Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2020, 7(6):45-56**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

The Inequality for the Coefficients of the Inverse of Analytic Functions Defined by *q*- Derivative

Nizami Mustafa, Semra Korkmaz

Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, TURKEY
nizamimustafa@gmail.com Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey
korkmazsemra36@outlook.com

Abstract The object of the present paper is to give sharp estimates for the some initial coefficients of the inverse of certain subclass of analytic functions defined by q-derivative operator with respect to symmetric points. In this study, the Fekete-Szegö problem for the inverse of this function class is also examined. In addition, here given upper bound estimate for the second Hankel determinant of the inverse of this class.

Keywords Univalent function, Analytic function, Coefficient problem, q-derivative operator
Subject Classification 30C45, 30C50, 30C55

1. Introduction

Let A represented the class of analytic functions f on the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ in the complex plane in the form

$$f(z)=z+a_2z^2+a_3z^3+...+a_nz^n+...=z+\sum_{n=2}^{\infty}a_nz^n, a_n\in\mathbb{C}. \tag{1}$$

We denote by S the subclass of A consisting of the functions which are also univalent.

The coefficient problem of certain subclasses of analytic functions and of the inverse of certain analytic functions are one of the important problems in the theory of analytic functions. The sharp estimates for the coefficients of the functions belonging certain subclass of analytic functions and inverses are still an open problem (see, for example [14, 17]).

As well known that, one of the important tools in theory of analytic functions is the functional $H_2(1)=a_3-a_2^2$, which is known as the Fekete-Szegö functional and one usually considers the further generalized functional $a_3-\mu a_2^2$, where μ is some real or complex number (see [7]). Estimating the upper bound of $|a_3-\mu a_2^2|$ is known as the Fekete-Szegö problem.

In 1969, Keogh and Merkes [13] solved the Fekete-Szegö problem for the classes starlike and convex functions. Someone can see the Fekete-Szegö problem for the classes of starlike and convex functions of order α at special cases in the paper of Orhan et al.[19]. On the other hand, recently, Çağlar and Aslan (see[3]) have obtained Fekete-Szegö inequality for a subclass of bi-univalent functions. Also, Zaprawa (see [23, 24]) have studied on Fekete-Szegö problem for some subclasses of bi-univalent functions. In special cases, they solve the Fekete-Szegö problem for the subclasses bi-starlike and bi-convex functions of order α .

It is well-known that, the upper bound of the expression $|H_2(2)| = |a_2a_4 - a_3^2|$ is one important problem in theory of analytic functions. Recently, the upper bound of $|H_2(2)| = |a_2a_4 - a_3^2|$ for the bi-starlike and bi-convex functions classes $S_\Sigma^*(\alpha)$ and $C_\Sigma(\alpha)$ were obtained by Deniz et al.[5]. Very soon, Orhan et al. [20] reviwed the study on the bound of the second Hankel determinant for the subclass $M_\Sigma^\alpha(\beta)$ of bi-univalent functions.

It is well known that (see, for example, [8]) every function $f \in S$ has an inverse function f^{-1} defined in the disk $D = \{w : |w| < r_0(f)\}, r_0(f) \geq \frac{1}{4}$ as follows

$$f^{-1}(w) = w + A_2w^2 + A_3w^3 + A_4w^4 + \ldots, w \in D, \tag{2}$$
$$A_2 = -a_2, A_3 = 2a_2^2 - a_3, A_4 = -5a_2^3 + 5a_2a_3 - a_4$$

For $m \in \mathbb{N}$, we define the m th Hankel determinant of the inverse f^{-1} of the function $f \in S$ as follows:

$$\overline{H}_m(n) = \begin{vmatrix} A_n & \ldots & A_{n+m-1} \\ . & \ldots & . \\ A_{n+m-1} & \ldots & A_{n+2m-2} \end{vmatrix}.$$

The functional $\overline{H}_2(1) = A_3 - A_2^2$, we will call Fekete-Szegö functional and one usually considers the further generalized functional $\overline{H}_2(1) = A_3 - \mu A_2^2$ of inverse f^{-1} of the function $f \in S$, where μ is some real or complex number. Estimating the upper bound of $|A_3 - \mu A_2^2|$, we will say the Fekete-Szegö problem for the inverse function.

Also, we define by $\overline{H}_2(2) = A_2A_4 - A_3^2$ second Hankel determinant of inverse f^{-1} of the function $f \in S$.

Incomplete bound estimates for the coefficients A_2, A_3 and A_4 were given in [4] for α logarithmically convex function class. Very soon by D. K. Thomas [22] were given the complete solution of this problem for a subclass of analytic functions.

