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Radius of Convexity of Certain Subclass of Analytic Functions

Nizami Mustafa*, Oğuzhan Derya

Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey
E-mail: nizamimustafa@gmail.com

Abstract In this paper, we introduce and investigate some subclasses of analytic functions in the open unit disk centred on origin. Here, various geometric properties of these classes are examined. Coefficient bounds, distortion bound and growth theorems for the functions belonging to these classes are also given. Further, radius of the convexity is determined for one of these classes.

Keywords Coefficient bound, starlike function, convex function, radius of convexity
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1. Introduction

Let A be the class of analytic functions in the open unit disk $U=\{z\in\mathbb{C}:\ |z|<1\}$ in the complex plane, normalized by $f(0)=0=f'(0)-1$ of the for

$$f(z)=z+a_2z^2+a_3z^3+\cdots+a_nz^n+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\quad a_n\in\mathbb{C} \tag{1.1}$$

and S denote the class of all functions in A which are also univalent in U.

Let T denote the subclass of all functions in A of the form

$$f(z)=z-a_2z^2-a_3z^3-\cdots-a_nz^n-\cdots=z-\sum_{n=2}^{\infty}a_nz^n,\quad a_n\geq 0. \tag{1.2}$$

For $\alpha\in[0,1)$, we denote by $S^*(\alpha)$, $C(\alpha)$ and $K(\alpha)$ the subclasses of S that are, respectively, starlike, convex and close-to-convex with respect to starlike function g (need not be normalized) of order α in the disk U.

By definition, we have (see for details, [4, 5], also [8])

$$S^*(\alpha)=\left\{f\in S:\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\ z\in U\right\}, \tag{1.3}$$

$$C(\alpha)=\left\{f\in S:\ \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\ z\in U\right\} \tag{1.4}$$

and

$$K(\alpha)=\left\{ f\in S:\ \mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right)>\alpha,\ z\in U,\ g\in S^{*}\right\}$$

for $\alpha\in[0,1)$. For convenience, $S^{*}=S^{*}(0),\ C=C(0)$ and $K=K(0)$ are, respectively, well-known starlike, convex and close-to-convex functions in U. For details on these classes, one could refer to the monograph by Goodman [5].

For $\alpha\in[0,1)$, $\beta\in[0,1]$ an interesting generalization of the class $K(\alpha)$ is the class

$$K(\alpha,\beta;g)=\left\{ f\in S:\ \mathrm{Re}\left(\frac{zf'(z)+\beta z^{2}f''(z)}{g(z)}\right)>\alpha,\ z\in U,\ g\in S^{*}\right\}.$$

We will denote $K(\alpha,\beta;z)=K(\alpha,\beta).$

Note 1.1. For $\alpha\in[0,1)$, $\beta\in[0,1]$ the class $K(\alpha,\beta;g)$ is the first time introduced in this paper. Clearly, in the case $\beta=0$ we have $K(\alpha,0;g)=K(\alpha)$.

Note that, we will use $TS^{*}(\alpha),\ TK(\alpha)$ and $TC(\alpha)$ instead $S^{*}(\alpha),\ K(\alpha)$ and $C(\alpha)$, respectively, if $f\in T$. Also, we will denote $TK(\alpha,\beta)$ instead of $K(\alpha,\beta)$ when $f\in T$.

For $\alpha\in[0,1)$, $\beta\in[0,1]$ an generalization of the classes $S^{*}(\alpha)$ and $C(\alpha)$ is the class

$$A(\alpha,\beta)=\left\{ f\in S:\mathrm{Re}\left(\frac{zf'(z)+\beta z^{2}f''(z)}{\beta zf'(z)+(1-\beta)f(z)}\right)>\alpha,z\in U\right\}. \tag{1.5}$$

In special case, we have $A(\alpha,0)=S^{*}(\alpha)$ and $A(\alpha,1)=C(\alpha)$.

We also denote $T(\alpha,\beta)$ instead $A(\alpha,\beta)$ if $f\in T$. Thus, for $\alpha\in[0,1)$, $\beta\in[0,1]$

$$T(\alpha,\beta)=\left\{ f\in T:\mathrm{Re}\left(\frac{zf'(z)+\beta z^{2}f''(z)}{\beta zf'(z)+(1-\beta)f(z)}\right)>\alpha,z\in U\right\}. \tag{1.6}$$

The class $T(\alpha,\beta)$ was investigated by Altintaş *et al.* [2] and [3] (in a more general way $T_{n}(p,\alpha,\beta)$) and (subsequently) by Irmak *et al.* [6]. In particular, the class $T_{n}(1,\alpha,\beta)$ was considered earlier by Altintaş [1].

