Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2020, 7(6):30-39**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Coefficient Bound Estimates for Certain Subclass of Analytic Functions of Complex Order

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Abstract In this paper, introduced and investigated two new subclasses of analytic functions in the open unit disk in the complex plane. Several interesting properties of the functions belonging to these classes are examined. Also, sufficient, and necessary and sufficient conditions for the functions belonging to these classes are given. Moreover, coefficient bound estimates for the functions belonging to these classes are obtained.

Keywords Analytic function, Coefficient bound, Starlike function, Convex function
Subject Classification 30C45, 30C50, 30C55

1. Introduction

Let A be the class of analytic functions f in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots = z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathbb{C}, \tag{1}$$

We denote by S the subclass of all functions in $A,$ which are also univalent in U.

Also, let us define by T the subclass of all functions f in A of the form

$$f(z) = z - a_2 z^2 - a_3 z^3 - \ldots - a_n z^n - \ldots = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{2}$$

Furthermore, we will denote by $S^*(\alpha)$, $C(\alpha)$ and $K(\alpha)$ the subclasses of S that are, respectively, starlike, convex and close-to-convex with respect to starlike function g (need not be normalized) of order $\alpha \in [0,1)$ in the open unit disk U.

By definition (see for details, [4, 5], also [9])

$$S^*(\alpha) = \left\{ f \in A : Re\left(\frac{zf'(z)}{f(z)} \right) > \alpha, z \in U \right\}, \alpha \in [0,1), \tag{3}$$

$$C(\alpha) = \left\{ f \in A : Re\left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in U \right\}, \alpha \in [0,1) \tag{4}$$

and

$$K(\alpha)=\left\{ f\in A: Re\left(\frac{zf^{'}(z)}{g(z)}\right)>\alpha, z\in U, g\in S^{*}\right\}, \alpha\in[0,1). \quad (5)$$

For convenience, $S^{*}=S^{*}(0), C=C(0)$ and $K=K(0)$ are, respectively, well-known starlike, convex and close-to-convex functions in U.

It is well known that close-to-convex functions are univalent in U, but not necessarily the converse. It is easy to verify that $C\subset S^{*}\subset K\subset S$. For details on these classes, one could refer to the monograph by Goodman [5].

An interesting generalization of the function class $K(\alpha)$ is provided by the class $K(\alpha,\beta;g)$ of functions $f\in A$, which satisfies the following condition

$$K(\alpha,\beta;g)=\left\{ f\in A: Re\left(\frac{zf^{'}(z)+\beta z^{2}f^{''}(z)}{g(z)}\right)>\alpha, z\in U, g\in S^{*}\right\}, \quad (6)$$
$$\alpha\in[0,1), \beta\in[0,1]$$

with respect to function g (need not be normalized).

It is clear that $K(\alpha,0;g)=K(\alpha;g)$ is well-known close-to-convex functions order $\alpha\in[0,1)$ in U.

We will denote $K(\alpha,\beta;z)=K(\alpha,\beta)$

Thus,

$$K(\alpha,\beta)=\left\{ f\in A: Re\left(f^{'}(z)+\beta zf^{''}(z)\right)>\alpha, z\in U\right\}, \alpha\in[0,1), \beta\in[0,1].$$

Notation 1 *The class* $K(\alpha,\beta), \alpha\in[0,1), \beta\in[0,1]$ *is the first time introduced and examined in this paper [7].*

We will use $TS^{*}(\alpha), TK(\alpha)$ and $TC(\alpha)$ instead $S^{*}(\alpha), K(\alpha)$ and $C(\alpha)$, respectively, also $TK(\alpha,\beta)$ instead $K(\alpha,\beta)$ if $f\in T$.