Also, in [22] Thomas gives the sharp estimates for the some initial coefficients of the inverse of certain analytic functions.

In this study, we will examine coefficient bound estimates, Fekete-Szegö problem and upper bound estimate for $|\overline{H}_2(2)| = |A_2A_4 - A_3^2|$ of the inverse for a new subclass of analytic functions.

Now, let us give some preliminary information that we need throughout the study.

In the fundamental paper [12] by Jackson introduced $q-$derivative operator of a function f as follows

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & \text{if} \quad z \neq 0, \\ f'(0), & \text{if} \quad z = 0 \end{cases} \tag{3}$$

for $q \in (0,1)$.

The $q-$ derivative operator used to investigate several subclasses of analytic functions in different ways by many researchers. In [2] by using the properties of the $q-$derivative shown that $q-$Szász Mirakyan operators are convex if the function involved is convex, generalizing well known results for $q = 1$. Moreover, in [2] shown that $q-$derivatives of these operators converge to $q-$derivatives of approximated functions. The effect of the $q-$derivative functions operator on the generalized hypergeometric series $r\varphi s(a_1,...a_r;b_1,...b_s;q,z)$ with respect to parameters $a_1,...a_r;b_1,...b_s$ are discussed in [11].

For the function $f\in A$ given by (1), we can easily show that

$$D_q f(z)=1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}, \tag{4}$$

where $[n]_q=\dfrac{1-q^n}{1-q}=\sum_{k=1}^{n-1} q^k$ for $q\in(0,1)$. It is clear that $[0]_q=0,\ [1]_q=1$ and $\lim_{q\to1^-}[n]_q=n$.

It follows from (4) that $\lim_{q\to1^-}D_q f(z)=f'(z)$.

Let S_s^* be the subclass of S consisting of the functions f given by (1) satisfying the condition

$$Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)>0, z\in U.$$

Similarly identifiable, $S_s^*(\alpha)$ the subclass of S consisting of the functions given by (1) satisfying the condition

$$Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)>\alpha, z\in U$$

for $\alpha\in[0,1)$.

It is clear that $S_s^*(\alpha)\subset S_s^*$ for $\alpha\in[0,1)$.

In [9] by Goel and Mehrok introduced a subclass of S_s^* as follows

$$S_s^*(A,B)=\left\{f\in S:\frac{2zf'(z)}{f(z)-f(-z)}\prec\frac{1+Az}{1+Bz}, z\in U\right\}, -1\le B<A\le 1$$

Inspired by the studies mentioned above, we introduce the following function class.

Definition 1 *A function $f\in S$ given by (1) is said to be in the class $S_{q,s}^*(\alpha), q\in(0,1)$, $\alpha\in[0,1)$, if the following condition is satisfied*

$$Re\left(\frac{2zD_q f(z)}{f(z)-f(-z)}\right)>\alpha, z\in U.$$

Remark 1 *Choose $\alpha=0$ in the Definition 1, we have the function class $S_{q,s}^*=S_{q,s}^*(0), q\in(0,1)$.*

It is clear that $S_{q,s}^*(\alpha)\subset S_{q,s}^*$ for $\alpha\in[0,1)$.

Remark 2 *Choose $q\to1^-$ in the Definition 1, we have function class $S_s^*(\alpha)=\lim_{q\to1^-}S_{q,s}^*(\alpha), \alpha\in[0,1)$.*

Remark 3 *Choose $q\to1^-$ and $\alpha=0$ in the Definition 1, we have function class $S_s^*=S_s^*(0)$*

In this paper, given sharp estimates for the some initial coefficients of the inverse for the function belonging in the subclass $S_{q,s}^*(\alpha)$. The Fekete-Szegö problem for the inverse of this function class is also examined. In addition, in the study given upper bound estimate for the second Hankel determinant for the inverse of this function class.

To prove our main results, we shall need the following lemmas concerning functions with real part (see e. g. [1, 6, 10, 15, 21]).

Denote by P the set of functions p analytic in U with expansion $p(z)=1+\sum_{n=1}^{\infty} p_n z^n$ and satisfying $Rep(z)>0$ for $z\in U$.

Lemma 1 *Let* $p\in P,$ *then* $|p_n|\le 2$ *is sharp for each* $n=1,2,3,\ldots$ *and*

$$2p_2 = p_1^2+(4-p_1^2)x,$$

$$4p_3 = p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2+2(4-p_1^2)(1-|x|^2)w$$

for some complex valued x *and w with* $|x|\le 1$ *and* $|w|\le 1$.