For $\alpha\in[0,1),\ \beta,\gamma\in[0,1]$, we introduce an generalization of the classes $A(\alpha,\beta)$ and $K(\alpha,\beta)$ provided by $A(\alpha,\beta;\gamma)$ as follows

$$A(\alpha,\beta;\gamma)=\left\{ f\in A:\mathrm{Re}\left(\frac{zf'(z)+\beta z^{2}f''(z)}{\gamma\left[\beta zf'(z)+(1-\beta)f(z)\right]+(1-\gamma)z}\right)>\alpha,z\in U\right\}. \tag{1.7}$$

In special case, we have $A(\alpha,\beta;1)=A(\alpha,\beta)$ and $A(\alpha,\beta;0)=K(\alpha,\beta)$.

Note that, we will use $T(\alpha,\beta;\gamma)$ instead $A(\alpha,\beta;\gamma)$ if $f\in T$.

Thus for $\alpha\in[0,1),\ \beta,\gamma\in[0,1]$

$$T(\alpha,\beta;\gamma)=\left\{f\in T:\operatorname{Re}\left(\frac{zf'(z)+\beta z^2f''(z)}{\gamma\left[\beta zf'(z)+(1-\beta)f(z)\right]+(1-\gamma)z}\right)>\alpha, z\in U\right\}. \tag{1.8}$$

In special case, we have $T(\alpha,\beta;1)=T(\alpha,\beta)$.

In this paper, two new subclasses $A(\alpha,\beta;\gamma)$ and $T(\alpha,\beta;\gamma)$ of the analytic functions in U are introduced. Coefficient bounds, distortion bound and growth theorems for the functions belonging to these classes are given. Radius of convexity for these classes are also determined.

2. Coefficient bounds for the classes $A(\alpha,\beta;\gamma)$ and $T(\alpha,\beta;\gamma)$

In this section, we examine some inclusion results of the classes $A(\alpha,\beta;\gamma)$ and $T(\alpha,\beta;\gamma)$. Firstly, we give a sufficient condition for the class $A(\alpha,\beta;\gamma)$ by the following theorem.

Theorem 2.1. Let $f\in A$. Then, the function f belongs to the class $A(\alpha,\beta;\gamma)$ if the following condition is satisfied

$$\sum_{n=2}^{\infty}(n-\alpha\gamma)\left[1+\beta(n-1)\right]|a_n|\le 1-\alpha. \tag{2.1}$$

The result is sharp for the function

$$f_n(z)=z+\frac{1-\alpha}{(n-\alpha\gamma)\left[1+\beta(n-1)\right]}z^n,\ z\in U \tag{2.2}$$

for each $n=2,3,\ldots$.

Proof. From the definition of the class $A(\alpha,\beta;\gamma)$, for $\alpha\in[0,1),\ \beta,\gamma\in[0,1]$ a function $f\in A(\alpha,\beta;\gamma)$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)+\beta z^2f''(z)}{\gamma\left[\beta zf'(z)+(1-\beta)f(z)\right]+(1-\gamma)z}\right\}>\alpha. \tag{2.3}$$

To show that the condition (2.3) is fulfilled, it is sufficient to show that

$$\left|\frac{zf'(z)+\beta z^2f''(z)}{\gamma\left[\beta zf'(z)+(1-\beta)f(z)\right]+(1-\gamma)z}-1\right|\le 1-\alpha. \tag{2.4}$$

Using the series expansion (1.1) of the function f and triangle inequality, we write

$$\left|\frac{zf'(z)+\beta z^2f''(z)}{\gamma\left[\beta zf'(z)+(1-\beta)f(z)\right]+(1-\gamma)z}-1\right|$$
$$=\left|\frac{\sum_{n=2}^{\infty}(n-\gamma)\left[1+\beta(n-1)\right]a_nz^n}{z+\sum_{n=2}^{\infty}\gamma\left[1+\beta(n-1)\right]a_nz^n}\right|\le\frac{\sum_{n=2}^{\infty}(n-\gamma)\left[1+\beta(n-1)\right]|a_n|}{1-\sum_{n=2}^{\infty}\gamma\left[1+\beta(n-1)\right]|a_n|}.$$

As can be easily seen that the necessary and sufficient condition for the fraction on the right hand side of the last inequality to be bounded with the number $1-\alpha$ is the realization of the following inequality

$$\sum_{n=2}^{\infty}(n-\gamma)\left[1+\beta(n-1)\right]|a_n| \le (1-\alpha)\left\{1-\sum_{n=2}^{\infty}\gamma\left[1+\beta(n-1)\right]|a_n|\right\}.$$

On the other hand, this inequality is equivalent to (2.1).

