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A Large Class in Köthe-Toeplitz Duals of Cesàro Difference Sequence Spaces with Fixed Point Property for Asymptotically Nonexpansive Mappings

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Abstract In 1970, Cesàro Sequence Spaces was introduced by Shiue. In 1981, Kızmaz defined difference sequence spaces for ℓ^∞, c_0 and c. Then, in 1983, Orhan introduced Cesàro Difference Sequence Spaces. In this study, first we discuss the fixed point property for these spaces. Then, we recall thatGoebel and Kuczumow showed that there exists a very large class of closed, bounded, convex subsets in Banach space of absolutely summable scalar sequences, ℓ^1 with fixed point property for nonexpansive mappings. In 2004, Kaczor and Prus investigated if similar result could be done for asymptotically nonexpansive mappings and they saw that there exists a large class of closed, bounded, convex subsets in ℓ^1 with fixed point property for affine asymptotically nonexpansive mappings. So we consider a Kaczor and Prus analogue result for a Köthe-Toeplitz Dual of a Cesàro Difference Sequence Space. We show that there exists a large class of closed, bounded and convex subsets of these spaces with fixed point property for affine asymptotically nonexpansive mappings.

Keywords Fixed point property, asymptotically nonexpansive mapping, Cesàro difference sequences, Köthe-Toeplitz dual
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1. Introduction

There is a strong relation between reflexivity and fixed point property for non-expansive mappings. It is an open question whether or not every non-reflexive fails the fixed point property for non-expansive mappings but it was shown by Lin [13] that a non-reflexive Banach space failing to have the fixed point property for non-expansive mappings can be renormed to have the fixed point property for non-expansive mappings. Lin showed this fact by setting an equivalent norm on Banach space of absolutely summable scalar sequences, ℓ^1. Because of sharing many common properties, it is natural to ask if , Banach space of scalar sequences converging to 0, c_0 can be renormed to have the fixed point property for non-expansive mappings as another well known classical non-reflexive Banach space. Maria and Hernandes Lineares [14] obtained the first example for the class of nonreflexive Banach spaces which can be renormed to have the fixed point property for affine nonexpansive mappins and their space was the Banach space of Lebesgue integrable functions on [0,1], $L_1[0,1]$. It can be said that all these works are inspired by the work of Goebel and Kuczumow [9]. Goebel and Kuczumow showed that there exists very large class of non-weakly compact, closed, bounded and convex subsets of ℓ^1 respect to weak* topology of ℓ^1 with fixed point property for non-expansive mappings. Later, Kaczor and Prus [10] investigated if similar result could be done for asymptotically nonexpansive mappings and they saw that there exists a large class of closed, bounded, convex subsets in ℓ^1 with fixed point property for affine asymptotically non-expansive mappings. Previously, Everest, in his Ph.D. thesis [7], written under supervision of Chris Lennard, considered large classes in ℓ^1 with fixed point property for affine asymptotically non-expansive mappings by generalizing Kaczor and Prus' work.

In this study, we work on Kaczor and Prus analogy for a Banach space contained in ℓ^1, which is a smaller space than ℓ^1. The space we consider is a Köthe-Toeplitz Dual of a Cesàro Difference Sequence Space. We show that there exists a very large class of closed, bounded and convex subsets of the space with the fixed point property for asymptotically non-expansive mappings under affinity condition.