Lemma 2 *Let* $p\in P,$ *then* $|p_n|\le 2$ *is sharp for each* $n=1,2,3,\ldots$ *and*

$$\left|p_2-\frac{v}{2}p_1^2\right|\le \max\{2,2|v-1|\}=\begin{cases}2, & if\ 0\le v\le 2,\\ 2|v-1|, & \text{elsewhere.}\end{cases}$$

If $v<0$ or $v>2$, equality holds if and only if $p(z)=(1+\varepsilon z)/(1-\varepsilon z), |\varepsilon|=1.$

If $0<v<2$ then equality holds if and only if $p(z)=(1+\varepsilon z^2)/(1-\varepsilon z^2), \quad |\varepsilon|=1.$

For $v=0$ equality holds if and only if

$$p(z):=\lambda\frac{1+\varepsilon z}{1-\varepsilon z}+(1-\lambda)\frac{1-\varepsilon z}{1+\varepsilon z}, \lambda\in[0,1], |\varepsilon|=1.$$

For $\mu=2,$ equality holds if and only if p is the reciprocal of p_2.

Lemma 3 *Let* $p\in P,$ *then*

$$\left|p_3-(1+\mu)p_1p_2+\mu p_1^3\right|\le \max\{2,2|2\mu-1|\}=\begin{cases}2, & if\ \mu\in[0,1],\\ 2|2\mu-1|, & \text{elsewhere,}\end{cases}$$

2. The Coefficient Inequalities for the Inverse Function

In this section, we give the following theorem on the sharp estimates for the some initial coefficients of the inverse f^{-1} of function f belonging in the class $S_{q,s}^*(\alpha)$.

Theorem 1 *Let* $f\in S_{q,s}^*(\alpha)$ *and* f^{-1} *be the inverse of the function* f *. Then,*

$$|A_2|\le\frac{2(1-\alpha)}{[2]_q}, |A_3|\le\frac{2(1-\alpha)}{[3]_q-1},$$

$$|A_4|\le\frac{2(1-\alpha)}{[4]_q}\begin{cases}1+4\left|\dfrac{5(1-\alpha)[4]_q}{[2]_q^3}-\mu\right| & if\alpha\in[a_1,a_2],\\ |2\mu-1|+4\left|\dfrac{5(1-\alpha)[4]_q}{[2]_q^3}-\mu\right| & if\alpha\in[0,1]/[a_1,a_2],\end{cases}$$

where $\mu=[(1-\alpha)(5[4]_q-[2]_q)-[2]_q([3]_q-1)]/[2]_q([3]_q-1)$, $a_1=1-\dfrac{2([3]_q-1)[2]_q}{5[4]_q-[2]_q}$ *and*

$a_2=1-\dfrac{1-a_1}{2}$. *The inequalities obtained for* $|A_2|$ *and* $|A_3|$ *are sharp.*

Proof. Assume that $f\in S_{q,s}^*(\alpha), \alpha\in[0,1), q\in(0,1)$ and f^{-1} be the inverse of the function f . Then,

$$Re\left(\frac{2zD_q f(z)}{f(z)-f(-z)}\right) > \alpha, z \in U;$$

that is,

$$\frac{2zD_q f(z)}{f(z)-f(-z)} = \alpha + (1-\alpha)p(z), z \in U \tag{5}$$

where $p \in P$.

By simple computation from (5), we have

$$z + \sum_{n=2}^{\infty} [n]_q a_n z^n = \left(z + \sum_{n=2}^{\infty} a_{2n-1} z^n\right)\left(\alpha + (1-\alpha)\left(1 + \sum_{n=1}^{\infty} p_n z^n\right)\right). \tag{6}$$

As a result of simple simplification from (6) comparing the coefficients of the like power of z in the both sides, we get

$$[2]_q a_2 = (1-\alpha)p_1, [3]_q a_3 = a_3 - (1-\alpha)p_2, [4]_q a_4 = (1-\alpha)(p_3 + a_3 p_1);$$

that is,

$$a_2 = \frac{1-\alpha}{[2]_q} p_1, \quad a_3 = \frac{1-\alpha}{[3]_q - 1} p_2, \ a_4 = \frac{1-\alpha}{[4]_q}\left(p_3 + \frac{(1-\alpha)}{[3]_q - 1} p_1 p_2\right). \tag{7}$$

Also, from (2) for the initial three coefficients of the inverse function f^{-1}, we have

$$A_2 = -a_2, A_3 = 2a_2^2 - a_3, A_4 = -5a_2^3 + 5a_2 a_3 - a_4 \tag{8}$$

In this case, the inequality for $|A_2|$ is obvious on using the first equality of (7) and applying the inequality for $|a_2|$ obtained in [[16], Theorem 1].