Thus, the inequality (2.4) is true if the condition (2.1) is satisfied. That is, the condition (2.3) is provided. Hence, $f \in A(\alpha,\beta;\gamma)$.

Now let's see that this inequality occurs as an equality for the functions given by the formula (2.2). Really if we take $a_n = \dfrac{1-\alpha}{(n-\alpha\gamma)\left[1+\beta(n-1)\right]}$, $n=2,3,\dots$ in the inequality (2.1), we can easily see that the following equality is provided

$$\sum_{n=2}^{\infty}(n-\alpha\gamma)[1+\beta(n-1)]\frac{1-\alpha}{(n-\alpha\gamma)[1+\beta(n-1)]} = 1-\alpha.$$

Thus the proof of Theorem 2.1 is completed.

If we take $\gamma = 1$ in Theorem 2.1, we can readily deduce the following corollary.

Corollary 2.1. The function f given by (1.1) belongs to the class $A(\alpha,\beta)$ if the following condition is satisfied

$$\sum_{n=2}^{\infty}(n-\alpha)\left[1+\beta(n-1)\right]|a_n| \le 1-\alpha.$$

The result obtained here is sharp.

By setting $\beta = 0$ in Corollary 2.1, we have the following result.

Corollary 2.2. (see [7, p. 110, Theorem 1]) The function f given by (1.1) belongs to the class $S^*(\alpha)$ if the following condition is satisfied

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \le 1-\alpha.$$

The result obtained here is sharp.

By taking $\beta = 1$ in Corollary 2.1, we arrive at the following result.

Corollary 2.3. (see [7, p. 110, Corollary of Theorem1]) The function f given by (1.1) belongs to the class $C(\alpha)$ if the following condition is satisfied

$$\sum_{n=2}^{\infty}n(n-\alpha)|a_n| \le 1-\alpha.$$

The result obtained here is sharp.

If we take $\gamma = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.4. The function f given by (1.1) belongs to the class $K(\alpha,\beta)$ if the following condition is satisfied

$$\sum_{n=2}^{\infty}n\left[1+\beta(n-1)\right]|a_n| \le 1-\alpha.$$

The result obtained here is sharp.

By taking $\beta = 0$ in Corollary 2.4, we have the following result.

Corollary 2.5. The function f given by (1.1) belongs to the class $K(\alpha)$ if the following condition is satisfied

$$\sum_{n=2}^{\infty} n|a_n| \le 1-\alpha .$$

The result obtained here is sharp.

For the function in class $T(\alpha,\beta;\gamma)$, the converse of Theorem 2.1 is also true.

Theorem 2.2. Let $f \in T$. Then, the function f belongs to the class $T(\alpha,\beta;\gamma)$ if and only if

$$\sum_{n=2}^{\infty} (n-\alpha\gamma)\left[1+\beta(n-1)\right]|a_n| \le 1-\alpha . \tag{2.1}$$

The result obtained here is sharp.

Proof. Since the proof of the sufficiency of the theorem is the same as the proof of Theorem 2.1, it is sufficient to prove the necessary part of the theorem.

Assume that $f \in T(\alpha,\beta;\gamma)$, then

$$\operatorname{Re}\left\{\frac{zf'(z)+\beta z^2 f''(z)}{\gamma\left[\beta zf'(z)+(1-\beta)f(z)\right]+(1-\gamma)z}\right\} > \alpha,\ z \in U . \tag{2.5}$$

Using the series expansion (1.2) of the function f, by simple computation the inequality (2.5) we can write as follows

$$\operatorname{Re}\left\{\frac{z-\sum_{n=2}^{\infty} n\left[1+\beta(n-1)\right]|a_n|z^n}{z-\sum_{n=2}^{\infty} \gamma\left[1+\beta(n-1)\right]|a_n|z^n}\right\} > \alpha ,\ z \in U . \tag{2.6}$$

