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Language of Incidence matrices of X-labeled graphs

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Abstract The aim of this work is to give the definition of the language of the model of incidence matrices of X-labeled connected graphs and then the up – down language of this model. We deduced that the universal language of the up-down language is a free group generated by the up-down language and then has length function. Moreover the up-down language is an up-down pregroup and their universal language is isomorphic to the universal group of the up-down pregroup of the model.

Keywords Up-down language, universal language of up-down language, length function of universal language and pregroup of up-down language

1. Introduction

We continue to give more applications of the model of incidence matrix of *X*- labeled connected graph. The basic concepts of the model of incidence matrix of *X*- labeled connected graph and its applications have been given in [1-4]. This model is a new description of *X*- labeled connected graphs, to let us write down algorithms and then write computer programs for those algorithms as we have done in [1-4]. Therefore we give a new concept for this model which is called the Language of the incidence matrix of *X*-labeled connected graph and their up-down language. Moreover the universal language of the up-down language, length function and the up-down pregroup and it's universal language will be isomorphic to the universal group of the up-dowon pregroups of the incidence matrix of *X*-labled connected graph. Therefore this work divides into six sections; In section one we give an introduction, in section two we give basic definitions of graphs free groups and incidence matrices of *X*-labeled connected graphs that will be use in the rest of this project. In section three we give the definition of language of incidence matrix of *X* – labeled graph and it's universal language . Moreover we give the definition of up-down language and it's universal language. In section four we define a length function on the universal language of up-down language. In section five we give the definition of an up-down language and it's universal. In section six we give the conclusion.

2. Basic Concepts

Let *F* be a group and *X* be a subset of *F*; then *F* is said **free group** on *X* if and only if the following two conditions hold:

i) *X* generates *F*, ii) there is no non-trivial relation between the elements of *X*.

A directed graph Γ is called a ***X*- labeled graph**, if each directed edge *e* of Γ is labeled by an element *x* of the set *X*.

Let Γ be any *X* – Labeled connected graph without loops (where $X = \{a,b\}$), then in [1] we gave the definition of **incidence matrix** of *X* – Labeled connected graph Γ which is an $n \times m$ incidence matrix $[x_{ij}]$, where $1 \le i \le n, 1 \le j \le m$) with x_{ij} entries such that

$$x_{ij} = \begin{cases} x \ \ if\ v_i = \ \ i(e_j)\ \ and \ \ e_j \ \ lables \ \ x \in X \\ 0\ if\ v_i \ \ is \ \ not\ incident \ \ with \ \ e_j \\ x^{-1}\ if\ v_i = \ \ \tau(e_j)\ \ and \ \ e_j \ \ labeles \ \ x \in X \end{cases}$$

N.B. i) Incidence matrices of X – Labeled graphs Γ will be denoted by $M_X(\Gamma)$.

ii) If $X = \{a, b\}$ and the X – Labeled connected graph Γ has loops with labeling *a* or *b*, then choose a mid point on all edges labeled *a* or *b* to make all of them two edges labeled *aa* or *bb* respectively .

iii) in the rest of this work we will assume that all X – Labeled graphs Γ are without loops.

Now let $M_X(\Gamma)$ be an $n \times m$ incidence matrix $[x_{ij}]$ of X – Labeled graph Γ and let r_i and c_j be a row and a column in $M_X(\Gamma)$ respectively. If x_{ij} is a non – zero entry in the row r_i, then r_i is called an **incidence row** with the column c_j at the non – zero entry $x_{ij} \in X \cup X^{-1}$ and if $x_{ij} \in X$, then the row r_i is called the **starting row** (denoted by $s(c_j)$) of the column c_j and the row r_i is called the **ending row** (denoted by $e(c_j)$) of the column c_j if $x_{ij} \in X^{-1}$. If the rows r_i and r_k are incidence with column c_j at the non – zero entries x_{ij} and x_{kj} respectively, then we say that the rows r_i and r_k are **adjacent**. If c_j and c_h are two distinct columns in $M_X(\Gamma)$ such that the row r_i is incidence with the columns c_j and c_h at the non – zero entries x_{ij} and x_{ih} respectively (where $x_{ij}, x_h \in X \cup X^{-1}$), then we say that c_j and c_h are **adjacent columns**. For each column *c* there is an inverse column denoted by $\bar{c}$ such that $s(\bar{c}) = e(c), e(\bar{c}) = s(c)$ and $\bar{\bar{c}} = c$.

A **scale** in $M_X(\Gamma)$ is a finite sequence of form $S = r_1, c_1^{\in_1}, r_2, c_2^{\in_2}, \ldots, r_{k-1}, c_{k-1}^{\in_{k-1}}, r_k$, where $k \geq 1$, $\in = \mp$, $s(c_j^{\in_j}) = r_j$, and $e(c_j^{\in_j}) = r_{j+1} = s(c_{j+1}), 1 \leq j \leq k-1$. The starting row of a scale $S = r_1, c_1^{\in_1}, r_2, c_2^{\in_2}, \ldots, r_{k-1}, c_{k-1}^{\in_{k-1}}, r_k$ is the starting row r_1 of the column c_1 and the **ending row** of the scale *S* is the ending row r_k of the column c_{k-1} and we say that *S* is a scale from r_1 to r_k and *S* is a scale of length *k* for $1 \leq j \leq k-2$. If $s(S) = e(S)$, then the scale is called **closed scale**. If the scale S is reduced and closed , then S is called a **circuit** or a **cycle.** Two rows r_i and r_k in $M_X(\Gamma)$ are called **connected** if there is a scale *S* in $M_X(\Gamma)$ containing r_i and r_k. More over $M_X(\Gamma)$ is called **connected** if any two rows r_i and r_k in $M_X(\Gamma)$ are connected by a scale *S*. If $M_X(\Gamma)$ is a connected and without any closed scale, then $M_X(\Gamma)$ is called a **tree** incidence matrix of X – Labeled graph Γ. Let Ω be a subgraph of Γ, then $M_X(\Omega)$ is called a **sub incidence matrix** of $M_X(\Gamma)$, if the set of rows and columns of $M_X(\Omega)$ are subsets of $M_X(\Gamma)$ and if *c* is a column in $M_X(\Delta)$, then $s(c), e(c)$ and $\bar{c}$ have the same meaning in $M_X(\Gamma)$ as they do in $M_X(\Omega)$.