By replace a_2 and a_3 with expressions in (7) in the second equality (8), for A_3 we write the following expression

$$A_3 = \frac{2(1-\alpha)^2}{[2]_q^2} p_1^2 - \frac{1-\alpha}{[3]_q - 1} p_2.$$

From this for $|A_3|$, we write

$$|A_3| = \frac{1-\alpha}{[3]_q - 1}\left|p_2 - \frac{2([3]_q - 1)(1-\alpha)}{[2]_q^2} p_1^2\right|;$$

that is,

$$|A_3| = \frac{1-\alpha}{[3]_q - 1}\left|p_2 - \frac{v}{2} p_1^2\right|,$$

with $v = 4([3]_q - 1)(1-\alpha)/[2]_q^2$.

Since $4([3]_q - 1)(1-\alpha)/[2]_q^2 \le 2$ for each $q \in (0,1)$ and $\alpha \in [0,1)$, applying Lemma 2 easily gives desired inequality for $|A_3|$.

Using (7) and third equality of (8), we can write the expression for A_4 as

$$A_4 = -\frac{1-\alpha}{[4]_q}\left[p_3 - \frac{(1-\alpha)(5[4]_q - [2]_q)}{([3]_q - 1)[2]_q} p_1 p_2 + \frac{5(1-\alpha)^2[4]_q}{[2]_q^3} p_1^3\right];$$

that is,

$$A_4 = -\frac{1-\alpha}{[4]_q}\left[p_3-(1+\mu)p_1p_2+\mu p_1^3+\left(\frac{5(1-\alpha)^2[4]_q}{[2]_q^3}-\mu\right)p_1^3\right],$$

with $\mu = [(1-\alpha)(5[4]_q-[2]_q)-[2]_q([3]_q-1)]/[2]_q([3]_q-1)$

Since $\mu\in[0,1]$ for $\alpha\in[a_1,a_2]$ and $\mu\notin[0,1]$ for $\alpha\in[0,1]/[a_1,a_2]$ for every $q\in(0,1)$, using firstly triangle inequality, then applying Lemma 3 gives the following inequality

$$|A_4|\le\frac{2(1-\alpha)}{[4]_q}\begin{cases}1+4\left|\dfrac{5(1-\alpha)[4]_q}{[2]_q^3}-\mu\right| & if\,\alpha\in[a_1,a_2],\\ |2\mu-1|+4\left|\dfrac{5(1-\alpha)[4]_q}{[2]_q^3}-\mu\right| & if\,\alpha\in[0,1]/[a_1,a_2],\end{cases}$$

where

$$\alpha_1 = 1-\frac{2([3]_q-1)[2]_q}{5[4]_q-[2]_q}\ \text{ and }\ a_2 = 1-\frac{1-a_1}{2}.$$

Thus, the required inequalities for $|A_2|$, $|A_3|$ and $|A_4|$ are proved.

Note that equality is attained in the inequality for $|A_2|$ and $|A_3|$, respectively, when $p_1=2$ and $p_1=0=p_2-2$.

Thus, the proof of Theorem 1 is completed.

From the Theorem 1, we arrive at the following results.

Corollary 1 *Let* $f\in S_{q,s}^*$ *and* f^{-1} *be the inverse of the function* f*. Then,*

$$|A_2|\le\frac{2}{[2]_q},\ |A_3|\le\frac{2}{[3]_q-1}\ \text{ and }\ |A_4|\le\frac{2}{[4]_q}\left[6\mu-\frac{20[4]_q+[2]_q^3}{[2]_q^3}\right],$$

where $\mu=[5[4]_q-[2]_q[3]_q]/[2]_q([3]_q-1)$. All the inequalities obtained here are sharp.

Corollary 2 *Let* $f\in S_s^*(\alpha)$ *and* f^{-1} *be the inverse of the function* f*. Then,*

$$|A_n|\le 1-\alpha, n=2,3\text{and}$$

$$|A_4|\le\frac{1-\alpha}{2}\begin{cases}10-17\alpha & \text{if }\alpha\in\left[0,\frac{1}{2}\right],\\ 2+3\alpha & \text{if }\alpha\in\left(\frac{1}{2},\frac{5}{9}\right),\\ 8\alpha-3 & \text{if }\alpha\in\left[\frac{5}{9},\frac{7}{9}\right],\\ 17\alpha-10 & \text{if }\alpha\in\left(\frac{7}{9},1\right).\end{cases}$$

The inequalities obtained for $|A_2|$ *and* $|A_3|$ *here are sharp.*

Corollary 3 *Let* $f\in S_s^*$ *and* f^{-1} *be the inverse of the function* f*. Then,*

$$|A_n|\le 1, n=2,3\text{and}|A_4|\le 5.$$

The inequalities obtained for $|A_2|$ *and* $|A_3|$ *here are sharp.*

3. The Fekete-Szegö Problem for the function class $S_{q,s}^{*}(\alpha)$

In this section, we will prove the following theorem on the Fekete-Szegö problem of the inverse of the function $f \in S_{q,s}^{*}(\alpha)$.