An interesting unification of the function classes $S^{*}(\alpha)$, $C(\alpha)$ and $K(\alpha,\beta)$ is provided by the class $S^{*}CK(\alpha,\beta,\gamma)$ of functions $f\in S$, which also satisfies the following condition

$$Re\left\{\frac{zf^{'}(z)+\beta z^{2}f^{''}(z)}{\gamma\left[\beta zf^{'}(z)+(1-\beta)f(z)\right]+(1-\gamma)z}\right\}>\alpha, \alpha\in[0,1), \beta,\gamma\in[0,1], z\in U.$$

Also, we will use $TS^{*}CK(\alpha,\beta,\gamma)=T\cap S^{*}CK(\alpha,\beta,\gamma)$.

The class $S^{*}CK(\alpha,\beta,1)=TS^{*}C(\alpha,\beta)$ was investigated by Altintaş et al. [2] and [3] (in a more general way $T_{n}S^{*}C(p,\alpha,\beta)$) and (subsequently) by Irmak et al. [6]. In particular, the class $T_{n}S^{*}C(1,\alpha,\beta)$ was considered earlier by Altintaş [1].

Inspired by the aforementioned works, we define two new subclasses of analytic functions as follows.

Definition 1 *A function* $f\in S$ *given by (1) is said to be in the class* $S^{*}CK(\alpha,\beta,\gamma,\tau), \alpha\in[0,1), \beta,\gamma\in[0,1], \tau\in\mathbb{C}^{*}=\mathbb{C}-\{0\}$ *if the following condition is satisfied*

$$Re\left\{1+\frac{1}{\tau}\left[\frac{zf^{'}(z)+\beta z^{2}f^{''}(z)}{\gamma\left[\beta zf^{'}(z)+(1-\beta)f(z)\right]+(1-\gamma)z}-1\right]\right\}>\alpha, z\in U.$$

Definition 2 *A function $f \in T$ given by (2) is said to be in the class* $TS^*CK(\alpha,\beta,\gamma,\tau), \alpha \in [0,1), \beta,\gamma \in [0,1], \tau \in \mathbb{C}^* = \mathbb{C}-\{0\}$ *if the following condition is satisfied*

$$Re\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)+\beta z^2 f''(z)}{\gamma[\beta zf'(z)+(1-\beta)f(z)]+(1-\gamma)z}-1\right]\right\} > \alpha, z \in U.$$

That is, $TS^*CK(\alpha,\beta,\gamma,\tau) = T \cap S^*CK(\alpha,\beta,\gamma,\tau)$.

Remark 1 *Choose $\tau = 1$ in Definition 1 we have the function class $A(\alpha,\beta,\gamma)$ [7]*

Remark 2 *Choose $\tau = \gamma = 1$ in Definition 1, we have the function class $A(\alpha,\beta)$ [7]*

Remark 3 *Choose $\tau = \gamma = 1$ and $\beta = 0$ in Definition 1, we have the function class $S^*(\alpha)$.*

Remark 4 *Choose $\tau = \gamma = 1$ and $\beta = 1$ in Definition 1, we have the function class $C(\alpha)$*

Remark 5 *Choose $\gamma = 0 = \tau - 1$ in Definition 1, we have the function class $K(\alpha,\beta)$..*

Remark 6 *Choose $\gamma = \beta = 0 = \tau - 1$ in Definition 1, we have the function class $K(\alpha)$.*

Remark 7 *Choose $\gamma = 1$ in Definition 2, we have the function class $T(\alpha,\beta,\gamma)$ [7].*

Remark 8 *Choose $\tau = \gamma = 1$ in Definition 2, we have the function class $T(\alpha,\beta)$ [7] .*

Remark 9 *Choose $\tau = \gamma = 1$ and $\beta = 0$ in Definition 2, we have the function class $TS^*(\alpha)$.*

Remark 10 *Choose $\tau = \gamma = 1$ and $\beta = 1$ in Definition 2, we have the function class $TC(\alpha)$.*

Remark 11 *Choose $\gamma = 0 = \tau - 1$ in Definition 2, we have the function class $TK(\alpha,\beta)$*

Remark 12 *Choose $\gamma = \beta = 0 = \tau - 1$ in Definition 2, we have the function class $TK(\alpha)$*