We recall that the Cesàro sequence spaces

$$\mathrm{ces}_p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

were introduced by Shiue [17] in 1970, where $1 \le p < \infty$. It has been shown that $\ell^p \subset \mathrm{ces}_p$ for $1 < p \le \infty$. Moreover, it has been shown that Cesàro sequence spaces ces_p for $1 < p < \infty$ are seperable reflexive Banach spaces. Furthermore, it was also proved by Cui-Hudzik [2], Cui-Hudzik-Li [3] and Cui-Meng-Pluciennik [4] that Cesàro sequence spaces ces_p for $1 < p < \infty$ have the fixed point property. They prove this result using different methods. One method is to calculate Garcia-Falset coefficient. It is known that if Garcia-Falset coefficient is less than 2 for a Banach space, then it has the fixed point property for nonexpansive mappings [8]. Using this fact, since they calculate this coeeficient for ces_p as $2^{1/p}$ similary to what it is for ℓ^p, they point the result for the Cesàro sequence spaces. Another fact is that they see that the space has normal structure for $1 < p < \infty$. Then using the fact via Kirk [12] that reflexive Banach spaces with normal structure has the fixed point property, they easily deduce that the space has the fixed point property for $1 < p < \infty$. Their results on Cesàro sequence spaces as a survey can be seen in [1].

Later, in 1981, Kızmaz [11] introduced difference sequence spaces for ℓ^∞, cand c_0 where they are the Banach spaces of bounded, convergent and null sequences $x = (x_n)_n$, respectively. As it is seen below, his definitions for these spaces were given using difference operator applied to the sequence x, $\triangle x = (x_k - x_{k+1})_k$.

$$\ell^\infty(\triangle) = \{x = (x_n)_n \subset \mathbb{R} |\, \triangle x \in \ell^\infty\},$$
$$c(\triangle) = \{x = (x_n)_n \subset \mathbb{R} |\, \triangle x \in c\},$$
$$c_0(\triangle) = \{x = (x_n)_n \subset \mathbb{R} |\, \triangle x \in c_0\}.$$

Kızmaz investigated Köthe-Toeplitz Duals and some properties of these spaces.

Furthermore, Cesàro sequence spaces X^p of non-absolute type were defined by Ng-Lee [15] in 1977 as follows:

$$X^p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| < \infty \right\},$$

where $1 \le p < \infty$. They prove that X^p is linearly isomorphic and isometric to ℓ^p for $1 \le p \le \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Later, in 1983, Orhan [16] introduced Cesàro Difference Sequence Spaces by the following definitions:

$$C_p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right| < \infty \right\},$$

where $1 \le p < \infty$. He noted that their norms are given as below for any $x = (x_n)_n$:

$$\|x\|_p^* = |x_1| + \left(\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\triangle x_k\right|^p\right)^{1/p} \text{ and } \|x\|_\infty^* = |x_1| + \sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\triangle x_k\right|$$

respectively.

Orhan showed that there exists a linear bounded operator $S: C_p \to C_p$ for $1 \le p \le \infty$ such that Köthe-Toeplitz β −Duals of these spaces are given respectively as follows:

$$S(C_p)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^q\} \text{ where } 1 < p < \infty \text{ and } q = \frac{p}{p-1},$$

$$S(C_1)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^\infty\} \text{ and}$$

$$S(C_\infty)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^1\}.$$

It might be better to use the notation $X^p(\triangle)$ instead of C_p for $1 \le p \le \infty$ since we also recalled the difference sequence spaces and used similar type of notation.

We note that Orhan also proved that $X^p \subset X^p(\triangle)$ for $1 \le p \le \infty$ strictly. Also, one can clearly see that $X^p(\triangle)$ is linearly isomorphic and isometric to ℓ^p for $1 \le p \le \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Note also that Köthe-Toeplitz Dual for $p = \infty$ case in Orhan's study and ℓ^∞ case in Kızmaz study coincides.

Furthermore, Et and Çolak [5] generalized the spaces introduced in Kızmaz's work [11] in the following way for $m \in \mathbb{N}$.

$$\ell^\infty(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \ell^\infty\},$$

$$\mathrm{c}(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \mathrm{c}\},$$

$$\mathrm{c}_0(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \mathrm{c}_0\}$$

where $\triangle x = (\triangle x_k) = (x_k - x_{k+1})_k$, $\triangle^0 x = (x_k)_k$, $\triangle^m x = (\triangle^m x_k) = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1})_k$ and $\triangle^m x_k = \sum_{i=0}^{m}(-1)^i\binom{m}{i} x_{k+i}$.