Theorem 2 *Let the function* f *given by (1) be in the class* $S_{q,s}^{*}(\alpha)$, f^{-1} *be the inverse of the function* f *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|A_3 - \mu A_2^2\right| \le \frac{2(1-\alpha)}{[3]_q - 1}\begin{cases} 1 & \text{if } \left|\mu + k(\alpha)\right| \le \dfrac{[2]_q}{2(1-\alpha)([2]_q - 1)}, \\ \left|\dfrac{2(1-\alpha)([2]_q - 1)}{[2]_q}\mu + \dfrac{1+(4\alpha-3)([2]_q - 1)}{[2]_q}\right| & \text{if } \left|\mu + k(\alpha)\right| \ge \dfrac{[2]_q}{2(1-\alpha)([2]_q - 1)}, \end{cases}$$

where $k(\alpha) = \dfrac{1+(4\alpha-3)([2]_q - 1)}{2(1-\alpha)([2]_q - 1)}$.

Proof. Let $f \in S_{q,s}^{*}(\alpha), q \in (0,1)$, $\alpha \in [0,1)$ f^{-1} be the inverse of the function f and $\mu \in \mathbb{C}$.

Using firstly the equality (8), then the expression for the coefficients a_2 and a_3 from the first and second equality of (7), we find

$$A_3 - \mu A_2^2 = (2-\mu)a_2^2 - a_3 = \frac{1-\alpha}{[3]_q - 1}\left[\frac{(1-\alpha)([3]_q - 1)}{[2]_q^2}(2-\mu)p_1^2 - p_2\right]. \tag{9}$$

Substituting the expression $p_2 = \frac{1}{2}\left[p_1^2 + \left(4 - p_1^2\right)x\right]$ from Lemma 1, in (9) and using triangle inequality, putting $t = |p_1|$, $|x| = \eta$, we can easily obtain that

$$\left|A_3 - \mu A_2^2\right| \le \frac{1-\alpha}{2([3]_q - 1)}\left[d_1(t) + d_2(t)\eta\right] = \Psi(\eta), \tag{10}$$

where

$$d_1(t) = \left|\frac{(1-\alpha)([3]_q - 1)}{[2]_q^2}(2-\mu) - 1\right| t^2 \ge 0 \text{ and } d_2(t) = 4 - t^2 \ge 0.$$

It is clear that the maximum of the function $\Psi(\eta)$ occurs at $\eta = 1$. Therefore,

$$\Psi(\eta) \le \max\{\Psi(\eta) : \eta \in [0,1]\} = \Psi(1) = \frac{1-\alpha}{2([3]_q - 1)}\left[d_1(t) + d_2(t)\right]. \tag{11}$$

Let us define the function $H : [0,2] \to \mathbb{R}$ as follows

$$H(t) = d_1(t) + d_2(t). \tag{12}$$

Substituting the value $d_1(t)$ and $d_2(t)$ in (12), we obtain

$$H(t) = C(\alpha, q, \mu)t^2 + 4, \tag{13}$$

where

$$C(\alpha,q,\mu)=\left|1-\frac{2(1-\alpha)([3]_q-1)}{[2]_q^2}(2-\mu)\right|-1$$

It is clear that the function $H(t)$ is an increasing function if $C(\alpha,q,\mu)\geq 0$ and decreasing function if $C(\alpha,q,\mu)\leq 0$.

Therefore,

$$H(t)\leq \max\{H(t):t\in(0,2)\}=\begin{cases}4C(\alpha,q,\mu)+4 & \text{if } C(\alpha,q,\mu)\geq 0,\\ 4 & \text{if } C(\alpha,q,\mu)\leq 0.\end{cases} \tag{14}$$

From (10)-(14), we obtain the following inequality for $\left|A_3-\mu A_2^2\right|$

$$\left|A_3-\mu A_2^2\right|\leq \frac{2(1-\alpha)}{[3]_q-1}\begin{cases}C(\alpha,q,\mu)+1 & \text{if } C(\alpha,q,\mu)\geq 0,\\ 1 & \text{if } C(\alpha,q,\mu)\leq 0.\end{cases}$$

This completes the proof of Theorem 2.

