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Application of Homotopy Perturbation Method to Nonlinear Mathematical HIV/AIDS Model Equations

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Abstract In this paper, we obtained solutions to seven nonlinear known ordinary differential equations derived from a developed mathematical model of an HIV/AIDS. We apply Homotopy Perturbation Method to those equations. In each of the equations, we obtain the approximate analytical solution of the equations which are convergent in the desired domains.

Keywords HIV/AIDS, Homotopy Perturbation Method, Mathematical Model, Analytical solution, Approximate Solution

1. Introduction

In recent years, the application of the Homotopy Perturbation Method (HPM) in non-linear problems has been developed by scientists and engineers, because this method continuously deforms the difficult problem under study into a simple problem thereby making it easier to solve. The homotopy perturbation method was proposed first by He in [1] and was developed and improved by [2-3]. Homotopy, is an important part of differential topology. Actually the homotopy perturbation method is a coupling of the traditional perturbation method and the homotopy method in topology [4]. The homotopy perturbation method (HPM) provides an approximate analytical solution in a series form. HPM has been widely used by numerous researchers successfully for different physical systems such as, bifurcation, asymptotology, nonlinear wave equations, oscillators with discontinuities by [5-8], reaction-diffusion equation and heat radiation equation by [9-10] and Approximate Solution of SIR Infectious Disease Model by [11].

2. Model Formulation and Procedures

The flow transmission parameters and variables in the model building blocks are as follows; the recruitment into susceptible unvaccinated class, $S_u(t)$ by the parameter, α. This class is divided into two components; the susceptible unvaccinated and restricted, $S_{ur}(t)$, and susceptible unvaccinated that use condom $S_{uc}(t)$. These populations are recruited by the parameter, σ *and* $(1-\sigma)$ respectively. A proportion, g of the population, $S_{ur}(t)$ leaves into the susceptible unvaccinated who are restricted and also use condom, $S_{ucr}(t)$ and $(1-f_2)\lambda$ proportion of $S_{ur}(t)$ is recruited into $S_{ue}(t)$, certain fraction, h of the class, $S_{uc}(t)$ moves to the $S_{ucr}(t)$. A proportion leaves by $(1-f_1)\lambda$ of this class goes into the $S_{ue}(t)$ class. Another fraction, $\lambda(1-f)$ leaves the unvaccinated, restricted class that uses condom for exposed class. Fraction recruited into the infected class, $I(t)$ by ρ_2 from $S_{ue}(t)$ class. Population of the full blown AIDS class is recruited from the infected class at a rate, γ. There is the disease induced death rate, d_1 in the full blown AIDS class. The parameter. c is the campaign for the usage of condom and the restriction of free movement. The force of infection is denoted by σ. There is natural death rate, μ in all the compartments.

The following are the corresponding model equations emanating from the assumptions

$$\left.\begin{aligned}
\frac{dS_u}{dt} &= \alpha - \sigma c S_u - (1-\sigma)cS_u - \mu S_u \\
\frac{dS_{uc}}{dt} &= \sigma c S_u - (1-f_1)\lambda S_{uc} - hS_{uc} - \mu S_{uc} \\
\frac{dS_{ur}}{dt} &= (1-\sigma)cS_u - gS_{ur} - (1-f_2)\lambda S_{ur} - \mu S_{ur} \\
\frac{dS_{ucr}}{dt} &= hS_{uc} + gS_{ur} - (1-f)\lambda S_{ucr} - \mu S_{ucr} \\
\frac{dS_{ue}}{dt} &= (1-f_1)\lambda S_{uc} + (1-f)\lambda S_{ucr} + (1-f_2)\lambda S_{ur} - \rho_1 S_{ue} - \mu S_{ue} \\
\frac{dI}{dt} &= \rho_1 S_{ue} - \gamma I - \mu I \\
\frac{dA}{dt} &= \gamma I - dA - \mu A
\end{aligned}\right\} \quad (1)$$

