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The Fekete-Szegö Problem for Certain Subclass Bi-univalent Functions of Complex Order

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Abstract In this paper, we introduce and investigate a subclass of analytic and bi-univalent functions of complex order on the open unit disk in the complex plane. Here, we solve the Fekete-Szegö problem for this function class.

Keywords Analytic functions, bi-univalent functions, coefficient bounds, Fekete-Szegö functional
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1. Introduction

Let A be the class of the functions in the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}, \tag{1.1}$$

which are analytic on the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ in the complex plane.

We denote by S the subclass of A consisting of the functions which are also univalent in U. Some of the important subclass of S is the class $\Re(\alpha,\beta)$ that is defined as follows

$$\Re(\alpha,\beta)=\{f\in S:\mathrm{Re}(f'(z)+\beta zf''(z))>\alpha, z\in U\}, \alpha\in[0,1), \beta\geq 0.$$

Gao and Zhou [12] investigated the class $\Re(\alpha,\beta)$ and showed some mapping properties of this class.

Early, by Alintaş and Özkan [1] were investigated a subclass $\Re(\alpha,\beta,\tau)$, $\alpha\in(0,1]$, $\beta\in[0,1]$, $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$ of the analytic and bi-univalent functions consisting of functions $f\in S$ which satisfy the condition

$$f\in T,\ \left|\frac{1}{\tau}[f'(z)+\beta zf''(z)-1]\right|\leq\alpha, z\in U.$$

Here T is subclass of A consisting of the functions f in the form

$$f(z)=z-a_2z^2-a_3z^3-\cdots=z-\sum_{n=2}^{\infty}a_nz^n,\ a_n\geq 0.$$

They found necessary and sufficient conditions for the functions belonging to this class.

It is well-known that (see, for example, [8]) every function $f \in S$ has an inverse f^{-1} which is defined by

$$f^{-1}(f(z)) = z,\ z \in U,\ f(f^{-1}(w)) = w, w \in U_{r_0} = \{w : |w| < r_0(f)\}, r_0(f) \geq \frac{1}{4},$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots, w \in U_{r_0},$$
$$A_2 = -a_2,\ A_3 = 2a_2^2 - a_3,\ A_4 = -5a_2^3 + 5a_2 a_3 - a_4.$$

As known that a function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U and U_{r_0}, respectively. Let Σ denote the class of bi-univalent functions in U given (1.1).

In 1967, Lewin [18] showed that for every function $f \in \Sigma$ of the form (1.1) for the second coefficient true the inequality $|a_2| < 1.51$. In 1967, Brannan and Clunie [2] conjectured that $|a_2| < \sqrt{2}$ for $f \in \Sigma$. In 1984, Tan [26] obtained the inequality $|a_2| < 1.485$, which is the best known estimate for the functions in the class Σ. In 1985, Kedzierawski [15] proved the Brannan-Clunie conjecture for the bi-starlike functions. Brannan and Taha [3] obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for the functions in the classes of bi-starlike functions of order α and bi-convex functions of order α.

The study of the bi-univalent functions was revived by Srivastava et al. [23] and a considerably large number of sequels to the work of Srivastava et al. [23] have appeared in the literature. In particular, several results on coefficient estimates for the initial coefficients $|a_2|, |a_3|$ and $|a_4|$ were proved for various subclasses of Σ (see, for example, [6, 10, 14, 21, 24, 25, 27 and 28]).

Recently, Deniz [7] and Kumar et al. [17] both extended and improved the results of Brannan and Taha [3] by generalizing their classes by means of the principle of subordination between analytic functions.

Despite the numerous studies mentioned above, the problem of estimating the coefficients $|a_n|$ $(n = 2, 3, \ldots)$ for the general class functions Σ is still open (see also [25] in this connection).