From the Theorem 2, we obtain the following results.

Corollary 4 *Let the function* f *given by (1) be in the class* $S_{q,s}^*$, f^{-1} *be the inverse of the function* f *and* $\mu\in\mathbb{C}$. *Then,*

$$\left|A_3-\mu A_2^2\right|\leq \frac{2(1-\alpha)}{[3]_q-1}\begin{cases}1 & \text{if } |\mu+k|\leq \dfrac{[2]_q}{2([2]_q-1)},\\ \left|\dfrac{2([2]_q-1)}{[2]_q}\mu+\dfrac{1-3([2]_q-1)}{[2]_q}\right| & \text{if } |\mu+k|\geq \dfrac{[2]_q}{2([2]_q-1)},\end{cases}$$

where $k=\dfrac{1-3([2]_q-1)}{2([2]_q-1)}$

Corollary 5 *Let the function* f *given by (1) be in the class* $S_s^*(\alpha)$, f^{-1} *be the inverse of the function* f *and* $\mu\in\mathbb{C}$. *Then,*

$$\left|A_3-\mu A_2^2\right|\leq (1-\alpha)\begin{cases}1 & \text{if } \left|\mu+\dfrac{2\alpha-1}{1-\alpha}\right|\leq \dfrac{1}{1-\alpha},\\ \left|(1-\alpha)\mu+(2\alpha-1)\right| & \text{if } \left|\mu+\dfrac{2\alpha-1}{1-\alpha}\right|\geq \dfrac{1}{1-\alpha}.\end{cases}$$

Corollary 6 *Let the function* f *given by (1) be in the class* S_s^* , f^{-1} *be the inverse of the function* f *and* $\mu\in\mathbb{C}$. *Then,*

$$\left|A_3-\mu A_2^2\right|\leq\begin{cases}1 & \text{if } |\mu-1|\geq 1,\\ |\mu-1| & \text{if } |\mu-1|\leq 1.\end{cases}$$

In the case $\mu\in\mathbb{R}$, Theorem 2 can be given as follows.

Theorem 3 *Let the function* f *given by (1) be in the class* $S_{q,s}^*(\alpha)$, f^{-1} *be the inverse of the function* f *and* $\mu\in\mathbb{R}$. *Then,*

$$\left|A_3-\mu A_2^2\right|\le\frac{2(1-\alpha)}{[3]_q-1}\begin{cases}\dfrac{2(2-\mu)(1-\alpha)([2]_q-1)}{[2]_q}-1 & if\quad \mu\le\dfrac{(1-2\alpha)([2]_q-1)-1}{(1-\alpha)([2]_q-1)},\\ 1 & if\ \dfrac{(1-2\alpha)([2]_q-1)-1}{(1-\alpha)([2]_q-1)}\le\mu\le 2,\\ 1-\dfrac{2(2-\mu)(1-\alpha)([2]_q-1)}{[2]_q} & if\quad 2\le\mu.\end{cases}\tag{15}$$

Proof. Let $f\in S_{q,s}^{*}(\alpha), q\in(0,1), \alpha\in[0,1)$, f^{-1} be the inverse of the function f and $\mu\in\mathbb{R}$. Then,.

From (2.4) and (2.3), we find that

$$A_3-\mu A_2^2=\frac{1-\alpha}{[3]_q-1}\left[\frac{(1-\alpha)([3]_q-1)}{[2]_q^2}(2-\mu)p_1^2-p_2\right];$$

that is,

$$A_3-\mu A_2^2=\frac{1-\alpha}{[3]_q-1}\left[-\left(p_2-\frac{\nu}{2}p_1^2\right)\right]\tag{16}$$

with $\nu=2(2-\mu)(1-\alpha)([3]_q-1)/[2]_q^2$.

Using Lemma 2 to equality (16), we obtain the following inequality for $\left|A_3-\mu A_2^2\right|$

$$\left|A_3-\mu A_2^2\right|\le\frac{2(1-\alpha)}{[3]_q-1}\begin{cases}1 & if\ 0\le\dfrac{(2-\mu)(1-\alpha)([2]_q-1)}{[2]_q}\le 1,\\ \left|\dfrac{2(2-\mu)(1-\alpha)([2]_q-1)}{[2]_q}-1\right| & if\quad \dfrac{(2-\mu)(1-\alpha)([2]_q-1)}{[2]_q}\mu\notin[0,1]\end{cases}$$

This completes the proof of the inequality (15).

Now, let's see that obtained result is sharp.