3.0 Basic Idea of He's Homotopy Perturbation Method

To demonstrate the basic idea of He's homotopy perturbation method, we consider the non linear differential equation, [3].

$$A(u) - f(r) = 0 \qquad r \in \Omega \qquad (2)$$

Subject to the boundary condition of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \qquad (3)$$

Where;

A is the general differential operator,

B is the boundary operator

$f(r)$; a known analytical solution and

Γ is the boundary of the domain Ω , [12]

The general operator , A can be divided into two parts namely; L and N where L is linear part and N is the non linear part. Hence (2) can therefore be written as;

$$L(u) + N(u) - f(r) = 0 \qquad r \in \Omega \qquad (4)$$

We shall now construct a homotopy $V(r,p)$ such that

$V(r,p): \Omega \times [0,1] \to R$ which satisfies

$$H(r,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \qquad (5)$$

$P \in [0,1], \quad r \in \Omega$

OR

$$H(r,p) = L(v) - L(u_0) + pL(u_0) + [N(v) - f(r)] = 0 \qquad (6)$$

Where

$L(u)$ is the linear part

$L(u) = L(v) - L(u_0) + pL(u_0)$ and $N(u)$ is the non-linear part.

$N(u) = pN(v)$

$P \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of equation (2) which satisfies the boundary conditions.

Obviously, considering equations(5) and (6), we have

$$H(v,0) = L(v) - L(u_0) = 0 \qquad (7)$$

$$H(v,1) = A(v) - f(r) = 0 \qquad (8)$$

The changing process of p from zero to unity is just that of $V(r,p)$ from u_0 to $u(r)$. In topology, this is called deformation while $L(v) - L(u_0)$, $A(v) - f(r)$ are called homotopy.

According to Homotopy perturbation method (HPM), we can first use the embedding parameter, p as a small parameter and assume solution for equation (5) and (6) which can be expressed as;

$$V = v_0 + pv_1 + p^2 v_2 + \cdots \qquad (9)$$

If we let $p = 1$, we will obtain an approximate solution of equation (9) as

$U = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots$ (10)

Equation (10) is the analytical solution of (2) by homotopy perturbation method.

He (2003), (2006) makes the following suggestion for convergence of (10)

(1). The second derivative of $N(v)$ wrt V must be small because parameter, p must be relatively large i.e $p \to 1$

(2). The norm of $L^{-1}\frac{\partial N}{\partial V}$ must be smaller than one so that the series converge.

Considering the following systems of non-linear ordinary differential equations in (1)

We let,

$k_2 = (1-\sigma)c,\ \ k_3 = (k_2 + \sigma c + \mu),\ k_4 = (1-f_1)\lambda,$

$k_5 = (k_4 + h + \mu), k_6 = (1-f_2)\lambda,\ k_7 = (k_6 + g + \mu),$

$k_8 = (1-f)\lambda\,, k_9 = (k_8 + \mu),\ \ k_{10} = (\rho_1 + \mu), k_{11} = (\gamma + \mu), k_{12} = (d + \mu)$

Rewriting (1) in a more compact form we get

$$\left.\begin{aligned}
\frac{dS_u}{dt} &= \alpha S - k_3 S_u \\
\frac{dS_{uc}}{dt} &= \sigma c S_u - k_5 S_{uc} \\
\frac{dS_{ur}}{dt} &= k_2 S_u - k_7 S_{ur} \\
\frac{dS_{ucr}}{dt} &= hS_{uc} + gS_{ur} - k_9 S_{ucr} \\
\frac{dS_{ue}}{dt} &= k_4 S_{uc} + k_8 S_{ucr} + k_6 S_{ur} - k_{10} S_{ue} \\
\frac{dI}{dt} &= \rho_1 S_{ue} - k_{11} I \\
\frac{dA}{dt} &= \gamma I - k_{12} A
\end{aligned}\right\} \quad (11)$$