A **component** of $M_X(\Gamma)$ is a maximal connected **sub incidence matrix** of $M_X(\Gamma)$. If $M_X(\Omega)$ is a **sub incidence matrix** of $M_X(\Gamma)$, and every two rows r_i and r_k in $M_X(\Gamma)$ are joined by at least one scale *S* in $M_X(\Omega)$, then $M_X(\Omega)$ is called **spanning incidence matrix** of $M_X(\Gamma)$ and $M_X(\Omega)$ is called **spanning tree** of $M_X(\Gamma)$ if $M_X(\Omega)$ is spanning and tree incidence matrix. The inverse of $M_X(\Gamma)$ is incidence matrix of X^{-1} - labeled graph Γ.

Lemma 2.1[4]: If Γ is a connected *X*- Labeled graph, then $M_X(\Gamma)$ is a connected incident matrix of *X*-Labeled graph Γ.

Definition 2.2. Let $M_X(\Gamma)$ be an incidence Matrix of *X*- labeled graph Γ. If $M_X(\Gamma)$ does not contain any row r_i with non zero entries x_{ij} and x_{ik} in $X \cup X^{-1}$ such that $x_{ij} = x_{ik}$, then $M_X(\Gamma)$ is called a **folded incidence matrix** of *X* – Labeled graphs Γ**.** Otherwise it is called **non- folded** incidence matrix of X-labeled graph.

Lemma 2.3 . If Γ is a folded *X*- Labeled graph, then $M_X(\Gamma)$ is a folded incident matrix of *X*-Labeled graph.

Proof. See [4].

3. Language of Incidence matrices of *X*-labeled connected graphs

In this section we will give the definition of language of Incidence matrix of *X*-labeled graphs, the up – down language of the incidence matrix of *X*- labeled graph. Moreover we give the incidence matrix of the universal language of the up-down language .

Definition 3.1: Let $M_X(\Gamma)$ be a folded incidence matrix of *X*-labeled connected graph Γ.

The **directed incidence matrix of *X*-labeled connected graph** Γ can be construct as follows,

i) choose a base row $r^* = r_1$;

ii) choose a maximal tree incidence matrix of *X*- labeled connected graph $M_X(T)$ from $M_X(\Gamma)$

iii) make the direction of all columns of $M_X(T)$ be away from the base row $r^* = r_1$, that if the direction of a column *c* in $M_X(T)$ is down, then make it up c^{-1} with non-zero entry x_c^{-1} at the starting row $r_c = s(c^{-1})$, $x_c \in X$ such that $s(c^{-1}) = x^{-1}$, $e(c^{-1}) = x$;

iv) the direction of all columns $c \in M_X(\Gamma)/M_X(T)$ be as in $M_X(\Gamma)$, away from the base row $r^* = r_1$.

Note: i) the directed incidence matrix of *X*-labeled graphs $M_X(\Gamma)$ with respect to the base row $r^* = r_1$ is denoted by $M_X(\Gamma, r^*)$.

ii) let $S = c_{j_1}, c_{j_2}, \cdots, c_{j_n}$ be an up reduce scale in $M_X(\Gamma, r^*)$ with non – zero entries $x_{c_{j_1}}, x_{c_{j_2}}, \cdots, x_{c_{j_n}}$, where $x_{c_{j_t}} \in X \cup X^{-1}$, $t = 1,2,\cdots,n$, then the non – zero entries $x_{c_{j_1}}, x_{c_{j_2}}, \cdots, x_{c_{j_n}}$, of the up reduce scale S is called **the up reduced word** of type S.

Therefore choose *U* to be the set of all up reduced words $w = x_{c_{j_1}}.x_{c_{j_2}} \cdots x_{c_{j_n}}$ of type S in $M_X(\Gamma, r^*)$ with non-zero entries $x_{c_j} \in X \cup X^{-1}$ and starting at the base row $r^* = r_1$.

Now let $w = x_{c_{j_1}}.x_{c_{j_2}} \cdots x_{c_{j_n}}$ be the up reduced word of type S in $M_X(\Gamma, r^*)$ with non-zero entries $x_{c_j} \in X \cup X^{-1}$ starting at the base row $r^* = r_1$ in $M_X(\Gamma, r^*)$. Since $M_X(\Gamma, r^*)$ is a finite incidence matrix of *X*-labeled graph Γ, so all up reduced words $u = x_{c_{j_1}}.x_{c_{j_2}} \cdots x_{c_{j_n}}$ of type S in $M_X(\Gamma, r^*)$ with non-zero entries $x_{c_j} \in X \cup X^{-1}$ are finite sequences of columns directed away from the base row $r^* = r_1$ such that the rows of the up reduced scale S are $r^* = r_1, r_2, \cdots, r_n$.

Note: the column c_{j_t} and the non-zero entry $x_{c_{j_t}}$ will be denote by c_t and x_{c_t} respectively.

Therefore a word of type S in $M_X(T)$ is a word of form $u = x_{c_1}x_{c_2}\cdots x_{c_n}$, where x_{c_j} is the non- zero entry of the starting row r_t of the column c_{j_t}, i.e. $x_{c_{j_t}} = s(c_{j_t})$,

$x_{c_{j_t}} \in X \cup X^{-1}$ and $t = 1,2,\cdots,n$. Therefore every word $u = x_{c_1}x_{c_2}\cdots x_{c_n}$ must be reduced in $M_X(\Gamma, r^*)$.

Definition 3.2. Let $S = c_{j_2}, c_{j_2}, \cdots, c_{j_n}$ and $S' = c'_{j_2}, c'_{j_2}, \cdots, c'_{j_n}$ be up-reduced scales in $M_X(\Gamma, r^*)$ such that both of them starting at $r^* = r_1$ and let $u = x_{c_1}x_{c_2}\cdots x_{c_n}$ and $u' = x'_{c_1}x'_{c_2}\cdots x'_{c_m}$ be up reduced words of types *S* and S' respectively, where $n \le m$. If $c_j = c'_j$ and $x_{c_j} = x'_{c'_j}$, for $1 \le j \le n$, then the word *u* is said to be **the initial subword** of the word u', and denoted by $u < u'$.

Definition3.3. Let $S = c_{j_2}, c_{j_2}, \cdots, c_{j_n}$ be an up reduced scale in $M_X(\Gamma, r^*)$, then *S* is called a **maximal up reduced scale** in $M_X(\Gamma, r^*)$ if $e(S) = e(c_{j_n}) = r_{i_n}$ is maximal row in $M_X(\Gamma, r^*)$.

Definition 3.4. The up reduced word $u = x_{c_1}x_{c_2}\cdots x_{c_n}$ of type S is said to be a **maximal up reduced word** in $M_X(\Gamma, r^*)$ if $e(u) = e(c_{j_n}) = x_{c_n}$ is a maximal row r_n at the non- zero entry $x_{c_{j_n}}$ in $M_X(\Gamma, r^*)$.