As known that one of the important tools in the theory of analytic functions is the functional $H_2(1) = a_3 - a_2^2$, which is known as the Fekete-Szegö functional and one usually considers the further generalized functional $a_3 - \mu a_2^2$, where μ is real or complex number (see [9]). Also, finding the upper bound of $|a_3 - \mu a_2^2|$ is known as the Fekete-Szegö problem in the literature. In 1969, Keogh and Merkes [16] solved the Fekete-Szegö problem for the classes starlike and convex functions. Someone can see the Fekete-Szegö problem for the classes of starlike functions of order α and convex functions of order α at special cases in the paper of Orhan et al. [20]. On the other hand, recently, Çağlar and Aslan [4] have obtained Fekete-Szegö inequality for a subclass of the bi-univalent functions. Also, Zaprawa [29, 30] have studied on Fekete-Szegö problem for some subclasses of the bi-univalent functions. In special cases, he solved the Fekete-Szegö problem for the subclasses of the bi-starlike functions of order α and of bi-convex functions of order α.

Motivated by the aforementioned works, we define a new subclass of bi-univalent functions Σ as follows.

Definition 1.1. *A function* $f \in \Sigma$ *given by (1.1) is said to be in the class* $\Im_\Sigma(\alpha, \beta, \tau)$, $\alpha \in [0,1)$, $\beta \geq 0$, $\tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$, *if the following conditions are satisfied*

$$\operatorname{Re}\left\{1 + \frac{1}{\tau}\left[f'(z) + \beta z f''(z) - 1\right]\right\} > \alpha, z \in U$$

and

$$\mathrm{Re}\left\{1+\frac{1}{\tau}\left[g'(w)+\beta wg''(w)-1\right]\right\}>\alpha, w\in U_{r_0},$$

where $g=f^{-1}$ *is inverse of the function* f .

Remark 1.1. *If we take* $\tau=1$ *in Definition 1.1 we have function class* $\mathfrak{I}_{\Sigma}(\alpha,\beta,1)=H_{\Sigma}(\alpha,\beta)$, $\alpha\in[0,1),\beta\geq 0$. *That is,*

$$f\in H_{\Sigma}(\alpha,\beta)\Leftrightarrow \mathrm{Re}\left(f'(z)+\beta zf''(z)\right)>\alpha, z\in U$$

and

$$\mathrm{Re}\left(g'(w)+\beta wg''(w)\right)>\alpha, w\in U_{r_0},$$

where $g=f^{-1}$ *is inverse of the function* f .

Remark 1.2. *Choose* $\beta=0$ *in Definition 1.1 we have function class* $\mathfrak{I}_{\Sigma}(\alpha,0,\tau)$, $\alpha\in[0,1)$, $\tau\in\mathbb{C}^{*}=\mathbb{C}-\{0\}$. *That is,*

$$f\in\mathfrak{I}_{\Sigma}(\alpha,0,\tau)\Leftrightarrow \mathrm{Re}\left\{1+\frac{1}{\tau}\left[f'(z)-1\right]\right\}>\alpha, z\in U$$

and

$$\mathrm{Re}\left\{1+\frac{1}{\tau}\left[g'(w)-1\right]\right\}>\alpha, w\in U_{r_0},$$

where $g=f^{-1}$ *is inverse of the function* f .

Remark 1.3. *Letting* $\beta=0$, $\tau=1$ *in Definition 1.1, we have function class* $\mathfrak{I}_{\Sigma}(\alpha,0,1)=\mathfrak{R}_{\Sigma}(\alpha,0)$, $\alpha\in[0,1)$. *That is,*

$$f\in\mathfrak{R}_{\Sigma}(\alpha,0)\Leftrightarrow \mathrm{Re}\left(f'(z)\right)>\alpha, z\in U \quad \text{and} \quad \mathrm{Re}\left(g'(w)\right)>\alpha, w\in U_{r_0},$$

where $g=f^{-1}$ *is inverse of the function* f .

Remark 1.4. *Setting* $\beta=1$ *in Definition 1.1, we have function class* $\mathfrak{I}_{\Sigma}(\alpha,1,\tau)$, $\alpha\in[0,1)$, $\tau\in\mathbb{C}^{*}=\mathbb{C}-\{0\}$. *That is,*

$$f\in\mathfrak{I}_{\Sigma}(\alpha,1,\tau)\Leftrightarrow \mathrm{Re}\left\{1+\frac{1}{\tau}\left[f'(z)+zf''(z)-1\right]\right\}>\alpha, z\in U$$

and

$$\mathrm{Re}\left\{1+\frac{1}{\tau}\left[g'(w)+wg''(w)-1\right]\right\}>\alpha, w\in U_{r_0},$$

where $g=f^{-1}$ *is inverse of the function* f .