The expression in the parentheses in the inequality (2.6) is real if choose z real. Thus, from the inequality (2.6) letting $z \to 1$ through real values, we obtain

$$\frac{1-\sum_{n=2}^{\infty} n\left[1+\beta(n-1)\right]|a_n|}{1-\sum_{n=2}^{\infty} \gamma\left[1+\beta(n-1)\right]|a_n|} \ge \alpha ;$$

that is,

$$1-\sum_{n=2}^{\infty} n\left[1+\beta(n-1)\right]|a_n| \ge \alpha\left\{1-\sum_{n=2}^{\infty} \gamma\left[1+\beta(n-1)\right]|a_n|\right\}$$

which equivalent to (2.1).

Thus, the proof of Theorem 2.2 is completed.

Special case of Theorem 2.2 has been proved by Altintaş *et al* [2], $\gamma = 1$ (here $p = n = 1$).

If we take $\gamma = 1$ in Theorem 2.2, we can readily deduce the following corollary.

Corollary 2.6. The function f given by (1.2) belongs to the class $T(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty} (n-\alpha)\left[1+\beta(n-1)\right]|a_n| \le 1-\alpha .$$

Remark 2.1. The result obtained in Corollary 2.6 verifies to Theorem 1 in [2].

By taking $\beta=0$ in Corollary 2.6, we have the following result.

Corollary 2.7. (see [7, p. 110, Theorem 2]) The function f given by (1.2) belongs to the class $TS^{*}(\alpha)$ if and only if

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\le 1-\alpha .$$

By taking $\beta=1$ in Corollary 2.6, we have the following result.

Corollary 2.8. (see [7, p. 111, Corollary 2]) The function f given by (1.2) belongs to the class $TC(\alpha)$ if and only if

$$\sum_{n=2}^{\infty}n(n-\alpha)|a_n|\le 1-\alpha .$$

By setting $\gamma=0$ in Theorem 2.2, we arrive at the following corollary.

Corollary 2.9. The function f given by (1.2) belongs to the class $TK(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}n\left[1+\beta(n-1)\right]|a_n|\le 1-\alpha .$$

By taking $\beta=0$ in Corollary 2.9, we have the following result.

Corollary 2.10. The function f given by (1.2) belongs to the class $TK(\alpha)$ if and only if

$$\sum_{n=2}^{\infty}n|a_n|\le 1-\alpha .$$

Now, on the coefficient bounds of the functions belonging in the class $T(\alpha,\beta;\gamma)$, we give the following lemma.

Lemma 2.1. Let the function f given by (1.2) belongs to the class $T(\alpha,\beta;\gamma)$. Then,

$$\sum_{n=2}^{\infty}|a_n|\le\frac{1-\alpha}{(1+\beta)(2-\alpha\gamma)}\quad\text{and}\quad\sum_{n=2}^{\infty}n|a_n|\le\frac{2(1-\alpha)}{(1+\beta)(2-\alpha\gamma)} .$$

Poof. Using Theorem 2.2, we write

$$(2-\alpha\gamma)(1+\beta)\sum_{n=2}^{\infty}|a_n|\le\sum_{n=2}^{\infty}(n-\alpha\gamma)\left[1+\beta(n-1)\right]|a_n|\le 1-\alpha .$$

From this, the first assertion of the lemma is obtained immediately.
Similarly to the above, we can write

$$(1+\beta)\sum_{n=2}^{\infty}(n-\alpha\gamma)|a_n|\le\sum_{n=2}^{\infty}(n-\alpha\gamma)\left[1+\beta(n-1)\right]|a_n|\le 1-\alpha ;$$

that is,

$$(1+\beta)\sum_{n=2}^{\infty}n|a_n|\le 1-\alpha+(1+\beta)\alpha\gamma\sum_{n=2}^{\infty}|a_n| .$$

Using the first assertion of lemma, we arrive at the following inequality

$$(1+\beta)\sum_{n=2}^{\infty} n|a_n| \le 1-\alpha+(1+\beta)\alpha\gamma\frac{1-\alpha}{(1+\beta)(2-\alpha\gamma)}=\frac{2(1-\alpha)}{2-\alpha\gamma}$$

which immediately yields the second assertion of Lemma 2.1.
Thus, the proof of Lemma 2.1 is completed.

