset of $E^{(m)}$, It is also easy to see that it is convex.

Now for any $h \in \Lambda$,
$s(h) = s(x) + \|h - x\| \leq s(x) + \|Th - x\| = s(Th)$ but this follows

$$\begin{aligned}&= \limsup_n \|Th - x^{(n)}\| \text{ then similarly to the inequality (2.1)} \\ &\leq \limsup_n \|Th - T(x^{(n)})\| + \limsup_n \|x^{(n)} - T(x^{(n)})\| \\ &\leq \limsup_n \|h - x^{(n)}\| + \limsup_n \|x^{(n)} - T(x^{(n)})\| \\ &\leq \limsup_n \|h - x^{(n)}\| + 0 \\ &= s(h).\end{aligned}$$

Hence, $s(h) \leq s(Th) \leq s(h)$ and so $s(Th) = s(h)$. Hence, $s(x) + \|Th - x\| = s(x) + \|h - x\|$. Therefore,

$$\|Th - x\| = \|h - x\|$$

and so $Th \in \Lambda$ but this means $T(\Lambda) \subseteq \Lambda$ and since T is continuous, Schauder's Fixed Point Theorem [18] tells us that T has a fixed point such that $Th = h$.
Therefore, $E^{(m)}$ has fpp(ne) as desired.

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