Also, Et [6] and Tripathy et. al. [18] generalized the space introduced by Orhan in the following way for $m \in \mathbb{N}$.

$$X^p(\triangle^m) = \left\{x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\triangle^m x_k\right|^p\right)^{1/p} < \infty\right\}$$

and

$$X^\infty(\triangle^m) = \left\{x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\triangle^m x_k\right| < \infty\right\},$$

Then, it is seen that that Köthe-Toeplitz Dual for$p = \infty$ case in Et's study [6] and ℓ^∞ case in Et and Çolak study [5] coincides such that Köthe-Toeplitz Dual was given as below for any $m \in \mathbb{N}$.

$$D_m := \{a = (a_n)_n \subset \mathbb{R} | (n^m a_n)_n \in \ell^1\}$$

$$= \left\{a = (a_k)_k \subset \mathbb{R} : \ \|a\| = \sum_{k=1}^{\infty} k^m |a_k| < \infty\right\}.$$

Note that $D_m \subset \ell^1$for any $m \in \mathbb{N}$.

One can see that corresponding function space for these duals can be given as below:

$$U_m := \left\{\begin{matrix} f: [0,1] \to \mathbb{R} \\ \text{measurable} \end{matrix} : \|f\| = \int_0^1 t^m |f(t)| dt < \infty \right\}.$$

Note that $L_1[0,1] \subset U_m$ and D_m is the space when counting measure is used for U_m.

As we have already stated, in this study, we consider Kaczor and Prus [10] analogy for a Köthe-Toeplitz Dual of a Cesàro Difference Sequence Space. We show that there exists a large class of closed, bounded and convex subsets of Köthe-Toeplitz Dual for $X^\infty(\triangle)$ with fixed point property for affine asymptotically nonexpansive mappings.

Now we provide some preliminaries before giving our main results.

Definition 1.1. *Let* $(X, \|\cdot\|)$ *be a Banach space and* C *is a non-empty closed, bounded, convex subset.*

1. If $T: C \to C$ is a mapping such that for all $\lambda \in [0,1]$ and for all $x, y \in C$, $T\big((1-\lambda)x + \lambda\, y\big) = (1-\lambda)T(x) + \lambda\, T(y)$ then T is said to be an affine mapping.

2. If $T: C \to C$ is a mapping such that $\| T(x) - T(y) \| \leq \| x - y \|$, for all $x, y \in C$ then T is said to be a nonexpansive mapping.

Also, if for every nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for nonexpansive mappings [fpp(ne)].

3. If $T: C \to C$ is a mapping such that there exists a sequence of scalars $(k_n)_{n\in N}$ decreasingly approach to 1 and $\| T^n(x) - T^n(y) \| \leq k_n \| x - y \|$, for all $x, y \in C$ and for all $n \in \mathbb{N}$ then T is said to be an asymptotically nonexpansive mapping.

Also, if for every asymptotically nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for asymptotically nonexpansive mappings [fpp(ane)].

Remark 1.1. *In 1979, Goebel and Kuczumow [9] showed there exists a large class of closed, bounded and convex subsets of ℓ^1 using a key lemma they obtained. Their lemma says that if $\{x_n\}$ is a sequence in ℓ^1 converging to x in weak-star topology, then for any $y \in \ell^1$,*

$$r(y) = r(x) + \|y - x\|_1 \ \ where\ \ r(y) = \limsup_n \|x_n - y\|_1 \ .$$

Since Köthe-Toeplitz Dual for $X^\infty(\triangle)$ is contained in ℓ^1 and in fact it is isometrically isomorphic to ℓ^1, Goebel and Kuczumow's lemma above (Lemma 1 in [9]) applies in Köthe-Toeplitz Dual for $X^\infty(\triangle)$. We will call this fact ∴.