Definition 3.5. Let $u = x_{c_1}x_{c_2}\cdots x_{c_n}$ be an up reduced word of type S in $M_X(\Gamma, r^*)$ and let $x_{c_{j_n}}$ be the non zero entry of the starting row $r_n = s(c_{j_n})$ of the column c_{j_n} in $M_X(\Gamma, r^*)/M_X(T, r^*)$, such that $s(c_{j_n}) = e(c_{j_{n-1}})$, then we define the set

$U^* = \{ux_{c_{j_n}}; u \in U, ux_{c_{j_n}} \in U$ if $c_{j_n} \in M_X(T, r^*)$ and $ux_{c_{j_{n+1}}} \notin U$ if

$c_{j_n} \in M_X(\Gamma^*, r^*)/M_X(T, r^*)$ with non-zero entry $x_{c_{j_n}} \in X \cup X^{-1}$.

Note: i) It is clear that $U \subseteq U^* \subseteq S(X)$ the set of all reduced word generated by $X = \{a, b\}$.

ii) in the rest of this work we will denote the column c_{j_n} and the non-zero entry $x_{c_{j_n}}$ of the starting row $r_n = s(c_{j_n})$ of the column c_{j_n} by c_n and x_{c_n} respectively.

Definition 3.6. Let u^* and v^* be any two up reduced words of types S in U^*, then we say that $u^* \le v^*$, if u^* is an **up subword** of v^*, $u^* < v^*$, if u^* is an **up proper subword** of v^* and $u^* \approx v^*$ if $u^* \le v^*$ and $v^* \le u^*$.

Lemma 3.7. The relation $\approx$ defined above is an equivalence relation.
Proof: By direct calculations the result follows.■

Lemma 3.8. Let U^* be defined as above, then U^* has exactly one up reduced word of type S of each element in U^* under the equivalence relation $\approx$ defined above.
Proof. Let u^* and v^* be any two elements in U^*, so $u^* = ux_c$ and $v^* = vx'_{c'}$ and suppose that $u^* \approx v^*$ .Since $U^* \subseteq S(X)$, so each up reduced word of type S is unique in U^*, $ux_c \le vx'_{c'}$ and $vx'_{c'} \le ux_c$. Hence $u = v$, $x_c = x'_{c'}$ and $c = c'$. Therefore $u^* = v^*$.■

Lemma 3.9. The elements of the set U^* form a tree like incidence matrix of *X*-labeled graph like, that if u^*, v^* and w^* are any elements in U^*, such that $u^* \le w^*$ and $v^* \le w^*$, then $u^* \le v^*$ or $v^* \le u^*$. Moreover the relation $\approx$ is transitive.

Proof: Let u^*, v^* and w^* be any up reduced words of types S in U^*, so $u^* = ux_c$, $v^* = vx'_{c'}$ and $w^* = wx''_{c''}$. Since $u^* \le w^*$ and $v^* \le w^*$, so either u^*, v^* are both of them in U or one of them is not in U. If u^*, $v^* \in U$ so $u^* < v^*$ or $v^* < u^*$. If $u^* \notin U$ or $v^* \notin U$, then $u^* \approx w^*$ or $v^* \approx w^*$ respectively. Therefore in both cases we get that $v^* \le u^*$ or $u^* \le v^*$ respectively. By the definition of the equivalence relation $\approx$ we get that $\approx$ is transitive relation.■

Note: i) the up reduced words of types S in U^* form a partially ordered tree incidence matrix of *X*- labeled connected graph with base row $r^* = r_1 = [1]$. It is denoted by $M^*_X(T^*, r^*)$. It is clear that $U \subseteq U^*$ and then $M_X(T, r^*) \subseteq M^*_X(T^*, r^*)$.

ii) since each up reduced word of type S in $M^*_X(T^*, r^*)$ is unique and the relation $\approx$ defined above is an equivalence relation, so each class is denoted by $u^* = [ux_c]$. Therefore the tree incidence matrix $M^*_X(T^*, r^*)$ will be construct as below;

i) Let the rows of $M^*_X(T^*, r^*)$ be the equivalence classes $u^* = [ux_c]$ of the set U^* and let the base row be the class $r^* = r_1 = [e]$;

ii) Join two rows $r = u^* = [ux_c]$ and $r' = v^* = [vx'_{c'}]$ by a column c' with non-zero entries x'_c and x'^{-1}_c, such that $x'_c \in X \cup X^{-1}$, if $u^* \prec v^*$ and $u^*, v^* = u^* x'_c$ are of heights $n-1$ and n respectively, such that x'_c is the non-zero entry of the starting row $r = u^*$ of the column c.

Definition 3.10. For any two reduced words $u^* = [ux_c]$, $w^* = [wx'_{c'}]$ of types S associated with two rows r_i and r_j respectively in $M^*_X(T^*, r^*)$, then we say that $u^* \cong w^*$ if and only if $u^* = ux_c \notin U$ and $w^* = wx'_{c'} \in U$, such that $e(c) = e(c')$ in $M_X(\Gamma, r^*)$.

Lemma 3.11. If $u^* = [ux_c]$, $w^* = [wx'_{c'}]$ are defined as above in $M^*_X(T^*, r^*)$, then $u^* \cong w^*$ if and only if $u^*.w^{*^{-1}}$ forms a cycle in $M_X(\Gamma, r^*)$.

Proof: $u^* \cong w^*$ if and only if $u^* = ux_c \notin U$ and $w^* = wx'_{c'} \in U$, such that $e(c) = e(c')$ in $M_X(\Gamma, r^*)$ if and only if $u^*.w^{*^{-1}}$ forms a cycle in $M_X(\Gamma, r^*)$.■

Lemma 3.12. The relation $\cong$ defined above is an equivalence relation .
Proof: It is clear that $\cong$ is an equivalence relation. ■

Lemma 3.13. For any reduced word $u^* = [ux_c]$ of type S in $M_X(T^*, r^*) / M_X(T, r^*)$, there is a unique reduced word $w^* = [wx'_{c'}]$ of type S in $M_X(T, r^*)$ such that $u^*.w^{*^{-1}}$ is a cycle and $u^* \cong w^*$.

Proof: Since $M_X(T,r^*)$ is Maximal tree incidence matrix of *X*-labeled connected graph Γ, with $X=\{a,b\}$, so each row *r* associated with a reduced word $w^*=[wx'_{c'}]$ of type S in $M_X(T,r^*)$ is of a different class. Since $u^*=[ux_c]$ is a reduced word of type S in $M_X(T^*,r^*)/M_X(T,r^*)$, so $u^*=[ux_c]$ is associated with a row *r* which is a terminal row of a column $c\notin M_X(T,r^*)$ with the labeled x_c. Hence $u^*\cong w^*$. We now suppose that there exists a n other redued word $z^*=[zx''_{c''}]$ in $M_X(T,r^*)$ such that $u^*\cong z^*$. Since $\cong$ is an equivalence relation , so $z^*\cong w^*$ and then z^*w^{*-1} forms a non trivial cycle in $M_X(\Gamma,r^*)$ a contradiction.■

Note: If $u^*\cong w^*$**,** then the reduced word $w^*=[wx'_{c'}]$ defined above will be denote by $\overline{ux_c}$.

