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On Cesàro Difference Sequence Spaces, their Köthe-Toeplitz Duals and Coefficient Estimate of Fixed Point Property for Uniform Lipschitz Mappings

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Abstract In 1970, Cesàro Sequence Spaces was introduced by Shiue. In 1981, Kızmaz defined difference sequence spaces for ℓ^∞, c_0 and c. Then, in 1983, Orhan introduced Cesàro Difference Sequence Spaces. In this study, first we discuss the fixed point property for these spaces. Then, we recall that Dowling, Lennard and Turett showed that if a Banach space contains an isomorphic copy of ℓ^1, then it fails the fixed point property for uniform Lipschitz mappings. So we work on a right shift mapping defined on a closed, bounded and convex subset of a Köthe-Toeplitz Dual of a Cesàro Difference Sequence Space so that the right shift mapping can be a uniform Lipschitz mapping. Thus, we investigate an upper bound estimate for the right shift mapping to be uniformly Lipschitz failing the fixed point property on a class of closed, bounded and convex subsets in those spaces.

Keywords Fixed point property, Uniform Lipschitz mapping, Cesàro Difference Sequences, Köthe-Toeplitz Dual
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1. Introduction and Preliminaries

A Banach space $(\mathrm{X}, \|.\|)$ is called to have the fixed point property for non-expansive mappings (fpp-ne) when any non-expansive self mappings defined on arbitrary non-empty closed, bounded and convex subset of the Banach space has a fixed point. If the above statement holds for arbitrary uniform Lipshcitz mappings, then we say $(\mathrm{X}, \|.\|)$ has the fixed point property for uniformly Lipschitz mappings (fpp-uL). Hence, if there exist a closed, bounded and convex subset C and a fixed point free uniformly Lipschitz mapping $\mathrm{T}: \mathrm{C} \to \mathrm{C}$ then we say X fails to have fpp-uL.

The Cesàro sequence spaces

$$\mathrm{ces}_p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

were introduced by Shiue [14] in 1970, where $1 \le p < \infty$. It has been shown that $\ell^p \subset \mathrm{ces}_p$ for $1 < p \le \infty$. Moreover, it has been shown that Cesàro sequence spaces ces_p for $1 < p < \infty$ are seperable reflexive Banach spaces. Furthermore, it was also proved by Cui-Hudzik [2], Cui-Hudzik-Li [3] and Cui-Meng-Pluciennik [4] that Cesàro sequence spaces ces_p for $1 < p < \infty$ have the fixed point property. They prove this result using different methods. One method is to calculate Garcia-Falset coefficient. It is known that if Garcia-Falset coefficient is less than 2 for a Banach space, then it has the fixed point property for nonexpansive mappings [9]. Using this fact, since they calculate this coeeficient for ces_p as $2^{1/p}$ similary to what it is for ℓ^p, they point the result for the Cesàro sequence spaces. Another fact is that they see that the space has normal structure for $1 < p < \infty$. Then using the fact via Kirk [11] that reflexive Banach spaces with normal structure has the fixed point property, they easily deduce that the space has the fixed point property for $1 < p < \infty$. Their results on Cesàro sequence spaces as a survey can be seen in [1].

Later, in 1981, Kızmaz [10] introduced difference sequence spaces for ℓ^∞, cand c_0 where they are the Banach spaces of bounded, convergent and null sequences $x = (x_n)_n$, respectively. As it is seen below, his definitions for these spaces were given using difference operator applied to the sequence x, $\triangle x = (x_k - x_{k+1})_k$.

$$\ell^\infty(\triangle) = \{x = (x_n)_n \subset \mathbb{R} | \triangle x \in \ell^\infty\},$$
$$\mathrm{c}(\triangle) = \{x = (x_n)_n \subset \mathbb{R} | \triangle x \in \mathrm{c}\},$$
$$\mathrm{c}_0(\triangle) = \{x = (x_n)_n \subset \mathbb{R} | \triangle x \in \mathrm{c}_0\}.$$

Kızmaz investigated Köthe-Toeplitz Duals and some properties of these spaces.

