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A Generalized Symmetric Result for Polynomial Function

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Abstract: Let $p(z)$ be a polynomial of degree n, $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$. In this paper we have been able to obtain a symmetric result concerning the maximum modulus of polynomial on two different radii. Our result provides generalizations of earlier proved results and opens new avenues for other results in the same field of research.

Keywords: Polynomials; Zeros; Inequalities; Complex domain, Maximum modulus.
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1-Introduction and Statement of Results

Concerning maximum modulus of polynomial functions we have several results so far, among which a few are as follows.

Theorem A. *If* $p(z)$ *is a polynomial of degree n, then for every* $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| . \tag{1.1}$$

The result is best possible and extremal polynomial is $p(z)=\lambda z^n$, $\lambda(\neq 0)$ *is a complex number.*

Inequality (1.1) is a simple deduction from the maximum modulus principle (for reference see [7] or [5]).

For the case $r \leq 1$ we have the following result.

Theorem B. *If* $p(z)$ *is a polynomial of degree n, then for* $r \leq 1$

$$\max_{|z|=r} |p(z)| \geq r^n \max_{|z|=1} |p(z)| . \tag{1.2}$$

The result is best possible andextremal polynomial is $p(z)=\lambda z^n$. $\lambda(\neq 0)$ *is a complex number.*

Inequality (1.2) is due to Zerrantonello and Varga [9].

If we restrict ourselves to the class of polynomials having no zeros in $|z|<1$, then the inequalities (1.1) and (1.2) can be sharpened. In this connection the following results are well known.

Theorem C. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<1$, then*

$$\max_{|z|=R\geq 1}|p(z)| \leq \frac{R^n+1}{2}\max_{|z|=1}|p(z)| . \tag{1.3}$$

The result is best possible and equality in (1.3) holds for $p(z)=\lambda+\mu z^n$, where $|\lambda|=|\mu|$.

Theorem D. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<1$, then for $r\leq 1$*

$$\max_{|z|=r}|p(z)| \geq \left(\frac{1+r}{2}\right)^n \max_{|z|=1}|p(z)|. \tag{1.4}$$

The result is best possible and equality in inequality (1.1.4) holds for $p(z)=\left(\frac{1+z}{2}\right)^n$.

The inequality (1.3) is due to Ankeny and Rivlin [1] and inequality (1.4) is due to Rivlin [8]. The following interesting result is due to Jain [3].

Theorem E. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z|\geq k$, $k>0$, then for $r\leq k\leq R$*

$$\frac{M(p,r)}{r^n+k\,r^{n-1}} \geq \frac{M(p,R)}{R^n+k\,R^{n-1}} . \tag{1.5}$$

Mir [4] generalized inequality (1.5) by introducing coefficients in it and proved the following

Theorem F. *If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, having all its zeros in $|z|\geq k$, $k>0$, then for $r\leq k\leq R$*

$$\frac{M(p,R)}{R^{n-1}(R^2+k^2)n|a_0|+2k^2R^n|a_1|} \leq \frac{M(p,r)}{r^{n-1}(r^2+k^2)n|a_0|+2k^2r^n|a_1|} \tag{1.6}$$

In this paper, our main aim is to generalize the above mentioned inequality (1.6) for lacunary type of polynomials. More specifically we prove the following

Theorem 1.1. *If $p(z)=a_0+\sum_{j=\mu}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z|\geq k$, $k\geq 1$, then for $r\leq k\leq R$,*

$$\frac{M(p,R)}{R^{n-1}(R^{\mu+1}+k^{\mu+1})n|a_0|+\mu|a_\mu|R^n k^{\mu+1}(R^{\mu-1}+k^{\mu-1})}$$
$$\le \frac{M(p,r)}{r^{n-1}(r^{\mu+1}+k^{\mu+1})n|a_0|+\mu|a_\mu|r^n k^{\mu+1}(r^{\mu-1}+k^{\mu-1})}. \qquad (1.7)$$

Remark 1.3. If we put $\mu=1$ in Theorem 1.1, we get Theorem F.

2. Lemmas

To prove the main result, we need the following lemmas.

Lemma 2.1. *If* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree n having no zeros in* $|z|<k$, $k\ge 1$, *then*

$$\max_{|z|=1}|p'(z)|\le n\frac{n|a_0|+k^2|a_1|}{(1+k^2)n|a_0|+2k^2|a_1|}\max_{|z|=1}|p(z)|. \qquad (2.1)$$

The result is best possible with extremal polynomial $p(z)=(z+k)^n$.