If we take $\gamma=1$ in Lemma 2.1, we arrive at the following corollary.

Corollary 2.11. Let the function f given by (1.2) belongs to the class $T(\alpha,\beta)$. Then,

$$\sum_{n=2}^{\infty}|a_n| \le \frac{1-\alpha}{(2-\alpha)(1+\beta)} \quad \text{and} \quad \sum_{n=2}^{\infty} n|a_n| \le \frac{2(1-\alpha)}{(2-\alpha)(1+\beta)}.$$

Remark 2.2. The result obtained in the Corollary 2.11 verifies to Lemma 2 (with $n=p=1$) of [2].
From Theorem 2.2, we have the following inequalities foe the coefficients.

Corollary 2.12. If $f\in T(\alpha,\beta;\gamma)$, then

$$|a_n| \le \frac{1-\alpha}{(n-\alpha\gamma)\left[1+\beta(n-1)\right]}, \; n=2,3,\dots .$$

Numerous consequences of Corollary 2.12 can indeed be deduced by specializing the various parameters involved. Many of these consequences were proved by earlier workers on the subject (cf., e.g., [1, 7, 9]).

3. Distortion bound and growth theorems for the class $T(\alpha,\beta;\gamma)$

In this section, we give distortion and growth theorems for the function class $T(\alpha,\beta;\gamma)$.
Our coefficient bound estimates we found, enable us to prove the following theorems.

Theorem 3.1. If $f\in T(\alpha,\beta;\gamma)$, then

$$r-\frac{1-\alpha}{(1+\beta)(2-\alpha\gamma)}r^2 \le |f(z)| \le r+\frac{1-\alpha}{(1+\beta)(2-\alpha\gamma)}r^2, \; |z|=r, \; r\le 1. \tag{3.1}$$

The result obtained here is sharp.
Proof. Using Theorem 2.2, we write

$$(2-\alpha\gamma)(1+\beta)\sum_{n=2}^{\infty}|a_n| \le \sum_{n=2}^{\infty}(n-\alpha\gamma)\left[1+\beta(n-1)\right]|a_n| \le 1-\alpha.$$

Hence,

$$\sum_{n=2}^{\infty}|a_n| \le \frac{1-\alpha}{(2-\alpha\gamma)(1+\beta)}. \tag{3.2}$$

If apply it, we obtain

$$|f(z)| \le r+\sum_{n=2}^{\infty}|a_n|\, r^n \le r+r^2\sum_{n=2}^{\infty}|a_n| \le r+\frac{1-\alpha}{(1+\beta)(2-\alpha\gamma)}r^2. \tag{3.3}$$

Also, using the inverse triangle inequality and the inequality (3.2), we can write

$$|f(z)| \ge r-\sum_{n=2}^{\infty}|a_n|\, r^n \ge r-r^2\sum_{n=2}^{\infty}|a_n| \ge r-\frac{1-\alpha}{(2-\alpha\gamma)(1+\beta)}r^2. \tag{3.4}$$

Unification of the inequalities (3.3) and (3.4), gives us the inequality (3.1).
Thus, the proof of Theorem 3.1 is completed.

If we take $\gamma = 1$ in Theorem 3.1, we arrive at the following corollary.

Corollary 3.1. If $f \in T(\alpha, \beta)$, then

$$r - \frac{1-\alpha}{(2-\alpha)(1+\beta)} r^2 \le |f(z)| \le r + \frac{1-\alpha}{(2-\alpha)(1+\beta)} r^2, \ |z| = r, \ r \le 1.$$

By taking $\beta = 0$ in Corollary 3.1, we have the following result.

Corollary 3.2. (see [7, p. 111, Theorem 4]) If $f \in TS^*(\alpha)$, then

$$r - \frac{1-\alpha}{2-\alpha} r^2 \le |f(z)| \le r + \frac{1-\alpha}{2-\alpha} r^2, \ |z| = r, \ r \le 1.$$

By setting $\beta = 1$ in Corollary 3.1, we have the following result.

Corollary 3.3. (see [7, p. 112, Corollary of Theorem 4]) If $f \in TC(\alpha)$, then

$$r - \frac{1-\alpha}{2(2-\alpha)} r^2 \le |f(z)| \le r + \frac{1-\alpha}{2(2-\alpha)} r^2, \ |z| = r, \ r \le 1.$$

If we take $\gamma = 0$ in Theorem 3.1, we arrive at the following corollary.