Furthermore, Cesàro sequence spaces X^p of non-absolute type were defined by Ng and Lee [12] in 1977 as follows:

$$X^p = \left\{x = (x_n)_n \subset \mathbb{R} \middle| \left(\sum_{n=1}^{\infty} \left|\frac{1}{n}\sum_{k=1}^{n} x_k\right|^p\right)^{1/p} < \infty\right\}$$

and

$$X^\infty = \left\{x = (x_n)_n \subset \mathbb{R} \middle| \sup_n \left|\frac{1}{n}\sum_{k=1}^{n} x_k\right| < \infty\right\},$$

where $1 \le p < \infty$. They prove that X^p is linearly isomorphic and isometric to ℓ^p for $1 \le p \le \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Later, in 1983, Orhan [13] introduced Cesàro Difference Sequence Spaces by the following definitions:

$$C_p = \left\{x = (x_n)_n \subset \mathbb{R} \middle| \left(\sum_{n=1}^{\infty} \left|\frac{1}{n}\sum_{k=1}^{n} \triangle x_k\right|^p\right)^{1/p} < \infty\right\}$$

and

$$C_\infty = \left\{x = (x_n)_n \subset \mathbb{R} \middle| \sup_n \left|\frac{1}{n}\sum_{k=1}^{n} \triangle x_k\right| < \infty\right\},$$

where $1 \le p < \infty$. He noted that their norms are given as below for any $x = (x_n)_n$:

$$\|x\|_p^* = |x_1| + \left(\sum_{n=1}^{\infty} \left|\frac{1}{n}\sum_{k=1}^{n} \triangle x_k\right|^p\right)^{1/p} \quad \text{and} \quad \|x\|_\infty^* = |x_1| + \sup_n \left|\frac{1}{n}\sum_{k=1}^{n} \triangle x_k\right|$$

respectively.

Orhan showed that there exists a linear bounded operator $S: C_p \to C_p$ for $1 \le p \le \infty$ such that Köthe-Toeplitz β −Duals of these spaces are given respectively as follows:

$$S(C_p)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^q\} \text{ where } 1 < p < \infty \text{ and } q = \frac{p}{p-1},$$
$$S(C_1)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^\infty\} \text{ and}$$
$$S(C_\infty)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^1\}.$$

It might be better to use the notation $X^p(\triangle)$ instead of C_p for $1 \le p \le \infty$ since we also recalled the difference sequence spaces and used similar type of notation.

We note that Orhan also proved that $X^p \subset X^p(\triangle)$ for $1 \le p \le \infty$ strictly. Also, one can clearly see that $X^p(\triangle)$ is linearly isomorphic and isometric to ℓ^p for $1 \le p \le \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Note also that Köthe-Toeplitz Dual for $p = \infty$ case in Orhan's study and ℓ^∞ case in Kızmaz study coincides.

Furthermore, Et and Çolak [6] generalized the spaces introduced in Kızmaz's work [10] in the following way for $m \in \mathbb{N}$.

$$\ell^{\infty}(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \ell^{\infty}\},$$
$$c(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in c\},$$
$$c_0(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in c_0\}$$

where $\triangle x = (\triangle x_k) = (x_k - x_{k+1})_k$, $\triangle^0 x = (x_k)_k$, $\triangle^m x = (\triangle^m x_k) = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1})_k$ and $\triangle^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$.

Also, Et [7] and Tripathy et. al. [15] generalized the space introduced by Orhan in the following way for $m \in \mathbb{N}$.

$$X^p(\triangle^m) = \left\{x = (x_n)_n \subset \mathbb{R} \middle| \left(\sum_{n=1}^{\infty} \left|\frac{1}{n}\sum_{k=1}^{n} \triangle^m x_k\right|^p\right)^{1/p} < \infty\right\}$$

and

$$X^{\infty}(\triangle^m) = \left\{x = (x_n)_n \subset \mathbb{R} \middle| \sup_n \left|\frac{1}{n}\sum_{k=1}^{n} \triangle^m x_k\right| < \infty\right\},$$