In this paper, two new subclasses $S^*CK(\alpha,\beta,\gamma,\tau)$ and $TS^*CK(\alpha,\beta,\gamma,\tau)$ of the analytic functions in the open unit disk are introduced. Various characteristic properties of the functions belonging to these classes are examined. Sufficient conditions for the analytic functions belonging to the class $S^*CK(\alpha,\beta,\gamma,\tau)$ and necessary and sufficient conditions for those belonging to the class $TS^*CK(\alpha,\beta,\gamma,\tau)$, in the case $\tau \in \mathbb{R}^* = \mathbb{R}-\{0\}$ are also given.

2. Coefficient bound estimates for the classes $S^*CK(\alpha,\beta,\gamma,\tau)$ and $TS^*CK(\alpha,\beta,\gamma,\tau)$

In this section, we will examine some inclusion results for the subclasses $S^*CK(\alpha,\beta,\gamma,\tau)$ and $TS^*CK(\alpha,\beta,\gamma,\tau)$. Also, we give coefficient bound estimates for the functions belonging to these classes.

A sufficient condition for the functions in the class $S^*CK(\alpha,\beta,\gamma,\tau)$ is given by the following theorem.

Theorem 1 *Let $f \in A$. Then, the function f given by (1) belongs to the class $S^*CK(\alpha,\beta,\gamma,\tau)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty}\{n+[(1-\alpha)|\tau|-1]\gamma\}[1+(n-1)\beta]|a_n| \le (1-\alpha)|\tau|. \tag{7}$$

The result obtained here is sharp.

Proof. According to Definition ??, a function f is in the class $S^*CK(\alpha,\beta,\gamma,\tau), \alpha \in [0,1), \beta,\gamma \in [0,1], \tau \in \mathbb{C}^* = \mathbb{C}-\{0\}$ if and only if

$$Re\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)+\beta z^2 f''(z)}{\gamma\left[\beta zf'(z)+(1-\beta)f(z)\right]+(1-\gamma)z}-1\right]\right\}>\alpha.$$

It suffices to show that

$$\left|\frac{1}{\tau}\left[\frac{zf'(z)+\beta z^2 f''(z)}{\gamma\left[\beta zf'(z)+(1-\beta)f(z)\right]+(1-\gamma)z}-1\right]\right|\le 1-\alpha. \tag{8}$$

By simple computation, we write

$$\left|\frac{1}{\tau}\left[\frac{zf'(z)+\beta z^2 f''(z)}{\gamma\left[\beta zf'(z)+(1-\beta)f(z)\right]+(1-\gamma)z}-1\right]\right|=\left|\frac{1}{\tau}\frac{\sum_{n=2}^{\infty}(n-\gamma)[1+(n-1)\beta]a_n z^n}{z+\sum_{n=2}^{\infty}\gamma[1+(n-1)\beta]a_n z^n}\right|$$

$$\le\frac{\sum_{n=2}^{\infty}(n-\gamma)[1+(n-1)\beta]|a_n|}{|\tau|\left\{z-\sum_{n=2}^{\infty}\gamma[1+(n-1)\beta]|a_n|\right\}}.$$

Last expression in the last inequality is bounded above by $1-\alpha$ if and only if

$$\sum_{n=2}^{\infty}(n-\gamma)[1+(n-1)\beta]|a_n|\le|\tau|(1-\alpha)\left\{1-\sum_{n=2}^{\infty}\gamma[1+(n-1)\beta]|a_n|\right\},$$

which is equivalent to

$$\sum_{n=2}^{\infty}\left(n+\left[(1-\alpha)|\tau|-1\right]\gamma\right)[1+(n-1)\beta]|a_n|\le(1-\alpha)|\tau|. \tag{9}$$

Thus, the proof of inequality of (7) is completed.

Let's show that the inequality obtained in theorem is sharp.