2. Main Result

In this section, we consider Kaczor and Prus [10] analogy for a Köthe-Toeplitz Dual of a Cesàro Difference Sequence Space. We show that there exists a large class of closed, bounded and convex subsets of Köthe-Toeplitz Dual for $X^\infty(\triangle)$ with fixed point property for affine asymptotically nonexpansive mappings.

Now, we consider the following class of closed, bounded and convex subsets. Note that here we will be using the ideas similar to those in the section 3 of Ph.D. thesis of Everest [7], written under supervision of Chris Lennard.

Example 2.1. *Fix $b \in (0,1)$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1$, and $f_n := \frac{1}{n} e_n$, for all integers $n \geq 2$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Next, we can define a closed, bounded, convex subset $E = E_b$ of $S(C_\infty)^\beta$ by*

$$E := \left\{ \sum_{n=1}^{\infty} t_n f_n :\ \forall n \in \mathbb{N}, \quad t_n \geq 0 \ \ and \ \ \sum_{n=1}^{\infty} t_n = 1 \right\} .$$

Theorem 2.1. *For any $b \in (0,1)$, the set E defined as in the example above has the fixed point property for affine asymptotically $\|.\|$-nonexpansive mappings.*

Proof. Fix $b \in (0,1)$.Let $T: E \to E$ be an affine asymptotically nonexpansive mapping. Then, since T is affine, by Lemma 1.1.2 in the Ph.D. thesis of Everest [7] written under supervision of Lennard, there exists a sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}} \in E$ such that $\left\|Tx^{(n)} - x^{(n)}\right\| \underset{n}{\to} 0$. Without loss of generality, passing to a subsequence if necessary, there exists $x \in S(C_\infty)^\beta$ such that $x^{(n)}$ converges to x in weak* topology. Then, by Goebel Kuczumow analog fact ∴ given in the last part of the previous section, we can define a function $s: S(C_\infty)^\beta \to [0,\infty)$ by

$$s(y) = \limsup_n \left\|x^{(n)} - y\right\| \quad , \ \forall y \in S(C_\infty)^\beta$$

and so

$$s(y) = s(y) + \|x - y\| \ , \ \forall y \in S(C_\infty)^\beta.$$

Now define the weak* closure of the set E as it is seen below.

$$W := \overline{E}^{w^*} = \left\{ \sum_{n=1}^{\infty} t_n f_n :\ each\ \ t_n \geq 0 \ \ and \ \ \sum_{n=1}^{\infty} t_n \leq 1 \right\}$$

Since T is asymptotically nonexpansive mapping, there exists a decreasing sequence $(\mathrm{k_n})_{\mathrm{n\in N}}$ in $[1,\infty)$ converging to 1 such that $\forall \mathrm{x}, y \in \mathrm{E}$ and $\forall \mathrm{n} \in \mathbb{N}$,

$$\|T^n\mathrm{x} - T^n\mathrm{y}\| \leq \mathrm{k_n} \parallel \mathrm{x} - \mathrm{y} \parallel.$$

Case 1: $x \in E$.

Fix $\mathrm{m} \in \mathbb{N}$. Then, we have $s(T^m x) = s(x) + \|T^m x - x\|$ and $\forall \mathrm{n} \in \mathbb{N}$,

$$\begin{aligned} s(T^m x) &= \limsup_n \left\|T^m x - x^{(n)}\right\| \\ &\leq \limsup_n \left\|T^m x - T^m\left(x^{(n)}\right)\right\| + \limsup_n \left\|x^{(n)} - T^m\left(x^{(n)}\right)\right\| \qquad (2.1) \\ &\leq \mathrm{k_m} \limsup_n \left\|x - x^{(n)}\right\| + \limsup_n \left\|x^{(n)} - T^m\left(x^{(n)}\right)\right\| \\ &\leq \mathrm{k_m} \limsup_n \left\|x - x^{(n)}\right\| + \limsup_n \sum_{j=1}^{m} \left\|T^{j-1}\left(x^{(n)}\right) - T^j\left(x^{(n)}\right)\right\| \\ &\leq \mathrm{k_m} \limsup_n \left\|x - x^{(n)}\right\| + \limsup_n \sum_{j=1}^{m} \mathrm{k_{j-1}} \left\|x^{(n)} - T\left(x^{(n)}\right)\right\| \\ &= \mathrm{k_m} s(x). \end{aligned}$$