Then, it is seen that that Köthe-Toeplitz Dual for$p = \infty$ case in Et's study [7] and ℓ^{∞} case in Et and Çolak study [6] coincides such that Köthe-Toeplitz Dual was given as below for any $m \in \mathbb{N}$.

$$D_m := \{a = (a_n)_n \subset \mathbb{R} | (n^m a_n)_n \in \ell^1\}$$
$$= \left\{a = (a_k)_k \subset \mathbb{R} : \ \|a\| = \sum_{k=1}^{\infty} k^m |a_k| < \infty\right\}.$$

Note that $D_m \subset \ell^1$for any $m \in \mathbb{N}$.

One can see that corresponding function space for these duals can be given as below:

$$U_m := \left\{ \begin{matrix} f: [0,1] \to \mathbb{R} \\ \text{measurable} \end{matrix} : \|f\| = \int_0^1 t^m |f(t)| dt < \infty \right\}.$$

Note that $L_1[0,1] \subset U_m$ and D_m is the space when counting measure is used for U_m.

Dowling et al. [5] showed that if a Banach space contains an isomorphic copy of ℓ^1, then it fails the fixed point property for uniform Lipschitz mappings. In this study, we work on a right shift mapping defined on a closed, bounded and convex subset of a Köthe-Toeplitz Dual of a Cesàro Difference Sequence Space so that the right shift mapping can be a uniform Lipschitz mapping. Thus, we investigate an upper bound estimate for the right shift mapping to be uniformly Lipschitz failing the fixed point property on a class of closed, bounded and convex subsets in in Köthe-Toeplitz Dual for $X^{\infty}(\triangle)$, the space D_1.

Now we provide some preliminaries before giving our main results.

Definition 1.1. *Let $(X, \|\cdot\|)$ be a Banach space and C is a non-empty closed, bounded, convex subset.*

1. If $T: C \to C$ is a mapping such that for all $\lambda \in [0,1]$ and for all $x, y \in C$, $T\big((1-\lambda)x + \lambda\, y\big) = (1-\lambda)T(x) + \lambda\, T(y)$ then T is said to be an affine mapping.

2. If $T: C \to C$ is a mapping such that $\| T(x) - T(y) \| \leq \| x - y \|$, for all $x, y \in C$ then T is said to be a nonexpansive mapping.

Also, if for every nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for nonexpansive mappings [fpp(ne)].

3. If $T: C \to C$ is a mapping such that there exists a constant $k \in [1, \infty)$ and for all $n \in \mathbb{N}$ and $\| T^n(x) - T^n(y) \| \leq k \| x - y \|$, for all $x, y \in C$ then T is said to be a uniformly Lipschitz mapping and k is said to be uniform Lipshcitz constant.

Also, if for every uniformly Lipschitz mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for uniformly Lipschitz mappings [fpp-uL].

2. Main Result

In this section, we find an upper bound estimate for the right shift mapping to be uniformly Lipschitz failing the fixed point property on a class of closed, bounded and convex subsets in Köthe-Toeplitz Dual ofa Cesàro

Difference Sequence Space$X^\infty(\triangle)$, the space D_1 given above for $m = 1$, which is also $S(C_\infty)^\beta$. We investigate right shift mapping defined on a large class of closed, bounded and convex subsets of $S(C_\infty)^\beta$. We want them to fail the fixed point property for the right shift mapping while the mapping becomes uniformly Lipschitz since we know by the result of Dowling et al. [5] it was shown that if a Banach space contains an isomorphic copy of ℓ^1, then it fails the fixed point property for uniform Lipschitz mappings. Now, we consider the following class of closed, bounded and convex subsets. Note that here we will be using the ideas similar to those in the section 3.2 of Ph.D. thesis of Everest [8], written under supervision of Chris Lennard.