Remark 1.5. *Choose* $\beta=1$, $\tau=1$ *in Definition 1.1, we have function class* $\mathfrak{I}_{\Sigma}(\alpha,1,1)=\mathfrak{R}_{\Sigma}(\alpha,1)$, $\alpha\in[0,1)$. *That is,*

$$f \in \Re_{\Sigma}(\alpha,1) \Leftrightarrow \operatorname{Re}\left(f'(z)+zf''(z)\right) > \alpha, z \in U \text{ and } \operatorname{Re}\left(g'(w)+wg''(w)\right) > \alpha, w \in U_{r_0},$$

where $g = f^{-1}$ *is inverse of the function* f .

Recently, the class $\Im_{\Sigma}(\alpha,\beta,\tau)$ were investigated by Mustafa and Turaç [19]. They give the upper bound estimates for three initial coefficients of the functions belonging to this class.

The class $\Im_{\Sigma}(\alpha,0,1) = \Re_{\Sigma}(\alpha,0) = N_{\Sigma}(\alpha)$ were investigated by Srivastava et al. [24] and by Çağlar et al. [5].

Recently, by Frasin [11] investigated subclass $\Im_{\Sigma}(\alpha,\beta,1) = H_{\Sigma}(\alpha,\beta), \alpha \in [0,1), \beta > 0$ with condition $2(1-\alpha)\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{\beta n+1} \le 1$. He found estimates on two first coefficients for the functions in this class.

The object of the present paper is to find a upper bound estimate for the Fekete-Szegö functional of the functions belonging to the class $\Im_{\Sigma}(\alpha,\beta,\tau)$. In the study, results were also obtained for some subclasses from the results obtained.

To prove our main results, we need require the following lemmas.

Lemma 1.1. (See, [22]) *If* $p \in \mathrm{P}$ *, then the sharp estimates* $|p_n| \le 2, n = 1,2,3,\ldots$ *are provided, where* P *is the family of all functions* p *, analytic in* U *for which* $p(0)=1$ *and* $\operatorname{Re}(p(z)) > 0,\ z \in U$ *, and*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, z \in U. \tag{1.2}$$

Lemma 1.2. (See, [13]) *If* $p \in \mathrm{P}$ *, then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2\left(4-p_1^2\right)p_1 x - \left(4-p_1^2\right)p_1 x^2 + 2\left(4-p_1^2\right)\left(1-|x|^2\right)y$$

for some x, y *with* $|x| \le 1$, $|y| \le 1$.

Lemma 1.3. (See, [13]) *The power series given in (1.2) converges in* U *to the function* p *in* P *if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ p_{-1} & 2 & p_1 & \cdots & p_{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ p_{-n} & p_{-n+1} & p_{-n+2} & \cdots & 2 \end{vmatrix}, \ n = 1,2,3,\ldots$$

and $p_{-n} = \bar{p}_n$ *, are all nonnegative. They are strictly positive except for*

$$p(z) = \sum_{n=1}^{m} \rho_n p_0\left(e^{it_n z}\right),\ \rho_n > 0,\ t_n \ real$$

and $t_n \ne t_k$ *for* $n \ne k$ *in this case* $D_n > 0$ *for* $n < m-1$ *and* $D_n = 0$ *for* $n \ge m$.

Note 1.1. *If* $p \in \mathrm{P}$*, then according to Lemma 1.3* $p_n \geq 0$ *for each* $n = 1, 2, 3, \ldots$*. On the other hand, from the Lemma 1.1* $|p_n| \leq 2$ *for each* $n = 1, 2, 3, \ldots$*. For these reasons, for* p_1 *which is first coefficient in (1.2), we will assume that* $\left|4 - p_1^2\right| = \left|4 - |p_1|^2\right| = 4 - |p_1|^2$ *throughout our study.*

2. Fekete-Szegö problem for the function class $\Im_\Sigma(\alpha, \beta, \tau)$

In this section, we give solution of the Fekete-Szegö problem in two different $\mu \in \mathbb{C}$ and $\mu \in \mathbb{R}$ case for the function class $\Im_\Sigma(\alpha, \beta, \tau)$.