Definition3.14. For any two columns c and c' in $M_X(T^*,r^*)$, we say that $c\sim c'$
If and only if (i) c and c' have the same non-zero entrices , (ii) $i(c)\approx i(c')$ and $t(c)\approx t(c')$.

Lemma3.15. The relation $\sim$ defined above is an equivalence relation.■

Lemma 3.16. $M_X(T^*,r^*)$ has exactly one column of each column class under the relation $\sim$.

Proof. Let c and c' be any two columns in $M_X(T^*,r^*)$**,** such that $c\sim c'$**.**

Since $M_X(T,r^*)$ has exactly one row of each row class under the relation $\approx$, so c and c' are not in $M_X(T,r^*)$. Therefore either c and c' are in $M_X(T^*,r^*)/M_X(T,r^*)$ or one of them in $M_X(T^*,r^*)/M_X(T,r^*)$ and the other in $M_X(T,r^*)$.

Case 1. if c and c' are in $M_X(T^*,r^*)/M_X(T,r^*)$, then $i(c)\approx i(c')$. But $i(c),i(c')$ are in $M_X(T,r^*)$, hence $i(c)=i(c')$, $t(c)=t(c')$ and $x_c=x_{c'}$, otherwise we get an unfolded incidence matrices of *X*-labeled core graph. Hence $c=c'$.

Case 2. If $c\in M_X(T^*,r^*)/M_X(T,r^*)$ and $c'\in M_X(T,r^*)$, then $i(c)\approx i(c')$, $t(c)\approx t(c')$ and $x_c=x_{c'}$. Moreover $i(c),i(c')$ and $t(c')$ are in $M_X(T,r^*)$. Hence $i(c)=i(c')$, $x_c=x_{c'}$, and then $M_X(T^*,r^*)$ is an unfolded incidence matrices of *X*-labeled connected graph Which is a contradiction. Hence $c=c'$.■

Lemma 3.17. If $u^*=[ux_c]$ and $v^*=[vx_{c'}]$ are two reduced words of types *S* in $M_X(T^*,r^*)$, such that $[ux_c]<[vx_{c'}]$ and $[ux_c]\cong[vx_{c'}]$, then $v^*=[vx_{c'}]\notin M_X(T,r^*)$ and $x_{c'}$ is a non-zero entry of initial row of a column $c'\notin M_X(T,r^*)$ and x_c is a non –zero entry of initial row of a column $c\in M_X(T,r^*)$.

Proof. The proof will be by contradiction. Therefore suppose that $v^*=[vx_{c'}]\in M_X(T,r^*)$. Since $[ux_c]\cong[vx_{c'}]$, so. Since $[ux_c]<[vx_{c'}]$, so $[ux_c].[vx_{c'}]^{-1}$ forms a cycle and $u^*=[ux_c]\in M_X(T,r^*)$**.** Hence $u^*=[ux_c]$ and $v^*=[vx_{c'}]$ are both in $M_X(T,r^*)$ and form a cycle a contradiction. Therefore $v^*=[vx_{c'}]\in M_X(T^*,r^*)/M_X(T,r^*)$, $x_{c'}$ is a non-zero entry of initial row of a column $c'\notin M_X(T,r^*)$ and x_c is a non –zero entry of initial row of a column $c\in M_X(T,r^*)$. ■

Corollary 3.18. If $u^* = [ux_c]$ and $v^* = [vx_{c'}]$ are two reduced words of type S in $M_X(T, r^*)$, such that $u^* < v^*$ and then $v^* = [vx_{c'}]$ is a non – zero entry of a reduced word of type *S* not in $M_X(T, r^*)$.

Proof. By above lemma 3.17 the result follows. ■

Lemma 3.19. Let $u^* = x_{c_1} x_{c_2} \cdots x_{c_n}$ be a reduced word of type *S* in $M_X(T, r^*)$, with non-zero entries x_{c_j} in $X = \{a, b\}$, $n \geq 1$. If $u^* = x_{c_1} x_{c_2} \cdots x_{c_n}$ is a reduced word of type *S* in $M_X(T, r^*)$, then $v^* = x_{c_1} x_{c_2} \cdots x_{c_{n-1}}$ is a reduced word of type *S* in $M_X(T, r^*)$.(where x_{c_i} means $x_{c_{k_i}}$, $i = 1, 2, \cdots, n$).

Proof: Since $v^* = x_{c_1} x_{c_2} \cdots x_{c_{n-1}}$ is a subword of type S in $M_X(T, r^*)$ and $\ell(v^*) < \ell(u^*)$ so $v^* < u^*$. Since $v^* = x_{c_1} x_{c_2} \cdots x_{c_{n-1}}$ is a reduced word of type *S* in $M_X(T, r^*)$, so $v^* = x_{c_1} x_{c_2} \cdots x_{c_{n-1}}$ is a reduced word of type S in $M_X(T, r^*)$ and then in *U*. ■

Definition3.20: Let $M_X(\Gamma, r^*)$ be a directed incidence matrix of *X*-labeled connected graph Γ with the base row r^* of $M_X(\Gamma, r^*)$. **The language of** $M_X(\Gamma, r^*)$ **with respect to the base row** r^* is the set of all reduced words of type S which are starting and ending at the row r^*.

Note: The language of $M_X(\Gamma, r^*)$ with respect to the row r^* is denoted by $L(M_X(\Gamma, r^*))$ **The following example** is the incidence matrix of the *X*-labeled connected graph in Fig. 3 in [5] page 614.

	e_1	e_2	e_3
r_1	a	0	b
r_2	a^{-1}	c	b^{-1}
r_3	0	c^{-1}	0

Fig.1. *the incidence matrix of the X-labeled graph that in Fig. 3 in [Ilya] page 614.*

Therefore the Language of the directed Incidence matrix of *X*-labeled connected graph $L(M_X(\Gamma, r^*))$ is the set of all non-zero reduced words of type S at rows r_1, r_2 and r_3.