For this it will be sufficient to see that inequality (7) is provided as equality for the function given below

$$f_n(z)=z+\frac{(1-\alpha)|\tau|}{\left\{n+\left[(1-\alpha)|\tau|-1\right]\gamma\right\}[1+(n-1)\beta]},\ z\in U$$

for every $n=2,3,\ldots$.

Thus the proof of Theorem 1 is completed.

Setting $\tau=1$ in Theorem 1, we arrive at the following corollary.

Corollary 1 *The function* f *given by (1) belongs to the class* $A(\alpha,\beta;\gamma)$ *if the following condition is satisfied*

$$\sum_{n=2}^{\infty}(n-\alpha\gamma)[1+(n-1)\beta]|a_n|\le 1-\alpha.$$

The result is sharp for the function

$$f_n(z)=z+\frac{1-\alpha}{(n-\alpha\gamma)[1+(n-1)\beta]}z^n,\ z\in U,\ n=2,3,\ldots.$$

Remark 13 *The result obtained in Corollary 1 verifies to Theorem 1 in [7].*

Choose $\gamma=1$ in Corollary 1, we have the following result.

Corollary 2 *The function* f *given by (1) belongs to the class* $A(\alpha,\beta)$ *if the following condition is satisfied:*

$$\sum_{n=2}^{\infty}(n-\alpha)[1+(n-1)\beta]|a_n|\le 1-\alpha.$$

The result is sharp for the function

$$f_n(z)=z+\frac{1-\alpha}{(n-\alpha)[1+(n-1)\beta]}z^n,\ z\in U,\ n=2,3,\ldots.$$

Remark 14 *The result obtained in Corollary 2 verifies to Corollary 1 in [7].*

From Theorem 1, we get the following results at different values of the parameters.

Corollary 3 *(see [[7], Corollary 2]) The function* f *given by (1) belongs to the class* $S^*(\alpha)$ *if the following condition is satisfied:*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z) = z + \frac{1-\alpha}{n-\alpha} z^n, z \in U$$

for every $n = 2,3,...$.

Corollary 4 *(see [[7], Corollary 3])The function* f *given by (1) belongs to the class* $C(\alpha)$ *if the following condition is satisfied*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z) = z + \frac{1-\alpha}{n(n-\alpha)} z^n, z \in U$$

for every $n = 2,3,...$.

Corollary 5 *(see [[7], Corollary4]) The function* f *given by (1) belongs to the class* $K(\alpha,\beta)$ *if the following condition is satisfied*

$$\sum_{n=2}^{\infty} n[1+(n-1)\beta]|a_n| \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z) = z + \frac{(1-\alpha)}{n[1+(n-1)\beta]} z^n, z \in U$$

for every $n = 2,3,...$.

Corollary 6 *(see [[7], Corollary5]) The function* f *given by (1) belongs to the class* $K(\alpha)$ *if the following condition is satisfied*

$$\sum_{n=2}^{\infty} n|a_n| \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z) = z + \frac{(1-\alpha)}{n} z^n, z \in U$$

for every $n = 2,3,...$.

From the following theorem, we see that for the functions in the class $TS^*CK(\alpha,\beta,\gamma,\tau), \alpha \in [0,1), \beta,\gamma \in [0,1], \tau \in \mathbb{R}^* = \mathbb{R}-\{0\}$ the converse of Theorem 1 is also true.

Theorem 2 *Let* $f \in T$ *. Then, the function* $f(z)$ *belongs to the class* $TS^*CK(\alpha,\beta,\gamma,\tau), \tau \in \mathbb{R}^* = \mathbb{R}-\{0\}$ *if and only if*

$$\sum_{n=2}^{\infty}\{n+[(1-\alpha)|\tau|-1]\gamma\}[1+(n-1)\beta]a_n \leq (1-\alpha)|\tau|.$$

The result obtained here is sharp.

Proof. The proof of the sufficiency of Theorem 2 can be proved similarly to the proof of Theorem 1. Therefore, we will prove only the necessity of theorem.