Firstly, we will give the following theorem on the solution of the Fekete-Szegö problem in the case $\mu \in \mathbb{C}$.

Theorem 2.1. *Let the function* f *given by (1.1) be in the class* $\Im_\Sigma(\alpha, \beta, \tau)$ *and* $\mu \in \mathbb{C}$*. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{2|\tau|(1-\alpha)}{3(1+2\beta)}, & \text{if } |1-\mu| \in [0, \mu_0), \\[2ex] |1-\mu| \dfrac{|\tau|^2 (1-\alpha)^2}{(1+\beta)^2}, & \text{if } |1-\mu| \geq \mu_0, \end{cases}$$

where $\mu_0 = \dfrac{2(1+\beta)^2}{3|\tau|(1-\alpha)(1+2\beta)}$.

Proof. Let $f \in \Im_\Sigma(\alpha, \beta, \tau)$, $\alpha \in [0,1),\ \beta \in [0,1],\ \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$ and $\mu \in \mathbb{C}$. Then,

$$1 + \frac{1}{\tau}\left[f'(z) + \beta z f''(z) - 1\right] = \alpha + (1-\alpha) p(z) \tag{2.1}$$

and

$$1 + \frac{1}{\tau}\left[g'(w) + \beta w g''(w) - 1\right] = \alpha + (1-\alpha) q(w), \tag{2.2}$$

where functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \cdots$ are in the class P.

Comparing the coefficients in (2.1) and (2.2), we have

$$a_2 = \frac{\tau(1-\alpha)}{2(1+\beta)} p_1, \quad a_3 = \frac{\tau(1-\alpha)}{3(1+2\beta)} p_2 \tag{2.3}$$

and

$$-a_2 = \frac{\tau(1-\alpha)}{2(1+\beta)} q_1, \quad 2a_2^2 - a_3 = \frac{\tau(1-\alpha)}{3(1+2\beta)} q_2. \tag{2.4}$$

From the first equalities of (2.3) and (2.4), we can write

$$\frac{\tau(1-\alpha)}{2(1+\beta)} p_1 = a_2 = -\frac{\tau(1-\alpha)}{2(1+\beta)} q_1 \text{ and } p_1 = -q_1 \tag{2.5}$$

Subtracting the second equality of (2.4) from the second equality of (2.3) and considering (2.5), we obtain

$$a_3 = a_2^2 + \frac{\tau(1-\alpha)}{6(1+2\beta)}(p_2 - q_2) \tag{2.6}$$

From (2.6), we get

$$a_3 - \mu a_2^2 = (1-\mu)a_2^2 + \frac{\tau(1-\alpha)}{6(1+2\beta)}(p_2 - q_2) \tag{2.7}$$

Since $p_1 = -q_1$, in view of Lemma 1.2, we can write

$$\left.\begin{array}{l} 2p_2 = p_1^2 + (4-p_1^2)x, \\ 2q_2 = q_1^2 + (4-q_1^2)y \end{array}\right\} \Rightarrow p_2 - q_2 = \frac{4-p_1^2}{2}(x-y) \tag{2.8}$$

for some x, y with $|x| \le 1$, $|y| \le 1$.

Note that if we take $|p_1| = t$, we can write $|4-p_1^2| = |4-|p_1|^2| = |4-t^2| = 4-t^2$ (see, Note 1.1 at the end of the first section). That is, we may assume without restriction that $t \in [0,2]$. In that case, substituting the expression (2.8) in (2.7), setting $|x| = \xi,\ |y| = \eta$ and using triangle inequality, we write

$$|a_3 - \mu a_2^2| \le d_1(t) + d_2(t)(\xi + \eta) = \varphi(\xi, \eta), \tag{2.9}$$

where

$$d_1(t) = |1-\mu|\frac{|\tau|^2(1-\alpha)^2}{4(1+\beta)^2}t^2 \ge 0 \quad \text{and} \quad d_2(t) = \frac{|\tau|(1-\alpha)(4-t^2)}{12(1+2\beta)} \ge 0.$$

It is clear that the maximum of the function φ occurs at $(\xi, \eta) = (1,1)$.
Therefore,

$$\varphi(\xi,\eta) \le \max\{\varphi(\xi,\eta) : \xi, \eta \in [0,1]\} = \varphi(1,1) = d_1(t) + 2d_2(t).$$

That is,

$$|a_3 - \mu a_2^2| \le d_1(t) + 2d_2(t). \tag{2.10}$$

Let us define the function $G : [0,2] \to \mathbb{R}$ as follows

$$G(t) = d_1(t) + 2d_2(t) \tag{2.11}$$

for each fixed $\tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$.