Therefore, $\|T^m x - x\| \leq (\mathrm{k_m} - 1)s(x)$ and so by taking limit as $\mathrm{m} \to \infty$, we have $\lim_m \|T^m x - x\| = 0$ but then since $\lim_m \|TT^m x - \mathrm{T}x\| \leq \lim_m \mathrm{k_1} \|T^m x - x\| = 0$, $\lim_m \|T^{m+1} x - \mathrm{T}x\| = 0$ and so $T^m x$ converges to x and Tx. Thus, by the uniqueness of limits $Tx = x$.

Case 2: $x \in W \backslash E$.

Then, x is of the form $\sum_{n=1}^{\infty} \gamma_n f_n$ such that $\sum_{n=1}^{\infty} \gamma_n < 1$ *and* $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$.

Define $\delta := 1 - \sum_{n=1}^{\infty} \gamma_n$ and next define

$$\mathrm{h} := (\gamma_1 + \delta) f_1 + \sum_{n=2}^{\infty} \gamma_n f_n.$$

Then, $\|\mathrm{h} - x\|_1 = \|b\delta e_1\|_1 = b\delta$.

Now fix $y \in E$ of the form $\sum_{n=1}^{\infty} t_n f_n$ such that $\sum_{n=1}^{\infty} t_n = 1$ with $t_n \geq 0$, $\forall n \in \mathbb{N}$.

Then,

$$\begin{aligned} \|\mathrm{y} - x\| &= \left\|\sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k\right\| = \mathrm{b}|t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &= \mathrm{b}|t_1 - \gamma_1| + b\sum_{k=2}^{\infty} |t_k - \gamma_k| + (1-b)\sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &\geq \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \gamma_k\right| + (1-b)\sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &= \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \gamma_k\right| + (1-b)\sum_{k=2}^{\infty} |t_k - \gamma_k| \\ &= \mathrm{b}|1 - (1-\delta)| + (1-b)\sum_{k=2}^{\infty} |t_k - \gamma_k| \end{aligned}$$

Hence,

$$\|\mathrm{y} - x\| \geq b\delta + (1-b)\sum_{k=2}^{\infty} |t_k - \gamma_k| \geq \|\mathrm{h} - x\| .$$

Next, we have the following.

$$\begin{aligned} s(h) = s(x) + \|\mathrm{h} - x\| \leq s(x) + \|T^m\mathrm{h} - x\| &= s(T^m\mathrm{h}) \\ &= \limsup_n \left\|T^m h - x^{(n)}\right\| \end{aligned}$$
then similarly to the inequality (1)

$$\leq \limsup_n \|T^m h - T^m(x^{(n)})\| + \limsup_n \|x^{(n)} - T^m(x^{(n)})\|$$
$$\leq \mathrm{k_m} \limsup_n \|h - x^{(n)}\| + \limsup_n \|x^{(n)} - T^m(x^{(n)})\|$$
$$\leq \mathrm{k_m} \limsup_n \|h - x^{(n)}\| + \limsup_n \sum_{j=1}^{m} \|T^{j-1}(x^{(n)}) - T^j(x^{(n)})\|$$
$$\leq \mathrm{k_m} \limsup_n \|h - x^{(n)}\| + \limsup_n \sum_{j=1}^{m} \mathrm{k_{j-1}} \|x^{(n)} - T(x^{(n)})\|$$
$$= \mathrm{k_m} s(h).$$