Corollary 3.4. If $f \in TK(\alpha, \beta)$, then

$$r - \frac{1-\alpha}{2(1+\beta)} r^2 \le |f(z)| \le r + \frac{1-\alpha}{2(1+\beta)} r^2, \ |z| = r, \ r \le 1.$$

By taking $\beta = 0$ in Corollary 3.4, we have the following result.

Corollary 3.5. If $f \in TK(\alpha)$, then

$$r - \frac{1-\alpha}{2} r^2 \le |f(z)| \le r + \frac{1-\alpha}{2} r^2, \ |z| = r, \ r \le 1.$$

Theorem 3.2. If $f \in T(\alpha, \beta; \gamma)$, then

$$1 - \frac{2(1-\alpha)}{(1+\beta)(2-\alpha\gamma)} r \le |f'(z)| \le 1 + \frac{2(1-\alpha)}{(1+\beta)(2-\alpha\gamma)} r, \ |z| = r, \ r \le 1. \tag{3.5}$$

The result obtained here is sharp.

Proof. Using the series expansion (1.2) of the function f and triangle inequality, we obtain

$$|f'(z)| = \left| z + \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \le 1 + \sum_{n=2}^{\infty} n |a_n| |z^{n-1}| \le 1 + r \sum_{n=2}^{\infty} n |a_n|. \tag{3.6}$$

In view of Theorem 2.2, we can write

$$(1+\beta) \sum_{n=2}^{\infty} n |a_n| - (1+\beta)\alpha\gamma \sum_{n=2}^{\infty} |a_n| \le \sum_{n=2}^{\infty} (n - \alpha\gamma)[1 + \beta(n-1)] |a_n| \le 1 - \alpha.$$

From this, using the inequality (3.2) we obtain

$$\sum_{n=2}^{\infty} n |a_n| \le \frac{1-\alpha}{1+\beta} + \alpha\gamma \sum_{n=2}^{\infty} |a_n| \le \frac{1-\alpha}{1+\beta} + \alpha\gamma \frac{1-\alpha}{(1+\beta)(2-\alpha\gamma)} = \frac{2(1-\alpha)}{(1+\beta)(2-\alpha\gamma)}. \tag{3.7}$$

Considering (3.7) in the (3.6), we obtain

$$\left|f'(z)\right| \le 1+\frac{2(1-\alpha)}{(1+\beta)(2-\alpha\gamma)}r. \tag{3.8}$$

Similarly, we obtain

$$\left|f'(z)\right| \ge 1-\sum_{n=2}^{\infty} n\left|a_n\right|\left|z^{n-1}\right| \ge 1-r\sum_{n=2}^{\infty} n\left|a_n\right| \ge 1-\frac{2(1-\alpha)}{(1+\beta)(2-\alpha\gamma)}r. \tag{3.9}$$

Unification of the inequalities (3.8) and (3.9), immediately gives the inequality (3.5).
Thus, the proof of Theorem 3.2 is completed.

If we take $\gamma=1$ in Theorem 3.2, we arrive at the following corollary.

Corollary 3.6. If $f \in T(\alpha,\beta)$, then

$$1-\frac{2(1-\alpha)}{(1+\beta)(2-\alpha)}r \le \left|f'(z)\right| \le 1+\frac{2(1-\alpha)}{(1+\beta)(2-\alpha)}r,\ \ |z|=r,\ r\le 1.$$

By setting $\beta=0$ in Corollary 3.6, we have the following result.

Corollary 3.7 (see [7, p. 112, Theorem 6]). If $f \in TS^*(\alpha)$, then

$$1-\frac{2(1-\alpha)}{2-\alpha}r \le \left|f'(z)\right| \le 1+\frac{2(1-\alpha)}{2-\alpha}r,\ \ |z|=r,\ r\le 1.$$

By taking $\beta=1$ in Corollary 3.6, we have the following result.

Corollary 3.8 (see [7, p. 112, Corollary of Theorem 6]). If $f \in TC(\alpha)$, then

$$1-\frac{1-\alpha}{2-\alpha}r \le \left|f'(z)\right| \le 1+\frac{1-\alpha}{2-\alpha}r,\ \ |z|=r,\ r\le 1.$$

By setting $\gamma=0$ in Theorem 3.2, we arrive at the following corollary.