Assume that $f \in TS^*CK(\alpha,\beta,\gamma,\tau), \alpha \in [0,1), \beta,\gamma \in [0,1], \tau \in \mathbb{R}^* = \mathbb{R} - \{0\}$; that is,

$$Re\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)+\beta z^2 f''(z)}{\gamma\left[\beta z f'(z)+(1-\beta)f(z)\right]+(1-\gamma)z}-1\right]\right\} > \alpha, z \in U. \tag{10}$$

By simple computation, from (10) we can easily show that

$$Re\left\{\frac{1}{\tau}\frac{-\sum_{n=2}^{\infty}(n-\gamma)[1+(n-1)\beta]a_n z^n}{z-\sum_{n=2}^{\infty}\gamma[1+(n-1)\beta]a_n z^n}\right\} > \alpha-1. \tag{11}$$

The expression

$$\frac{-\sum_{n=2}^{\infty}(n-\gamma)[1+(n-1)\beta]a_n z^n}{\tau\left\{z-\sum_{n=2}^{\infty}\gamma[1+(n-1)\beta]a_n z^n\right\}}$$

is real if choose z real.

Thus, from the previous inequality (11) letting $z \to 1^-$ through real values, we obtain

$$\frac{-\sum_{n=2}^{\infty}(n-\gamma)[1+(n-1)\beta]a_n}{\tau\left\{1-\sum_{n=2}^{\infty}\gamma[1+(n-1)\beta]a_n\right\}} \geq \alpha-1. \tag{12}$$

We will examine of the last inequality depending on the different cases of the sign of the parameter τ as follows.

Let us $\tau > 0$. Then, from (12), we have

$$-\sum_{n=2}^{\infty}(n-\gamma)[1+(n-1)\beta]a_n \geq (\alpha-1)\tau\left\{1-\sum_{n=2}^{\infty}\gamma[1+(n-1)\beta]a_n\right\},$$

which is equivalent to

$$\sum_{n=2}^{\infty}\left\{n+\left[(1-\alpha)|\tau|-1\right]\gamma\right\}[1+(n-1)\beta]a_n \leq (1-\alpha)\tau = (1-\alpha)|\tau|. \tag{13}$$

Now, let us $\tau < 0$. Then, since $\tau = -|\tau|$, from (12), we get

$$\frac{\sum_{n=2}^{\infty}(n-\gamma)[1+(n-1)\beta]a_n}{|\tau|\left\{1-\sum_{n=2}^{\infty}\gamma[1+(n-1)\beta]a_n\right\}} \geq \alpha-1;$$

that is,

$$\sum_{n=2}^{\infty}(n-\gamma)[1+(n-1)\beta]a_n \geq (\alpha-1)|\tau|\left\{1-\sum_{n=2}^{\infty}\gamma[1+(n-1)\beta]a_n\right\}$$

Therefore,

$$\sum_{n=2}^{\infty}\left\{n+\left[(\alpha-1)|\tau|-1\right]\gamma\right\}[1+(n-1)\beta]a_n \geq -(1-\alpha)|\tau|.$$

Since $\alpha < 1$ (or $1-\alpha > \alpha-1$), from the last inequality, we have

$$\sum_{n=2}^{\infty}\left\{n+\left[(1-\alpha)|\tau|-1\right]\gamma\right\}[1+(n-1)\beta]a_n \geq -(1-\alpha)|\tau|. \tag{14}$$

Thus, from (14) and (13) the proof of the necessity is completed.

Now, to show that the result obtained is sharp it will be sufficient to see that inequality obtained in theorem is provided as equality for the function given below

$$f_n(z) = z - \frac{(1-\alpha)|\tau|}{\left\{n+\left[(1-\alpha)|\tau|-1\right]\gamma\right\}[1+(n-1)\beta]}z^n, z \in U, n = 2,3,....$$

for every $n = 2,3,...$.

Thus, the proof of Theorem 2 is completed.