Hence, $s(h) \leq s(T^m \mathrm{h}) \leq \mathrm{k_m} s(h)$ and so taking limit as $\mathrm{m} \to \infty$, we have $\lim_m s(T^m \mathrm{h}) = s(h)$ since $\lim_m \mathrm{k_m} = 1$. That is, $\lim_m s(x) + \|T^m \mathrm{h} - x\| = s(x) + \|\mathrm{h} - x\|$ which means $\lim_m \|T^m \mathrm{h} - x\| = \|\mathrm{h} - x\|$. (2)

Moreover, for any $y \in E$,

$$\|\mathrm{y} - h\| = \left\|\sum_{k=1}^{\infty} t_k f_k - (\gamma_1 + \delta) f_1 - \sum_{k=2}^{\infty} \gamma_k f_k\right\|$$
$$= \left\|\sum_{k=2}^{\infty} (t_k - \gamma_k) f_k - (\gamma_1 + \delta - t_1) f_1\right\|$$
$$= \sum_{k=2}^{\infty} |t_k - \gamma_k| + b|\gamma_1 + \delta - t_1|$$
$$= \sum_{k=2}^{\infty} |t_k - \gamma_k| + b\left|\gamma_1 + 1 - \sum_{k=1}^{\infty} \gamma_k - 1 + \sum_{k=2}^{\infty} t_k\right|$$
$$\leq \sum_{k=2}^{\infty} |t_k - \gamma_k| + \mathrm{b} \sum_{k=2}^{\infty} |t_k - \gamma_k|$$
$$= (1 + \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k|$$
$$= \frac{1+b}{1-b} (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k|$$
$$= \frac{1+b}{1-b} \left[b\delta - b\delta + (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| \right]$$
$$= \frac{1+b}{1-b} \left[b\big(1 - (1 - \delta)\big) - b\delta + (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| \right]$$
$$= \frac{1+b}{1-b} \left[b\big(1 - (1 - \delta)\big) + (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| - b\delta \right]$$
$$= \frac{1+b}{1-b} \left[b\left(\sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \gamma_k \right) + (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| - b\delta \right]$$
$$\leq \frac{1+b}{1-b} \left[b \sum_{k=1}^{\infty} |t_k - \gamma_k| + (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| - b\delta \right]$$

Hence,

$$\|y-h\| \leq \frac{1+b}{1-b}\left[b|t_1-\gamma_1|+\sum_{k=2}^{\infty}|t_k-\gamma_k|-b\delta\right]=\frac{1+b}{1-b}[\|y-x\|-\|h-x\|]$$

Now, fix $\varepsilon > 0$ and recall that $b \in (0,1)$. Then, we can choose $\mu(\varepsilon) := \frac{1-b}{1+b}\varepsilon \in (0,\infty)$ such that for any $y = \sum_{k=1}^{\infty} t_k f_k \in E$,

$$|\|y-x\|-\|h-x\|| \leq \|y-x\|-\|h-x\| < \mu.$$

Then, $\|y-h\| < \frac{1+b}{1-b}\mu = \varepsilon$.

Hence, for every $\varepsilon > 0$, there exists $\mu = \mu(\varepsilon)$ such that if $|\|y-x\|-\|h-x\|| < \mu$ then $\|y-h\| < \varepsilon$ so this implies for any sequence $(z_n)_n$ in E with $\lim_n \|z_n - x\| = \|h-x\|$ implies $\lim_n \|z_n - h\| = 0$. But then since in (2) we obtained $\lim_m \|T^m h - x\| = \|h - x\|$, we have $\lim_m \|T^m h - h\| = 0$.

Furthermore,

$$\|h-Th\| \leq \lim_m \|T^m h - h\| + \lim_m \|T^m h - Th\|$$
$$\leq k_1 \lim_m \|T^{m-1} h - h\| = 0$$

Hence, $Th = h$ and so E has fpp(n.e.) as desired.

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