Example 2.1. *Fix $b \in (0,1)$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1$, and $f_n := \frac{1}{n} e_n$, for all integers $n \geq 2$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Next, define the closed, bounded, convex subset $E = E_b$ of $S(C_\infty)^\beta$ by*

$$E := \left\{ \sum_{n=1}^{\infty} t_n f_n : \forall n \in \mathbb{N}, \quad t_n \geq 0 \ \ and \ \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

Consider the right shift mapping $T: E \to E$ defined by

$$T(x) = T\left(\sum_{n=1}^{\infty} t_n f_n\right) = \sum_{n=1}^{\infty} t_n f_{n+1}.$$

Then, for any $x = \sum_{k=1}^{\infty} t_k f_k$ and $y = \sum_{k=1}^{\infty} \gamma_k f_k$ in E. It is easy to see that T is affine and fixed point free. Moreover, for every $m \in \mathbb{N}$,

$$\begin{aligned} \|T^m y - T^m x\| &= \left\|\sum_{k=1}^{\infty} t_k f_{k+m} - \sum_{k=1}^{\infty} \gamma_k f_{k+m}\right\| = |t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &= \left|1 - \sum_{k=2}^{\infty} t_k - 1 + \sum_{k=2}^{\infty} \gamma_k\right| + \sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &= \left|\sum_{k=2}^{\infty} t_k - \gamma_k\right| + \sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &\leq 2 \sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &\leq 2\left(b|t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k|\right). \end{aligned}$$

But since

$$\begin{aligned} \|y - x\| &= \left\|\sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k\right\| \\ &= b|t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k|, \end{aligned} \tag{2.1}$$

we get that for any $m \in \mathbb{N}$,

$$\|T^m y - T^m x\| \leq 2\|y - x\|.$$

Also, for any $m \in \mathbb{N}$,

$$\begin{aligned} \|T^m y - T^m x\| &= |t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &= \frac{1}{b}\left(b|t_1 - \gamma_1| + b\sum_{k=2}^{\infty} |t_k - \gamma_k|\right) \\ &\leq \frac{1}{b}\left(b|t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k|\right) \end{aligned}$$

$$= \frac{1}{b}\|y - x\| \ \ by\ the\ equality\ (2.1)$$

Hence, for any $m \in \mathbb{N}$, $\|T^m y - T^m x\| \le min\left\{2, \frac{1}{b}\right\}\|y - x\|$. *Therefore, the right shift* T *is uniformly Lipschitz with Lipschitz coefficient* $M_b = min\left\{2, \frac{1}{b}\right\}$. *Now, we find the smallest possible Lipschitz coefficient in the following theorem.*

In fact, as we see that we get exactly similar results and calculations to those of the section 3.2 in Everest's Ph.D. thesis [8] , the following theorem will be also obtained similarly to his proof method.

Theorem 2.1. *Fix* $b \in (0,1)$. *Define a sequence* $(f_n)_{n\in\mathbb{N}}$ *by setting* $f_1 := b\, e_1$, *and* $f_n := \frac{1}{n} e_n$, *for all integers* $n \ge 2$ *where the sequence* $(e_n)_{n\in\mathbb{N}}$ *is the canonical basis of both* c_0 *and* ℓ^1. *Next, define the closed, bounded, convex subset* $E = E_b$ *of* $S(C_\infty)^\beta$ *by*

$$E := \left\{\sum_{n=1}^{\infty} t_n f_n : \ \forall n \in \mathbb{N}, \qquad t_n \ge 0 \text{ and} \sum_{n=1}^{\infty} t_n = 1\right\}.$$

Consider the right shift mapping $T: E \to E$ *defined by*

$$T(x) = T\left(\sum_{n=1}^{\infty} t_n f_n\right) = \sum_{n=1}^{\infty} t_n f_{n+1}.$$

Then, for any $x = \sum_{k=1}^{\infty} t_k f_k$ *and* $y = \sum_{k=1}^{\infty} \gamma_k f_k$ *in* E *and* $\forall m \in \mathbb{N}$,

$$\|T^m y - T^m x\| \le \frac{2}{1+b}\|y - x\|$$

such that $\frac{2}{1+b}$ *is the smallest possible uniform Lipschitz constant satisfying the above condition.*