We now apply HPM on the system (11) by assuming the solution as;

$$\left.\begin{aligned}
S_u &= r_0 + Pr_1 + P^2 r_2 + \cdots \\
S_{uc} &= t_0 + Pt_1 + P^2 t_2 + \cdots \\
S_{ur} &= u_0 + Pu_1 + p^2 y_2 + \cdots \\
S_{ucr} &= v_0 + Pv_1 + P^2 v_2 + \cdots \\
S_{ue} &= w_0 + Pw_1 + P^2 w_2 + \cdots \\
I &= x_0 + Px_1 + P^2 x_2 + \cdots \\
A &= y_0 + Py_1 + P^2 y_2 + \cdots
\end{aligned}\right\} \quad (12)$$

From the first equation of (11),

$\frac{dS_u}{dt} = \alpha - k_3 S_u$,

The linear part is

$\frac{dS_u}{dt} = 0$ and the non-linear part is

$\alpha - k_3 S_u$

We now apply HPM

$\Rightarrow (1-P)\frac{dS_u}{dt} + P\left[\frac{dS_u}{dt} - \alpha + k_3 S_u\right] = 0$

Expanding, this gives

$\frac{dS_u}{dt} - P\frac{dS_u}{dt} + P\frac{dS_u}{dt} - P(\alpha - k_3 S_u) = 0$

$\Rightarrow \frac{dS_u}{dt} - P(\alpha - k_3 S_u) = 0$

$\Rightarrow \frac{dS_u}{dt} - P\alpha + Pk_3 S_u = 0$ (13)

Substituting the first and second equations of (12) into (13) gives

$(r_0' + Pr_1' + P^2 r_2' + \cdots +) - P\alpha + Pk_3(r_0 + Pr_1 + P^2 r_2 + \cdots)$

Collecting the coefficient of powers of P, we have;

$$\left.\begin{aligned}
P^0 &: r_0' = 0 \\
P^1 &: r_1' - \alpha + k_3 r_0 = 0 \\
P^2 &: r_2' - \alpha + k_3 r_1 = 0
\end{aligned}\right\} \quad (14)$$

Similarly we have;

$$\left.\begin{aligned} &P^0: t_0' = 0 \\ &P^1: t_1' - \sigma c r_0 + k_5 t_0 = 0 \\ &P^2: t_2' - \sigma c r_1 + k_5 t_1 = 0 \end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned} &P^0: u_0' = 0 \\ &P^1: u_1' - k_2 r_0 + k_7 u_0 = 0 \\ &P^2: u_2' - k_2 r_1 + k_7 u_1 = 0 \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} &P^0: v_0' = 0 \\ &P^1: v_1' - h t_0 - g u_0 + k_9 v_0 = 0 \\ &P^2: v_2' - h t_1 - g u_1 + k_9 v_1 = 0 \end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned} &P^0: w_0' = 0 \\ &P^1: w_1' - k_4 t_0 - k_6 u_0 - k_8 v_0 + k_{10} w_0 = 0 \\ &P^2: w_2' - k_4 t_1 - k_6 u_1 - k_8 v_1 + k_{10} w_1 = 0 \end{aligned}\right\} \quad (18)$$

$$\left.\begin{aligned} &P^0: x_0' = 0 \\ &P^1: x_1' - \rho_2 w_0 + k_{11} x_0 = 0 \\ &P^2: x_2' - \rho_2 w_1 + k_{11} x_1 = 0 \end{aligned}\right\} \quad (19)$$

$$\left.\begin{aligned} &P^0: y_0' = 0 \\ &P^1: y_1' - \gamma x_0 + k_{12} y_0 = 0 \\ &P^2: y_2' - \gamma x_1 + k_{12} y_1 = 0 \end{aligned}\right\} \quad (20)$$

From the first equation of (14),

$r_0' = 0$

$\frac{dr_0}{dt} = 0$

$\Rightarrow dr_0 = 0$

Integrating gives us

$\int dr_0 = S_{u0}$

$\therefore r_0 = d_1$

Where d_1 is constant of integration. Applying the initial condition we have

$r_0(0) = S_{u0}$

$\Rightarrow d_1 = S_{u0}$

$\therefore r_0 = S_{u0}$ (21)