Really, as seen from Lemma 2 the result is sharp for the function , which satisfies following condition

$$\frac{2zD_q f(z)}{f(z)-f(-z)}=\alpha+(1-\alpha)p(z)=\frac{1+(1-2\alpha)\varepsilon z}{1-\varepsilon z},|\varepsilon|=1$$

if $\mu<\dfrac{(1-2\alpha)([2]_q-1)-1}{(1-\alpha)([2]_q-1)}$ or $\mu>2$ and for the function f, which satisfies following condition

$$\frac{2zD_q f(z)}{f(z)-f(-z)}=\alpha+(1-\alpha)p(z)=\frac{1+(1-2\alpha)\varepsilon z^2}{1-\varepsilon z^2},|\varepsilon|=1$$

if $\dfrac{(1-2\alpha)([2]_q-1)-1}{(1-\alpha)([2]_q-1)}<\mu<2$.

For $\mu=2$,

$$\frac{2zD_q f(z)}{f(z)-f(-z)}=\lambda\frac{1+\varepsilon z}{1-\varepsilon z}+(1-\lambda)\frac{1-\varepsilon z}{1+\varepsilon z}:=p_2(z),\lambda[0,1],|\varepsilon|=1$$

For $\mu=\dfrac{(1-2\alpha)([2]_q-1)-1}{(1-\alpha)([2]_q-1)}$, equality holds if and only if p is the reciprocal of p_2.

Thus, the proof of Theorem 3 is completed.

Notation 1 *It should be noted that Theorem 3 could also be given as a direct result of Theorem 2. But, here we give shorter proof in the case* $\mu \in \mathbb{R}$.

Choose $\mu = 0$ in Theorem 3, we obtain the following inequality for $|A_3|$, which confirm the inequality obtained in Theorem1 and Corollaries 1,2 and 3, respectively.

Corollary 7 *The following inequalities are provided:*

If $f \in S^*_{q,s}(\alpha)$ then,

$$|A_3| \le \frac{2(1-\alpha)}{[3]_q - 1}.$$

If $f \in S^*_{q,s}$ then,

$$|A_3| \le \frac{2}{[3]_q - 1}.$$

If $f \in S^*_s(\alpha)$ then,

$$|A_3| \le 1-\alpha.$$

If $f \in S^*_s$ then,

$$|A_3| \le 1.$$

4. The Second Hankel Determinant of the function class $S^*_{q,s}(\alpha)$

In this section, we prove the following theorem on upper bound of the second Hankel determinant for the inverse of the function $f \in S^*_{q,s}(\alpha)$.

Theorem 4 *Let the function* f *given by (1) be in the class* $S^*_{q,s}(\alpha)$, f^{-1} *be the inverse of the function* f *. Then,*

$$|A_2A_4 - A_3^2| \le \frac{4(1-\alpha)^2}{[2]_q}\left[\frac{1}{[4]_q} + \frac{[2]_q + 2([2]_q - 1)(1-\alpha)}{([3]_q - 1)^2}\right].$$

Proof. Assume that $f \in S^*_{q,s}(\alpha), q \in (0,1), \alpha \in [0,1)$ and f^{-1} be the inverse of the function f. Then, from the equality (8) and (7) we have

$$A_2A_4 - A_3^2 = a_2^2(a_2^2 - a_3) + (a_2a_4 - a_3^2).$$

Using triangle inequality, then applying inequality obtained for $|a_2|$ in the Theorem 1, inequality obtained for $|a_3 - a_2^2|$ in the Theorem 3 (when $\mu = 1$) and inequality obtained for $|a_2a_4 - a_3^2|$ in the Theorem 4 in the paper [16] to the last equality, we obtain

$$|A_2A_4 - A_3^2| \le \frac{4(1-\alpha)^2}{[2]_q}\frac{2(1-\alpha)}{[3]_q - 1} + \frac{4(1-\alpha)^2}{[2]_q[4]_q}\left[1 + \frac{[2]_q[4]_q}{([3]_q - 1)^2}\right];$$

that is,

$$|A_2A_4 - A_3^2| \le \frac{4(1-\alpha)^2}{[2]_q}\left[\frac{1}{[4]_q} + \frac{[2]_q + 2([2]_q - 1)(1-\alpha)}{([3]_q - 1)^2}\right].$$

Thus, the proof of Theorem 4 is completed.

From the Theorem 4, we obtain the following results.