Definition 3.21: If ux_c is a reduced word of type *S* in $M_X(T^*, r^*)$ and $\overline{ux_c}$ is a reduced word of type *S* in $M_X(T, r^*)$, such that $ux_c \overline{ux_c}^{-1}$ is a cycle starting and ending at the row $r^* = r_1$,. Then the set ${}^{\uparrow}U_{\downarrow}^* = \{ux_c \overline{ux_c}^{-1}; u$ is a reduced word of type *S* in $M_X(T, r^*)$ }is called **the set of up-down languages of type S in** $M_X(\Gamma, r^*)$ .It's denoted by $\overline{L}({}^{\uparrow}U_{\downarrow}^*, r^*)$. **Therefore** $M_X \overline{L}({}^{\uparrow}U_{\downarrow}^*, r^*)$ **is called the directed incidence matrix of the up-down language *of X*- labeled connected graph.**

Definition 3.22. For any two elements u^*, v^* in $M_X \overline{L}({}^{\uparrow}U_{\downarrow}^*, r^*)$, such that $u^* = ux_c \overline{ux_c}^{-1}$ and $v^* = vx_{c'} \overline{vx_{c'}}^{-1}$, then we say that $u^*.v^*$ is **defined** , whenever $u^*.v^* = u'x'_c v'x'_{c'} \overline{v'x'_{c'}}^{-1}$ is of form up-down language of the directed incidence matrix of *X*- labeled graph in reduced form. It's denoted by u^*v^* and then $u^*v^* \in \overline{L}({}^{\uparrow}U_{\downarrow}^*, r^*)$.

Note: Since the product of the elements of $\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*})$ is a partially product, so $\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*})$ is not a group in general.

Theorem 3.23. If ux_c is a reduced word of type *S* in $M_X(T^*,r^*)$ and $\overline{ux_c}$ is a reduced word of type *S* in $M_X(T,r^*)$, such that $u^* = ux_c\overline{ux_c}^{-1} \in M_X\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*})$. Let $\bigcup(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*})) = \{u_1^*.u_2^*.\cdots.u_n^*; u_i^* \in \overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}), 1\le i\le n\}$ be the set of all reduced words of up-down languages in $\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*})$, then $\bigcup(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$ **is a group generated by** $\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*})$.

Proof. It is easy to show that $\bigcup(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$ **is a group.**

We now show that the group $\bigcup(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$ **generates** by $\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*})$.

Let $x = x_{c_1}.x_{c_2}\cdots.x_{c_n}$ be a reduced word in $\bigcup(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$ starting and ending at the base row $r^* = r_1$ with non- zero entries $x_{c_i} \in X\cup X^{-1}$, $1\le i\le n$.

Now, for each element of type *S* in $M_X(T,r^*)$ starting at $r^* = r_1$, such that $u_{j+1} = \overline{u_j x_{c_j}}$, if $u_1^*.u_2^*.\cdots.u_n^*$ is a reduced up-down scales in $M_X(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$, where $u_j^* = u_j x_{c_j}\overline{u_j x_{c_j}}^{-1}$, $1\le j\le n$ and u_j^* in $M_x(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$ for all j, $1\le j\le n$.

Hence $u_1^*.u_2^*.\cdots.u_n^* = u_1x_{c_1}\overline{u_1x_{c_1}}^{-1}.u_2x_{c_2}\overline{u_2x_{c_2}}^{-1}.u_3x_3\overline{u_3x_{c_3}}^{-1}.\cdots.u_nx_{c_n}\overline{u_nx_{c_n}}^{-1}$

$= u_1x_{c_1}\overline{u_1x_{c_1}}^{-1}.\overline{u_1x_{c_1}}x_{c_2}\overline{u_2x_{c_2}}^{-1}.\overline{u_2x_{c_2}}x_{c_3}.\overline{u_3x_{c_3}}^{-1}.\cdots.\overline{u_{n-1}x_{c_{n-1}}}^{-1}\overline{u_{n-1}x_{c_{n-1}}}x_{c_n}.\overline{u_nx_{c_n}}^{-1}$

$= u_1x_{c_1}.x_{c_2}.x_{c_3}.x_{c_3}.\cdots.x_{c_n}\overline{u_nx_{c_n}}^{-1} = u_1x_{c_1}x_{c_2}.\cdots.x_{c_n}u_{n+1}^{-1}$.

Since $e(u_{n+1}^{-1}) = r^* = r_1 = 1$, $\overline{e(c_1)} = r^* = r_1 = \overline{e(u_nx_{c_n})}$ and $s(c_1) = r_n = s(u_1x_{c_1}) = s(u_1)$ so the maximal common reduced word of type S between u_1 and c_{j_1} is $r^* = r_1$, and also the maximal common reduced word of type *S* between c_{j_1} and u_{n+1} is $r^* = r_1$. Therefore $u_1 = 1$ and $u_{n+1} = 1$. Hence $x = u_1^*.u_2^*.\cdots.u_n^*$ is a reduced word generated by the set of all up-down laguages $M_X(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$ of incidence matrix of *X*-labeled graph■

Note: $M_X(\bigcup(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*})))$ is called **the directed incidence matrix of the universal language of the up – down languages** in $\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*})$ **of *X*- labeled graph.**

Lemma 3.24. If $x = u_1^*.u_2^*.\cdots.u_n^*$ is a reduced word of the universal language of the up – down languages in $M_X((\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*})))$ of type *S* in $M_X(T,r^*)$ and $x_c \in X\cup X^{-1}$ is a non zero entry of column *c* in $M_X(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$, then

(i) $ux_c\overline{ux_c}^{-1} = e$ if and only if $ux_c \in M_X(T,r^*)$ (ii) $u = \overline{\overline{ux_c}\,x_c^{-1}}$.

Proof: Since $M_X(T,r^*)$ has exactly one row of each row class, so $\overline{ux_c}$ is the only reduced word of type S of the row *r* in $M_X(T,r^*)$, such that $ux_c\overline{ux_c}^{-1}$ is a cycle in $M_X(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$, so $ux_c\overline{ux_c}^{-1} = e$ if and only if $ux_c\overline{ux_c}^{-1}$ is the trivial cycle in $M_X(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$ if and only if $ux_c=\overline{ux_c}$ if and only if $ux_c\in M_X(T,r^{*})$.

ii) Since $ux_c\in M_X(T^{*},r^{*})$ and $\overline{ux_c}\in M_X(T,r^{*})$, so $\overline{ux_c}x_c^{-1}$ is an up - down reduced subword of type S of the reduced word $\overline{ux_c}x_c^{-1}u^{-1}$ of type S in $M_X(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$, such that $t(\overline{\overline{ux_c}\,x_c^{-1}})=t(u)$, therefore u is the unique reduced word of type S in $M_X(T,r^{*})$, such that $\overline{ux_c}x_c^{-1}u^{-1}$ is a cycle in $M_X(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$. Hence u = $\overline{\overline{ux_c}\,x_c^{-1}}$.■

Lemma 3.25. If ux_c and $vx_{c'}$ are two reduced words of types S in $M_X(T^{*},r^{*})/M_X(T,r^{*})$, then either (i) $x_c\overline{ux_c}^{-1}vx_{c'}=e$ in which case $v=\overline{ux_c}$, $x_{c'}=x_c^{-1}$ and $u=vx_{c'}$ or

(ii) $x_c(\overline{ux_c})^{-1}vx_{c'}$ is a reduced word of type S of length at least 2 such that $s(x_c(\overline{ux_c})^{-1}vx_{c'})=s(x_c)$ and $e(x_c(\overline{ux_c})^{-1}vx_{c'})=e(x_c)$.