Similarly, we have that;

$\therefore t_0 = S_{uc0}$ (22)

$\therefore u_0 = S_{ur0}$ (23)

$\therefore v_0 = S_{ucr0}$ (24)

$\therefore w_0 = S_{ue0}$ (25)

$\therefore x_0 = I_0$ (26)

$\therefore y_0 = A_0$ (27)

From the second equation of (14),

$r_1' - \alpha + k_3 r_0 = 0,$

$r_1' = \alpha - k_3 r_0$

$\Rightarrow \frac{dr_1}{dt} = \alpha - k_3 r_0$

$\Rightarrow dr_1 = (\alpha - k_3 r_0)dt$ (28)

Substituting (21) into (28) we obtain;

$dr_1 = (\alpha - k_3 S_{u0})dt$

Integrating with respect to t, we have;

$r_1 = (\alpha - k_3 S_{u0})t + d_9$

Where d_9 is constant of integration. Applying the initial condition we have;

$r_1(0) = 0, \quad \Rightarrow d_9 = 0$

$\therefore r_1 = (\alpha - k_3 S_{u0})t$ (29)

Similarly,

$\therefore t_1 = (\sigma c S_{u0} - k_5 S_{uc\,0})t$ (30)

$\therefore u_1 = (k_2 S_{u0} - k_7 S_{ur\,0})t$ (31)

$\therefore v_1 = (h S_{uc\,0} + g S_{ur\,0} - k_9 S_{ucr\,0})t$ (32)

$\therefore w_1 = (k_4 S_{uc\,0} + k_6 S_{ur\,0} + k_8 S_{ucr\,0} - k_{10} S_{ue\,0})t$ (33)

$\therefore x_1 = (\rho_2 S_{ue\,0} - k_{11} I_0)t$ (34)

$\therefore y_1 = (\gamma I_0 - k_{12} A_0)t$ (35)

From the third equation of (14),

$r_2' - \alpha + k_3 r_1 = 0$

$r_2' = \alpha - k_3 r_1$

$\Rightarrow \frac{dr_2}{dt} = \alpha - k_3 r_1$

$\Rightarrow dr_2 = (\alpha - k_3 r_1)\, dt$ (36)

Substituting (29) into (36) we obtain;

$dr_2 = [\alpha - k_3(\alpha - k_3 S_{u0})t]dt$

$dr_2 = [\alpha - k_3(\alpha - k_3 S_{u0})]tdt$

$dr_2 = [\alpha - \alpha k_3 + k_3^2 S_{u0}]tdt$

Integrating both sides with respect to t, we have;

$r_2 = [\alpha - \alpha k_3 + k_3^2 S_{u0}]\frac{t^2}{2} + d_{17}$

Where d_{17} is constant of integration. Applying the initial condition we have;

$r_2(0) = 0, \quad \Rightarrow d_{17} = 0$

$\therefore r_2 = [\alpha - \alpha k_3 + k_3^2 S_{u0}]\frac{t^2}{2}$ (37)

Similarly,

$\therefore t_2 = [\sigma c \alpha - c\sigma k_5 S_{u0} + k_5^2 S_{uc\,0}]\frac{t^2}{2}$ (38)

$\therefore u_2 = [\alpha - (k_2 k_3 + k_2 k_7) S_{u0} + k_7^2 S_{ur\,0}]\frac{t^2}{2}$ (39)

$\therefore v_2 = [(h\sigma c - k_2 g) S_{u0} - (hk_3 + hk_9) S_{uc\,0} - (gk_7 + gk_9) S_{ur\,0} + k_9^2 S_{ucr\,0}]\frac{t^2}{2}$ (40)

$\therefore W_2 = [(k_4 \sigma c + k_2 k_6) S_{u0} - (k_4 k_5 + k_4 k_{10} - hk_8) S_{uc\,0} - (k_6 k_7 + k_6 k_{10} - gk_8) S_{ur\,0} - (k_8 k_{10} + gk_9) S_{ucr\,0} + k_{10}^2 S_{ue\,0}]\frac{t^2}{2}$ (41)