Substituting the value d_1 and d_2 in (2.11), we obtain

$$G(t) = C(\alpha, \beta, \mu, \tau)t^2 + D(\alpha, \beta, \tau),$$

where

$$C(\alpha,\beta,\mu,\tau) = \frac{(1-\alpha)^2|\tau|^2}{4(1+\beta)^2}\left[|1-\mu| - \frac{2(1+\beta)^2}{3(1-\alpha)|\tau|(1+2\beta)}\right], \quad D(\alpha,\beta,\tau) = \frac{2|\tau|(1-\alpha)}{3(1+2\beta)}$$

Now, we must investigate the maximum of the function G on the interval $[0,2]$.

By simple computation, we have

$$G'(t)=2C(\alpha,\beta,\mu,\tau)t.$$

We will examine the sign of the function $G'(t)$ depending on the different cases of the sign of $C(\alpha,\beta,\mu,\tau)$ as follows.

1. Let us $C(\alpha,\beta,\mu,\tau)\geq 0$. Then $G'(t)\geq 0$, so G is an increasing function. Therefore,

$$G(t)\leq \max\{G(t):t\in(0,2)\}=G(2)=d_1(2)=|1-\mu|\frac{|\tau|^2(1-\alpha)^2}{(1+\beta)^2}. \quad (2.12)$$

2. Let us $C(\alpha,\beta,\mu,\tau)<0$. Then $G'(t)<0$; that is, G is a decreasing function. Therefore,

$$G(t)\leq \max\{G(t):t\in(0,2)\}=G(0)=2d_2(0)=\frac{2|\tau|(1-\alpha)}{3(1+2\beta)}. \quad (2.13)$$

From (2.12) and (2.13), we conclude that

$$G(t)\leq \max\{G(t):t\in(0,2)\}=|1-\mu|\frac{|\tau|^2(1-\alpha)^2}{(1+\beta)^2} \quad (2.14)$$

if $|1-\mu|\geq \mu_0$ and

$$G(t)\leq \max\{G(t):t\in(0,2)\}=\frac{2|\tau|(1-\alpha)}{3(1+2\beta)} \quad (2.15)$$

if $|1-\mu|<\mu_0$, where $\mu_0=\dfrac{2(1+\beta)^2}{3|\tau|(1-\alpha)(1+2\beta)}$.

Thus, from (2.9), (2.10) and (2.14), (2.15), the proof of Theorem 2.1 is completed.

In the special cases from Theorem 2.1, we arrive at the following results.

Corollary 2.1. *Let the function* f *given by (1.1) be in the class* $\Im_\Sigma(\alpha,\beta,1)=H_\Sigma(\alpha,\beta)$ *and* $\mu\in\mathbb{C}$. *Then,*

$$|a_3-\mu a_2^2|\leq\begin{cases}\dfrac{2(1-\alpha)}{3(1+2\beta)}, & \text{if } |1-\mu|\in[0,\mu_0),\\[2mm] |1-\mu|\dfrac{(1-\alpha)^2}{(1+\beta)^2}, & \text{if } |1-\mu|\geq\mu_0,\end{cases}$$

where $\mu_0=\dfrac{2(1+\beta)^2}{3(1-\alpha)(1+2\beta)}$.

Corollary 2.2. *Let the function* f *given by (1.1) be in the class* $\Im_\Sigma(\alpha,0,\tau)$ *and* $\mu\in\mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{2|\tau|(1-\alpha)}{3}, & \text{if } |1-\mu|\in[0,\mu_0),\\ |1-\mu||\tau|^2(1-\alpha)^2, & \text{if } |1-\mu|\ge\mu_0,\end{cases}$$

where $\mu=\dfrac{2}{3|\tau|(1-\alpha)}$.