The above lemma is due to Govil, Rahman and Schmeisser [2].
Qazi [6] generalized Lemma 1.1 and proved the following

Lemma 2.2. *If* $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\le\mu\le n$ *is a polynomial of degree n not vanishing in* $|z|<k$, $k\ge 1$, *then*

$$\max_{|z|=1}|p'(z)|\le n\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}\max_{|z|=1}|p(z)|. \qquad (2.2)$$

3. Proof of the Main Theorem

Proof of Theorem 1.1. Let $0<r\le k$. Since $p(z)$ has all its zeros in $|z|\ge k$, $k\ge 1$. Therefore the polynomial $T(z)=p(rz)$ has all its zeros in $|z|\ge\frac{k}{r},\frac{k}{r}\ge 1$.

Applying Lemma 2.2 to the polynomial $T(z)$, we get

$$\max_{|z|=1}|T'(z)| \le n \frac{1+\frac{\mu}{n}\left|\frac{a_\mu r^\mu}{a_0}\right|\left(\frac{k}{r}\right)^{\mu+1}}{1+\left(\frac{k}{r}\right)^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu r^\mu}{a_0}\right|\left\{\left(\frac{k}{r}\right)^{\mu+1}+\left(\frac{k}{r}\right)^{2\mu}\right\}}\max_{|z|=1}|T(z)|,$$

which implies

$$\max_{|z|=1} r|p'(rz)| \le n \frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\frac{k^{\mu+1}}{r}}{1+\left(\frac{k}{r}\right)^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left\{\frac{k^{\mu+1}}{r}+\frac{k^{2\mu}}{r^\mu}\right\}}\max_{|z|=1}|p(rz)|$$

Or

$$\max_{|z|=r}|p'(z)| \le n \frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\frac{k^{\mu+1}}{r}}{r+\frac{k^{\mu+1}}{r^\mu}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left\{k^{\mu+1}+\frac{k^{2\mu}}{r^{\mu-1}}\right\}}\max_{|z|=r}|p(z)|. \tag{3.1}$$

As $p'(z)$ is a polynomial of degree at most $n-1$, we have by maximum modulus principle [5, p. 158, problem III 269] we have

$$\frac{M(p',t)}{t^{n-1}} \le \frac{M(p',r)}{r^{n-1}}, \text{ for } t \ge r. \tag{3.2}$$

Combining inequality (3.1) and (3.2) we have

$$M(p',t) \le \frac{t^{n-1}}{r^{n-1}} M(p',r)$$

$$M(p',t) \le \frac{t^{n-1}}{r^{n-1}} n \frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\frac{k^{\mu+1}}{r}}{r+\frac{k^{\mu+1}}{r^\mu}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left\{k^{\mu+1}+\frac{k^{2\mu}}{r^{\mu-1}}\right\}} M(p,r).$$

Now, we have for $0 \le \theta < 2\pi$

$$\left|p(Re^{i\theta})-p(re^{i\theta})\right|\le\int_r^R\left|p'(te^{i\theta})\right|dt$$

$$\le\int_r^R\frac{nt^{n-1}}{r^{n-1}}\left(\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\frac{k^{\mu+1}}{r}}{r+\frac{k^{\mu+1}}{r^\mu}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left\{k^{\mu+1}+\frac{k^{2\mu}}{r^{\mu-1}}\right\}}\right)M(p,r)\,dt$$

$$\le\frac{(R^n-r^n)}{r^{n-1}}\left[\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\frac{k^{\mu+1}}{r}}{r+\frac{k^{\mu+1}}{r^\mu}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left\{k^{\mu+1}+\frac{k^{2\mu}}{r^{\mu-1}}\right\}}\right]M(p,r)\,,$$

which implies

$$\left|p(Re^{i\theta})\right|\le\left[1+\frac{R^n-r^n}{r^{n-1}}\left\{\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\frac{k^{\mu+1}}{r}}{r+\frac{k^{\mu+1}}{r^\mu}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left\{k^{\mu+1}+\frac{k^{2\mu}}{r^{\mu-1}}\right\}}\right\}\right]M(p,r)$$

$$M(p,R)\le\left[1+\frac{R^n-r^n}{r^{n-1}}\left\{\frac{n|a_0|r^\mu+\mu|a_\mu|k^{\mu+1}r^{\mu-1}}{n|a_0|(r^{\mu+1}+k^{\mu+1})+\mu|a_\mu|\{k^{\mu+1}r^\mu+rk^{2\mu}\}}\right\}\right]\times M(p,r). \tag{3.3}$$

Inequality (3.3) on simplification and using the fact that $R\ge r$ reduces to

$$M(p,R)\le\left[\frac{R^{n-1}(R^{\mu+1}+k^{\mu+1})n|a_0|+\mu|a_\mu|R^nk^{\mu+1}(R^{\mu-1}+k^{\mu-1})}{r^{n-1}(r^{\mu+1}+k^{\mu+1})n|a_0|+\mu|a_\mu|r^nk^{\mu+1}(r^{\mu-1}+k^{\mu-1})}\right]M(p,r).$$

This is equivalent to inequality (1.7) and thus proof of the Theorem 1.1 is completed.

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