Proof: Since ux_c and $vx_{c'}$ are two reduced words of types S in $M_X(T^{*},r^{*})/M_X(T,r^{*})$ and x_c and $x_{c'}$ are the non-zero entries of columns c and c' respectively. Therefore there are unique reduced words $\overline{ux_c}$ and $\overline{vx_{c'}}$ of types S in $M_X(T,r^{*})$ such that $ux_c(\overline{ux_c})^{-1}$ and $vx_c(\overline{vx_{c'}})^{-1}$ are non-trivial cycles in $M_X(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$. Thus the maximal common reduced word of type S between $\overline{ux_c}$ and v is w, therefore either

(1) $\overline{ux_c}=w=v$, (2) $w=\overline{ux_c}$, $w<v$, (3) $w=v$, or (4) $w<v$, $w<\overline{ux_c}$ holds.

If (1) holds, then either $x_cx_{c'}=e$, then $x_c=x_{c'}^{-1}$, $\overline{ux_c}=v$, $u=\overline{vx_{c'}}$, and hence $x_c\overline{ux_c}^{-1}vx_{c'}=e$, or $x_cx_{c'}\neq e$, then $x_cx_{c'}$ is a reduced word of type S and of length 2, and then $x_x(\overline{ux_c})^{-1}vx_{c'}$ is a reduced word of type S and it is of length at least 2. Now if (2) , (3) or (4) holds, then $\overline{ux_c}^{-1}v\neq e$, hence $x_x(\overline{ux_c})^{-1}vx_{c'}$ is a reduced word of type S and it's of length at least 2, such that $s(x_c(\overline{ux_c})^{-1}vx_{c'})=s(c)$ which is the non-zero entry x_c of the column c and $e(x_c(\overline{ux_c})^{-1}vx_{c'})=e(c)$ which is the non-zero entry x_c of the column c.■

Lemma 3.26. If ux_c is a reduced word of type S in $M_X(T^{*},r^{*})/M_X(T,r^{*})$ then all reduced up – down words $ux_c(\overline{ux_c})^{-1}$ of types S are distinct and the set of them is equal to the disjoined union of the set $L^{*}=\{ux_c\overline{ux_c}^{-1};u$ is a reduced word of type S in $M_X(T)$ and x_c is a non-zero entry in the starting row of the column c in $M_X(T^{*},r^{*})/M_X(T,r^{*})$, $x_c\in X\}$ and $L^{*^{-1}}=\{(ux_c\overline{ux_c}^{-1})^{-1};ux_c\overline{ux_c}^{-1}\in L^{*}\}$.

Proof: Since $ux_c\in M_X(T^{*},r^{*})/M_X(T,r^{*})$, x_c is a non-zero entry in the starting row of the column c in $M_X(T^{*},r^{*})/M_X(T,r^{*})$, so by lemma 3.14 there exists a unique reduced word of type S in $M_X(T)$ such that $ux_c\overline{ux_c}^{-1}$ is a cycle, so $ux_c\overline{ux_c}^{-1}$ is an element in $M_X(\overline{L}(^{\uparrow}U_{\downarrow}^{*},r^{*}))$. Since all columns c with non-zero entries x_c are distinct in $M_X^{*}(T^{*})/M_X(T)$, so all reduced words $ux_c\overline{ux_c}^{-1}$ of type S in L^{*} are distinct. Since

$L^{*^{-1}} = \{(ux_c\overline{ux_c}^{-1})^{-1}; ux_c\overline{ux_c}^{-1} \in L^*\}$ and $(ux_c\overline{ux_c}^{-1})^{-1}$ is the inverse of $ux_c\overline{ux_c}^{-1}$, so $(ux_c(\overline{ux_c})^{-1})^{-1} = \overline{ux_c}(ux_c)^{-1} = \overline{ux_c}x_c^{-1}u^{-1}$ is a non-trivial cycle in $M_X(\overline{L}(^{\uparrow}U_{\downarrow}^*, r^*))$. Hence all elements of $L^{*^{-1}}$ are distinct, and then all elements of $L^* \cup L^{*-1}$ are distinct.■

4. Length function of universal language of the up- down language

In this section we show that the universal language $M_X(\bigcup(\overline{L}(^{\uparrow}U_{\downarrow}^*, r^*)))$ of the up-down language $M_X(\overline{L}(^{\uparrow}U_{\downarrow}^*, r^*))$ of incidence matrix of X – labeled graph has length function. Therefore we start with the basice definition of length function of a group.

In [6] Lyndon gave the definition of integer - valed length function on a group H to be a function $\ell : H \to Z$ satisfying the following axioms:

$A1' : \ell(e) = 0$, where e is the identity element of H;

$A2 : \ell(x) = \ell(x^{-1}),\ \forall x \in H;$

$A4 :$ if $\alpha(x, y) < \alpha(y, z)$, then $\alpha(x, y) = \alpha(x, z)$, $\forall x, y, z \in H$, where

$2\alpha(x, y) = \ell(x) + \ell(y) - \ell(xy^{-1})$

We now define a length on the reduced words of $M_X(\bigcup(\overline{L}(^{\uparrow}U_{\downarrow}^*, r^*)))$ as below.

Definition 4.1: For any reduced word $g = u_1^*.u_2^*.\cdots.u_n^*$ of type S in $M_X(\bigcup(\overline{L}(^{\uparrow}U_{\downarrow}^*, r^*)))$, defines a length

$\ell(g) = \ell(u_1^*.u_2^*.\cdots.u_n^*) = \sum_{i=1}^{n+1}(\#C(\overline{u_{i-1}x_{c_{i-1}}}^{-1}) + \#C(u_i x_{c_i}) - 2\#C(w_{i-1}))$, where

$u_i^* = u_i x_{c_i}\overline{u_i x_{c_i}}^{-1}$, #C is the number of columns, x_{c_i} is a non-zero entry in a column c_i, $x_{c_i} \in X \cup X^{-1}$, $x_{c_0} = e = x_{c_{n+1}}$, $\forall i,\ 1 \le i \le n$, $\#C(w_{i-1})$ is the number of columns in the maximal common subword w_{i-1} between $\overline{u_{i-1}x_{c_{i-1}}}^{-1}$ and $u_i x_{c_i}$.