Corollary 2.3. *Let the function* f *given by (1.1) be in the class* $\Im_\Sigma(\alpha,0,1)=\Re_\Sigma(\alpha,0)=N_\Sigma(\alpha)$ *and* $\mu\in\mathbb{C}$*. Then,*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{2(1-\alpha)}{3}, & \text{if } |1-\mu|\in[0,\mu_0),\\ |1-\mu|(1-\alpha)^2, & \text{if } |1-\mu|\ge\mu_0,\end{cases}$$

where $\mu=\dfrac{2}{3(1-\alpha)}$.

Corollary 2.4. *Let the function* f *given by (1.1) be in the class* $\Im_\Sigma(\alpha,1,\tau)$ *and* $\mu\in\mathbb{C}$*. Then,*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{2(1-\alpha)|\tau|}{9}, & \text{if } |1-\mu|\in[0,\mu_0),\\ |1-\mu|\dfrac{(1-\alpha)^2|\tau|^2}{4}, & \text{if } |1-\mu|\ge\mu_0,\end{cases}$$

where $\mu_0=\dfrac{8}{9|\tau|(1-\alpha)}$.

Corollary 2.5. *Let the function* f *given by (1.1) be in the class* $\Im_\Sigma(\alpha,1,1)=\Re_\Sigma(\alpha,1)$*. Then,*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{2(1-\alpha)}{9}, & \text{if } |1-\mu|\in[0,\mu_0),\\ |1-\mu|\dfrac{(1-\alpha)^2}{4}, & \text{if } |1-\mu|\ge\mu_0,\end{cases}$$

where $\mu_0=\dfrac{8}{9(1-\alpha)}$

In the case $\mu\in\mathbb{R}$, Theorem 2.1 can be given as follows.

Theorem 2.2. *Let the function* f *given by (1.1) be in the class* $\Im_\Sigma(\alpha,\beta,\tau)$ *and* $\mu\in\mathbb{R}$*. Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} (1-\mu)\dfrac{|\tau|^2(1-\alpha)^2}{(1+\beta)^2}, & \text{if } \mu<1-\mu_0, \\ \dfrac{2|\tau|(1-\alpha)}{3(1+2\beta)}, & \text{if } 1-\mu_0\le\mu\le 1+\mu_0, \\ (\mu-1)\dfrac{|\tau|^2(1-\alpha)^2}{(1+\beta)^2}, & \text{if } 1+\mu_0\le\mu, \end{cases}$$

where $\mu_0=\dfrac{2(1+\beta)^2}{3|\tau|(1-\alpha)(1+2\beta)}$.

Proof. Since the proof of the Theorem 2.2 is similar to the proof of Theorem 2.1, we do not give this proof.

Taking $\mu=0$ and $\mu=1$ in Theorem 2.2, we can easily arrive at the following result.

Corollary 2.6. *Let the function* f *given by (1.1) be in the class* $\Im_\Sigma(\alpha,\beta,\tau)$, $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$. *Then,*

$$|a_3| \le \begin{cases} \dfrac{2(1-\alpha)|\tau|}{3(1+2\beta)}, & \text{if } |\tau|\in(0,\tau_0), \\ \dfrac{(1-\alpha)^2|\tau|^2}{(1+\beta)^2}, & \text{if } |\tau|\ge\tau_0, \end{cases}$$

where $\tau_0=\dfrac{2(1+\beta)^2}{3(1-\alpha)(1+2\beta)}$ *and*

$$\left|a_3-a_2^2\right|\le\frac{2|\tau|(1-\alpha)}{3(1+2\beta)}.$$

Note 2.1. *The first result of Corollary 2.6 confirms the second inequality of Theorem 1 in* [19].

Remark 2.1 *Numerous consequences of the results obtained in the Corollary 2.1 – 2.6 can indeed be deduced by specializing the various parameters involved.*

3. Results and Discussion

In this paper, we introduce and investigate a subclass of analytic and bi-univalent functions of complex order on the open unit disk in the complex plane. Here, the Fekete-Szegö problem solved for this function class. Consequence, we give the upper bound estimate for the coefficient $|a_3|$ of the functions belonging to this class.

4. Conclusion

In this paper we have given the Fekete-Szegö inequality for a new class of analytic and bi-univalent functions.

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