Proof. Fix $m \in \mathbb{N}$. Let $= \sum_{k=1}^{\infty} t_k f_k$ *and* $y = \sum_{k=1}^{\infty} \gamma_k f_k$ *in* E. *Let* $\tau \in [0,1]$.Then we have

$$\|T^m y - T^m x\| = \left\|\sum_{k=1}^{\infty} t_k f_{k+m} - \sum_{k=1}^{\infty} \gamma_k f_{k+m}\right\|$$

$$= |t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k|$$

$$= \tau|t_1 - \gamma_1| + (1-\tau)|t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k|$$

$$= \tau|t_1 - \gamma_1| + (1-\tau)\left|1 - \sum_{k=2}^{\infty} t_k - 1 + \sum_{k=2}^{\infty} \gamma_k\right| + \sum_{k=2}^{\infty} |t_k - \gamma_k|$$

$$= \tau|t_1 - \gamma_1| + (1-\tau)\left|\sum_{k=2}^{\infty} t_k - \gamma_k\right| + \sum_{k=2}^{\infty} |t_k - \gamma_k|$$

$$\le \tau|t_1 - \gamma_1| + (2-\tau)\sum_{k=2}^{\infty} |t_k - \gamma_k| \tag{2.2}$$

$$= \frac{\tau}{b}\left(b|t_1 - \gamma_1| + \frac{(2-\tau)b}{\tau}\sum_{k=2}^{\infty} |t_k - \gamma_k|\right) \tag{2.3}$$

$$= (2-\tau)\left(\frac{\tau}{(2-\tau)}|t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k|\right) \tag{2.4}$$

Here, to use (2.3) so that we can get uniform Lipschitz estimate, we need

$$\frac{(2-\tau)b}{\tau b} \le 1 \Leftrightarrow 2b - \tau b \le \tau b \Leftrightarrow \frac{2b}{b+1} \le \tau.$$

Then, to minimize coefficient in (2.3), which is $\frac{\tau}{b}$, we minimize τ, so by the above fact, minimum value for τ satisfying (2.3) would be $\frac{2b}{b+1}$. Thus, minimum coefficient $\frac{\tau}{b}$ is $\frac{2}{b+1}$ satisfying (2.3).

But to use (2.4) so that we can get uniform Lipschitz estimate, we need $\frac{\tau}{(2-\tau)} \leq b$ and we minimize $2-\tau$. Here we note that

$$\frac{\tau}{(2-\tau)} \leq b \Leftrightarrow \tau \leq 2b - \tau b \Leftrightarrow \tau \leq \frac{2b}{b+1}.$$

Then, to minimize $2-\tau$ we would maximize τ and that maximum value of τ would be $\frac{2b}{b+1}$ by the above fact. So minimum value for $2-\tau$ in (2.4) is $\frac{2}{b+1}$. That is, minimum coefficient $2-\tau$ in (2.4) is $\frac{2}{b+1}$.
Therefore, from both results, we can say that for any $m \in \mathbb{N}$,

$$\|T^m y - T^m x\| \leq M_b \left(b|t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k| \right) = M_b \|y - x\|$$

and M_b might be $\frac{2}{b+1}$. In fact, the following fact tells us that smallest possible coefficient M_b is $\frac{2}{b+1}$.
Indeed, consider $x := f_1$ and $y := f_2$. Then, $\|y - x\| = \left\|\frac{1}{2}e_2 - be_1\right\| = \left\|\left(-b, \frac{1}{2}, 0,0,0, \ldots\right)\right\| = b + 1.$
Then, $\|T^m y - T^m x\| = \|T^m f_2 - T^m f_1\| = \|f_{m+2} - f_{m+1}\| = \left\|\frac{1}{m+2}e_{m+2} - \frac{1}{m+1}e_{m+1}\right\| = \frac{m+2}{m+2} + \frac{m+1}{m+1} = 2 = \frac{2}{b+1}(b+1) = \frac{2}{b+1}\|y - x\| = M_b\|y - x\|.$

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