Corollary 8 *Let the function* f *given by (1) be in the class* $S^*_{q,s}$ *and* f^{-1} *be the inverse of the function* f *. Then,*

$$\left|A_2A_4 - A_3^2\right| \leq \frac{4}{[2]_q}\left[\frac{1}{[4]_q} + \frac{3[2]_q - 2}{\left([3]_q - 1\right)^2}\right].$$

Corollary 9 *Let the function* f *given by (1) be in the class* $S^*_s(\alpha)$ *and* f^{-1} *be the inverse of the function* f *. Then,*

$$\left|A_2A_4 - A_3^2\right| \leq \frac{(1-\alpha)^2}{2}(5-2\alpha).$$

Corollary 10 *Let the function* f *given by (1) be in the class* $S^*_s(\alpha)$ *and* f^{-1} *be the inverse of the function* f *. Then,*

$$\left|A_2A_4 - A_3^2\right| \leq \frac{5}{2}$$

References

[1]. Ali, R. M. (2003). Coefficients of the inverse of strongly starlike functions. Bull. Malays. Math, Sci. Soc., 26(1): 63-71.

[2]. Ali, A., Viyaj, G. (2006). The q-derivative and applications to Szasz Mirakyan operator. Calcolo, 43(3): 151-170.

[3]. Çağlar, M., Aslan, S. (2016). Fekete-Szegö inequalities for subclasses of bi-univalent functions satisfying subordinate conditions. AIP Conference Proceedings, 1726, 020078.

[4]. Darus, M. and Thomas, D. K. (1998). $\alpha-$logarithmically convex functions. Indian J. Pure Appl. Math, 29(1): 1049-1059.

[5]. Deniz, E. Çağlar, M. and Orhan, H. (2015). Second Hankel determinant for bi-stalike and bi-convex functions of order . Appl Math Comput, 271: 301-307.

[6]. Duren, P. L. (1983). Univalent Functions. Grundlehren der Mathematishen Wissenschaften, 259: Springer, New York.

[7]. Fekete, M., Szegö, G. (1933). Eine Bemerkung uber ungerade schlichte Funktionen. J London Math Soc, 8: 85-89.

[8]. Gasper, G. and Rahman, M. (1990). Basic hypergeometric series. Cambridge University Press, Cambridge.

[9]. Goel, R. M. and Mehrok, B. C. (1982). A subclass of starlike functions with respect to symmetric points. Tamkang J. Math., 13(1): 11-24.

[10]. Goodman, A. W. (1983). Univalent Functions. Volume I, Polygonal, Washington

[11]. Hossan, A. Ghany. (2009). q-Derivative of Basic Hypergeometric Series with Respect to Parameters. Int. Journal of Math. Analysis, 3(33): 1617-1632.

[12]. Jackson, F. H. (1908). On functions and a certain difference operator. Trans. Roy. Soc. Edin, 46: 253-281.

[13]. Keogh, F. R., Merkes, E. P. (1969). A coefficient inequality for certain classes of analytic functions. Proc Amer Math Soc, 20: 8-12.

[14]. Lewin, M. (1967). On a coefficient problem for bi-univalent functions. Proc. Amer. Math. Soc., 18: 63-68.

[15]. Libera, R. J. and Zlotkiewicz, E. J. (1982). Early coefficients of the inverse of a regular convex functions. Proc. Amer. Math. Soc., 85(2): 225-230.

[16]. Mustafa, N. and Korkmaz, S. (2020). The sharp inequality for the coefficients of certain subclass of analytic functions defined by q-derivative. Journal of Scientific and Engineering Research, 7(4): 209-218.
[17]. Netanyahu, E. (1969). The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in . Ach. Rational Mech. Anal., 32: 100-112.
[18]. Noonan, J. W. and Thomas, D. K. (1976). On the second Hankel determinant of a really mean p-valent functions. Trans Amer Math Soc, 223: 337-346.
[19]. Orhan, H., Deniz, E., Raducanu, D. (2010). The Fekete-Szegö problem for subclasses of analytic functions defined by a differential operator related to conic domains. Comput Math Appl, 59:283-295.
[20]. Orhan, H., Magesh, N., Yamini, J. (2016). Bounds for the second Hankel determinant of certain bi-univalent functions. Turkish J Math, 40: 679-687.
[21]. Srivastava, H. M. and Owa, S. (1992). Editors, Current Topics in Analytic Function Theory. World Scientific, Singapore.
[22]. Thomas, D. K. (2018). On the coefficients of gamma-starlike functions. J. Korean Math. Soc. 55(1): 175-184.
[23]. Zaprawa, P. (2014). Estimates of initial coefficients for bi-univalent functions. Abstr Appl Anal, Article ID 357480: 1-6.
[24]. Zaprawa, P. (2014). On the Fekete-Szegö problem for classes of bi-univalent functions. Bull Belg Math Soc Simon Stevin, 21: 169-178.