$\therefore x_2 = [\rho_2 k_4 S_{uc\,0} + (\rho_2 k_6 + \rho_2 k_8) S_{ucr\,0} - (\rho_2 k_{10} + \rho_2 k_{11}) S_{ue\,0} + k_{11}^2 I_0]\frac{t^2}{2}$ (42)

$\therefore y_2 = [\gamma \rho_2 S_{ue\,0} - (\gamma k_{11} + \gamma k_{12}) I_0 + k_{12}^2 A_0]\frac{t^2}{2}$ (43)

Substituting (21) , (29) and (37) into the first equation of (12), we obtain;

$S_u(t) = S_{u0} + P(\alpha - k_3 S_{u0})t + P^2[\alpha + k_3^2 S_{u0}]\frac{t^2}{2}$ (44)

Setting $p = 1$, the solution (52) becomes;

$S_u(t) = S_{u0} + (\alpha - k_3 S_{u0})t + [\alpha + k_3^2 S_{u0}]\frac{t^2}{2} + \ldots$ (45)

Similarly,

$S_{uc}(t) = S_{uc\,0} + (\sigma c S_{u0} - k_5 S_{uc\,0})t + [\sigma c \alpha S_0 - c\sigma k_5 S_{u0} + k_5^2 S_{uc\,0}]\frac{t^2}{2} + \ldots$ (46)

$S_{ur}(t) = S_{ur\,0} + (k_2 S_{u0} - k_7 S_{ur\,0})t + [-(k_2 k_3 + k_2 k_7) S_{u0} + k_7^2 S_{ur\,0}]\frac{t^2}{2} + \ldots$ (47)

$S_{ucr}(t) = S_{ucr\,0} + (h S_{uc\,0} + g S_{ur\,0} - k_9 S_{ucr\,0})t + [(h\sigma c - k_2 g) S_{u0} - (hk_3 + hk_9) S_{uc\,0} - (gk_7 + gk_9) S_{ur\,0} + k_9^2 S_{ucr\,0}]\frac{t^2}{2} + \ldots$ (48)

$S_{ue}(t) = S_{ue\,0} + (k_4 S_{uc\,0} + k_6 S_{ur\,0} + k_8 S_{ucr\,0} - k_{10} S_{e0})t + [(k_4 \sigma c + k_2 k_6) S_{u0} - (k_4 k_5 + k_4 k_{10} - hk_8) S_{uc\,0} - (k_6 k_7 + k_6 k_{10} - gk_8) S_{ur\,0} - (k_8 k_{10} + gk_9) S_{ucr\,0} + k_{10}^2 S_{ue\,0}]\frac{t^2}{2} + \ldots$ (49)

$$I(t) = I_0 + (\rho_2 S_{ue\,0} - k_{11} I_0)t + [\rho_2 k_4 S_{uc\,0} + (\rho_2 k_6 + \rho_2 k_8) S_{ucr\,0} - (\rho_2 k_{10} + \rho_2 k_{11}) S_{ue\,0} + k_{11}^2 I_0] \frac{t^2}{2} + \ldots \quad (50)$$

$$A(t) = A_0 + (\gamma I_0 - k_{12} A_0)t + [\gamma \rho_2 S_{ue\,0} - (\gamma k_{11} + \gamma k_{12}) I_0 + k_{12}^2 A_0] \frac{t^2}{2} + \ldots \quad (51)$$

Hence, equations (45) to (51) are our model equations in HPM.

4. Conclusion

In this paper, we solved some nonlinear time dependent ordinary differential equations using Homotopy Perturbation Method. We considered eight systems of nonlinear ordinary differential equations arising from the developed mathematical model. We applied He's same structure in handling the model equations when applying homotopy perturbation method, HPM and obtained approximate analytical solutions. The result entails the efficiency of homotopy perturbation method in solving nonlinear equations.

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