Lemma 4.2. ℓ define a function on $M_X(\cup(\overline{L}(^{\uparrow}U_{\downarrow}^*, r^*)))$.

Proof. Let $u^* = u_1^*.u_2^*.\cdots.u_n^*$, $v^* = v_1^*.v_2^*.\cdots.v_m^*$ be reduced words in $M_X(\cup(\overline{L}(^{\uparrow}U_{\downarrow}^*, r^*)))$. Suppose that $u^* = v^*$, so $u_1^*.u_2^*.\cdots.u_n^*, = v_1^*.v_2^*.\cdots.v_m^*$. Since $u_i^* = u_i x_{c_i}\overline{u_i x_{c_i}}^{-1}$, $v_j^* = v_j y_{c_j}\overline{v_j y_{c_j}}^{-1}$, $\forall i, j$, $1 \le i \le n$, and $1 \le j \le m$, and each class u_i^* is unique, so $u_i x_{c_i} = v_j y_{c_j}$ and $\overline{u_i x_{c_i}}^{-1} = \overline{v_j y_{c_j}}^{-1}$, $n = m$. Hence $\#C(u^*) = \#C(v^*)$ and then $\ell(u^*) = \ell(v^*)$.■

Theorem 4.3. ℓ is a length function on $M_X(\cup(\overline{L}(^{\uparrow}U_{\downarrow}^*, r^*)))$.

Proof. It is clear that $A1'$ and $A2$ hold. We now show that $A4$ holds.

let $u^* = u_1^*.u_2^*.\cdots.u_n^*$, $v^* = v_1^*.v_2^*.\cdots.v_m^*$ and $z^* = z_1^*.z_2^*.\cdots.z_t^*$,be reduced words in $M_X(\cup(\overline{L}(^{\uparrow}U_{\downarrow}^*, r^*)))$. Then $u^*v^{*^{-1}} = u_1^*.u_2^*.\cdots.u_{n-1}^*.u_n^* v_m^{-1}.v_{m-1}^{*-1}.\cdots.v_2^{*-1}.v_1^{*-1}$.

Since each reduced word is unique, so $u_i x_{c_i} \overline{u_i x_{c_i}}^{-1} = .(v_i y_{c_i} \overline{v_i y_{c_i}}^{-1})$, $u_i^* = v_i^* \ \forall i$, $i = 1,2,\cdots,j$. Then $u_i x_{c_i} \overline{u_i x_{c_i}}^{-1} .(v_i y_{c_i} \overline{v_i y_{c_i}}^{-1})^{-1} = e$, $\#C(w'_{i-1})$ is the number of columns in the maximal common subword w'_{i-1} between $\overline{u_{i-1} x_{c_{i-1}}}^{-1}$ and $u_i x_{c_i}$ for all $i = 1,2,3,\cdots,j$, plus the number of columns in maximal common subwords between $\overline{u_{j-1} x_{c_{j-1}}}^{-1}$ and $\overline{v_{j-1} y_{c_{j-1}}}$ will be delete.

Therefore $u^* v^{*^{-1}} = u_1^* . u_2^* . \cdots . u_{j-1}^{*'} . v_{j-1}^{*'-1} . \cdots . v_2^{*-1} . v_1^{*-1}$ in reduced form.

Now let w_j^* be the maximal common proper ending subword between u^* and $v^{*^{-1}}$.

Since $\ell(u^* v^{*^{-1}}) = \ell(u^*) + \ell(v^{*^{-1}}) - 2\#C(w_j^*)$ and $2\alpha(u^*, v^*) = \ell(u^*) + \ell(v^*) - \ell(u^* v^{*^{-1}})$, so $2\alpha(u^*, v^*) = 2\#C(w_j^*)$.

. Therefore w_j^* is the maximal proper ending subword of u^* and v^*.

Now suppose that $z^* = z_1^* . z_2^* . \cdots . z_t^*$, $u^* = u_1^* . u_2^* . \cdots . u_n^*$ and $v^* = v_1^* . v_2^* . \cdots . v_m^*$ are reduced words in $M_X(\cup(\overline{L}(^{\uparrow}U_{\downarrow}^*, r^*)))$, such that $\alpha(u^*, v^*) < \alpha(v^*, z^*)$.

We now show that $\alpha(u^*, v^*) = \alpha(u^*, z^*)$.

Similarly $2\alpha(v^*, z^*) = 2\#C(s_k^*)$, where s_k^* is the maximal common a proper ending subword between ending of v^* and z^*. Since $\alpha(u^*, v^*) < \alpha(v^*, z^*)$, so $\#C(w_j^*) < \#C(s_k^*)$. Since w_j^*, s_k^* are proper ending subwords of v^* ,so w_j^* is a proper subword of s_k^*. Since s_k^* is a proper subword of z^*, so w_j^* is a proper subword of z^*. Hence w_j^* is the maximal common proper ending subword between u^* and z^*, and then $2\#\alpha(u^*, z^*) = 2\#C(w_j)$. Therefore $\alpha(u^*, v^*) = \alpha(u^*, z^*)$ and then ℓ is length function on $M_X(\cup(\overline{L}(^{\uparrow}U_{\downarrow}^*, r^*)))$.■

5. Up-down Language and Pregroups

In this section we show that the up-down language is an up-down pregroup.

The definition of pregroup was given by Stallings in [7] that in 1971 as a generalizion of free product with amilagmation. In [8] Stallings defined the up-down pregroup of free groups and show that the universal group of up-dowm pregroup is isomorphic to free group generated by *X*. In [9] we proved that any group with length function comes from an up-down pregroup.

Definition 5.1.[7]. A pregroup *P* consists of :

a) set *P*,

b) An element 1 in *P*,

c) A map $P \to P$, denoted by $x \mapsto x^{-1}$,

d) A subset *D* of $P \times P$,

e) A map $D \to P$, denoted by $(x, y) \mapsto xy$,

(we shall say that *xy* is defined instead of $(x, y) \in D$),

such that the following axioms are true:

$P1$: for all $x \in P$, $x1 = 1x = x$,

$P2$: for all $x \in P$, $xx^{-1} = x^{-1}x = 1$,

$P4$: for all x, y and z in P, if xy and yz are defined, then $x(yz)$ is defined if and only if $(xy)z$ is defined in which case they are equal.