Corollary 3.9. If $f \in TK(\alpha,\beta)$ then

$$1-\frac{1-\alpha}{1+\beta}r \le \left|f'(z)\right| \le 1+\frac{1-\alpha}{1+\beta}r,\ \ |z|=r,\ r\le 1.$$

By taking $\beta=1$ in Corollary 3.9, we have the following result.

Corollary 3.10. If $f \in TK(\alpha)$, then

$$1-\frac{1-\alpha}{2}r \le \left|f'(z)\right| \le 1+\frac{1-\alpha}{2}r,\ \ |z|=r,\ r\le 1.$$

4. Radius of the convexity for class $T(\alpha,\beta;\gamma)$

In this section, we determine the radius of the convexity for class $T(\alpha,\beta;\gamma)$.

Theorem 4.1. The function $f \in T(\alpha,\beta;\gamma)$ is convex in the disk

$$U_{r(\alpha,\beta;\gamma)}=\left\{z\in\mathbb{C}:|z|<r=r(\alpha,\beta;\gamma)\right\},$$

where

$$r(\alpha,\beta;\gamma)=\inf\left\{\left(\frac{(n-\alpha\gamma)(1+\beta(n-1))}{(1-\alpha)n^2}\right)^{1/(n-1)}:\ n=2,3,\ldots\right\}.$$

Proof. To prove the theorem, it suffices to show that $\left|zf''(z)/f'(z)\right|\le 1$ for $z\in U_{r(\alpha,\beta;\gamma)}$.

Using the series expansion (1.2) of the function f, by simple computation we obtain

$$\left|\frac{zf''(z)}{f'(z)}\right|=\left|\frac{-\sum_{n=2}^{\infty}n(n-1)|a_n|z^{n-1}}{1-\sum_{n=2}^{\infty}n|a_n|z^{n-1}}\right|\le\frac{\sum_{n=2}^{\infty}n(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}n|a_n||z|^{n-1}}. \tag{4.1}$$

The fraction to the right hand of the inequality (4.1) is bounded by 1 if and only if

$$\sum_{n=2}^{\infty}n(n-1)|a_n||z|^{n-1}\le 1-\sum_{n=2}^{\infty}n|a_n||z|^{n-1}$$

which is equivalent to

$$\sum_{n=2}^{\infty}n^2|a_n||z|^{n-1}\le 1. \tag{4.2}$$

Also, according to Theorem 2.2 we write

$$\sum_{n=2}^{\infty}\frac{(n-\alpha\gamma)\left[1+\beta(n-1)\right]}{1-\alpha}|a_n|\le 1. \tag{4.3}$$

It follows from (4.2) and (4.3) that inequality (4.2) will be true if

$$n^2|z|^{n-1}\le\frac{(n-\alpha\gamma)\left[1+\beta(n-1)\right]}{1-\alpha},\ n=2,3,\ldots \tag{4.4}$$

Solving the inequality (4.4) for $|z|$, we obtain

$$|z|\le\left\{\frac{(n-\alpha\gamma)\left[1+\beta(n-1)\right]}{(1-\alpha)n^2}\right\}^{1/(n-1)},\ n=2,3,\ldots \tag{4.5}$$

From the inequality (4.5), obtained the result.
Thus, the proof of Theorem 4.1 is completed.

If we take $\gamma=1$ in Theorem 4.1, we arrive at the following corollary.

Corollary 4.1. İf $f\in T(\alpha,\beta)$, then f is convex in the disk

$$U_{r(\alpha,\beta)}=\left\{z\in\mathbb{C}:|z|<r=r(\alpha,\beta)\right\},$$

where

$$r(\alpha,\beta)=\inf\left\{\left(\frac{(n-\alpha)(1+\beta(n-1))}{(1-\alpha)n^2}\right)^{1/(n-1)}:\ n=2,3,\ldots\right\}.$$

By setting $\beta=0$ in Corollary 4.1, we have the following result.

Corollary 4.2. (see [7, p. 113, Theorem 8]). If $f\in S^*(\alpha)$, then f is convex in the disk

$$U_{r(\alpha)} = \{ z \in \mathbb{C} : |z| < r = r(\alpha) \},$$

where

$$r(\alpha) = \inf \left\{ \left(\frac{n-\alpha}{(1-\alpha)n^2} \right)^{1/(n-1)} : \ n = 2, 3, \ldots \right\}.$$

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