P5: For any w, x, y and z in P, wx, xy and yz are defined in P, then wxy or xyz is defined in P.

Hoare **[**10**]** showed that we could prove axiom P3 above by using the following proposition, P1, P2 and P4.

Proposition 5.2: If xy is defined, then $(xy)y^{-1}$ is defined and equal to x. ■

Definition 5.3 [10**]:** For any $x \in P$, put $L(x) = \{a \in P : ax$ is defined $\}$. We write $x \le y$ if $L(y) \subseteq L(x), x < y$ if $L(y) \subset L(x)$ and $L(x) \ne L(y)$, and $x \sim y$ if $L(x) \ne L(y)$. It is clear that ~ is an equivalence relation compatible with $\le$.

The following results are taken from Stallings [7] and Rimlinger [11]. (See [10] for shorter proofs).

Proposition 5.4:

(i) If $x \le y$ or $y \le x$, then $x^{-1}y$ and $y^{-1}x$ are defined.

(ii) If xa and $a^{-1}y$ are defined , then $(xa)(a^{-1}y)$ is defined if and only if xy is defined in which case they are equal. ■

By using axiom P5 above (will be denoted by P5(i)) Rimlinger [11] proved conditions P5(ii) and P5(iii) of Lemma5.5 below.

Lemma 5.5: [10**] .** The following conditions on elements of P are equivalent :

P(i). If wx,xy and yz are defined , then either wxy or xyz is defined .

P(ii). If $x^{-1}a$ and $a^{-1}y$ are defined but $x^{-1}y$ is not , then $a < x$ and $a < y$.

P(iii). If $x^{-1}y$ is defined , then $x \le y$ or $y \le x$.■

Therefore we will say P is a pregroup, if it satisfies axioms P1, P2. P4 and the conditions of Lemma 5.5. The universal group of pregroup P has the following presentation $< P; x.y = xy$ whenever xy is defined, for $x, y, \in P >$.

Definition 5.6: For any two elements $u^*, v^* \in \overline{L}(^{\uparrow}U_{\downarrow}^*, r_1)$, such that $u^* = u_i x_{c_i} \overline{u_i x_{c_i}}^{-1}$, $v^* = v_j y_{c_j} \overline{v_j y_{c_j}}^{-1}$, then **we say** that $u^* v^*$ is definend if and only if $\overline{u_i x_{c_i}}$ is a subword of $v_j y_{c_j}$ or $v_j y_{c_j}$ is a subwoed of $\overline{u_i x_{c_i}}$.

Lemma 5.7: Axioms P1, P2 and P4 hold in $\overline{L}(^{\uparrow}U_{\downarrow}^*, r_1)$.

Proof: Since $u^* = u_i x_{c_i} \overline{u_i x_{c_i}}^{-1} = e$ **if and only if** $\overline{u_i x_{c_i}} = u_i x_{c_i}$ **by lemma 3.25(i), so** $e \in \overline{L}(^{\uparrow}U_{\downarrow}^*, r_1)$. Hence P1 holds.

Since e is the empty word, so e subword of any subword $u_i x_{c_i}$ or $\overline{u_i x_{c_i}}$, so $eu^* = u^* = u^* e$ $\forall x \in \overline{L}(^{\uparrow}U_{\downarrow}^*, r_1)$.Hence P2 holds.

Since $\overline{L}(^{\uparrow}U_{\downarrow}^*, r_1)$ is a subset of $\bigcup(\overline{L}(^{\uparrow}U_{\downarrow}^*, r_1))$ and $\bigcup(\overline{L}(^{\uparrow}U_{\downarrow}^*, r_1))$ is a group, so P4 holds. Therefor P1, P2 and P4 hold in $\overline{L}(^{\uparrow}U_{\downarrow}^*, r_1)$ ■

We now prove P5 in the following lemma.

Lemma 5.8. for any u^*, v^*, w^* in $\overline{L}({}^{\uparrow}U_{\downarrow}^*, r_1)$, such that if, $u^{*^{-1}}w^*$, $w^{*^{-1}}v^*$ are defined and $u^{*^{-1}}v^*$ is not defined in $\overline{L}({}^{\uparrow}U_{\downarrow}^*, r_1)$, then $w^* < u^*$ and $w^* < v^*$.

Proof: Let $u^* = u_i x_{c_i} \overline{u_i x_{c_i}}^{-1}, v^* = v_j y_{c_j} \overline{v_j y_{c_j}}^{-1}$ and $w^* = wz_{c_t} \overline{wz_{c_t}}^{-1}$.

Since $u^{*^{-1}}w^*$ is defined, so either $wz_{c''}$ is a subword of ux_c $\cdots$(1) or

ux_c is a subword of $wz_{c''}$ $\cdots$(2)

Since $w^{*-1}v^*$ is defined, so either $wz_{c''}$ is a sub word of $vy_{c'}$ $\cdots$ (3) or

$vy_{c'}$ is a subword of $wz_{c''}$ $\cdots$ (4).

Since $u^{*^{-1}}v^*$ is not defined, so neither ux_c is a subword of $vy_{c'}$ nor $vy_{c'}$ is a subword of ux_c .Therefore we have four cases.

Case 1: If relation (1) and (3) hold,

then $w^* \le u^*$ and $w^* \le v^*$. Therefore $L(u^*) \subseteq L(w^*)$ and $L(v^*) \subseteq L(w^*)$.

Since neither ux_c is a subword of $vy_{c'}$ nor $vy_{c'}$ is a subword of ux_c, so $L(u^*) \not\subset L(v^*)$ and $L(v^*) \not\subset L(u^*)$. Therefore there exist $a, b \in \overline{L}({}^{\uparrow}U_{\downarrow}, r^*)$, such that $a \in L(u^*)$ and $a \notin L(v^*)$. Also $b \in L(v^*)$ and $b \notin L(u^*)$.

Hence $a \in L(w^*)$ and $a \notin L(v^*)$, and then $L(v^*) \subset L(w^*)$. Also $b \in L(w^*)$ and $b \notin L(u^*)$, then $L(v^*) \subset L(w^*)$. Hence $w^* < u^*$ and $w^* < v^*$.

Other cases give us contradictions. Hence P5 holds■

Theorem 5.9: $\overline{L}({}^{\uparrow}U_{\downarrow}^*, r_1)$ is an up-down pregroup.

Proof: By Lemmas 5.7 and 5.8 the result follows.■

6. Conclusion

This work and the previous works that we have done in [1-4] appear the flexibility of the model of incidence matrix of *X*- labeled graph. This model provides a powerful tool to write computer program for any X- labeled graph which appears that any *X*-labeled graph has an up-down pregroup and length function. Moreover this model compatible with group action on trees.

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