Special case of Theorem 2 has been proved by Altintaş et al [2], $\tau = 1$ (there $p = n = 1$).

Setting $\tau = 1$ in Theorem 2, we arrive at the following corollary.

Corollary 7 *The function* f *given by (2) belongs to the class* $T(\alpha, \beta, \gamma)$ *if and only if*

$$\sum_{n=2}^{\infty} (n-\alpha\gamma)[1+(n-1)\beta]a_n| \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z) = z - \frac{1-\alpha}{(n-\alpha\gamma)[1+(n-1)\beta]} z^n, z \in U, n = 2,3,....$$

Remark 15 *The result obtained in Corollary 7 verifies to Theorem 2 in [7].*

Choose $\gamma = 1$ in Corollary 7, we have the following result.

Corollary 8 *The function* f *given by (2) belongs to the class* $T(\alpha, \beta)$ *if and only if*

$$\sum_{n=2}^{\infty} (n-\alpha)[1+(n-1)\beta]a_n| \leq (1-\alpha).$$

The result is sharp for the function

$$f_n(z) = z - \frac{1-\alpha}{(n-\alpha)[1+(n-1)\beta]} z^n, z \in U, n = 2,3,....$$

Remark 16 *The result obtained in Corollary 8 verifies to Corollary 6 in [7].*

From Theorem 2 we get the following results at different values of the parameters.

Corollary 9 *(see [[7], Corollary7]) The function* f *given by (2) belongs to the class* $TS^*(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} (n-\alpha)a_n \leq 1-\alpha$$

The result is sharp for the function

$$f_n(z) = z - \frac{1-\alpha}{n-\alpha} z^n, z \in U$$

for every $n = 2,3,...$.

Corollary 10 *(see [[7], Corollary 8]) The function* f *given by (2) belongs to the class* $TC(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z) = z - \frac{1-\alpha}{n(n-\alpha)} z^n, z \in U$$

for every $n = 2,3,...$.

Corollary 11 *(see [[7], Corollary 9]) The function* f *given by (2) belongs to the class* $TK(\alpha, \beta)$ *if the following condition is satisfied*

$$\sum_{n=2}^{\infty} n[1+(n-1)\beta]a_n \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z) = z - \frac{1-\alpha}{n[1+(n-1)\beta]} z^n, z \in U$$

for every $n = 2,3,...$.

Corollary 12 *(see [[7], Corollary 10]) The function* f *given by (2) belongs to the class* $TK(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} na_n \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z) = z - \frac{1-\alpha}{n} z^n, z \in U$$

for every $n = 2,3,\dots$.

On the coefficient bound estimates of the functions belonging in the class $TS^*CK(\alpha, \beta, \gamma, \tau)$, we give the following result.

Theorem 3 *Let the function* f *given by (2) belongs to the class* $TS^*CK(\alpha, \beta, \gamma, \tau), \tau \in \mathrm{R}^* = \mathrm{R} - \{0\}$. *Then,*

$$ {}_{n=2}^{\infty} a_n \le \frac{(1-\alpha)|\tau|}{(1+\beta)\{2 + [(1-\alpha)|\tau| - 1]\gamma\}} \tag{15}$$

and

$$ {}_{n=2}^{\infty} na_n \le \frac{2(1-\alpha)|\tau|}{(1+\beta)\{2 + [(1-\alpha)|\tau| - 1]\gamma\}}. \tag{16}$$

Proof. Using Theorem 2, we obtain

$$\{2 + [(1-\alpha)|\tau| - 1]\gamma\}(1+\beta) {}_{n=2}^{\infty} a_n$$
$$\le {}_{n=2}^{\infty} \{n + [(1-\alpha)|\tau| - 1]\gamma\}[1 + (n-1)\beta]a_n \le (1-\alpha)|\tau|$$

From here, the inequality (15) is obtained immediately.
Similarly, we write

$$(1+\beta) {}_{n=2}^{\infty} \{n + [(1-\alpha)|\tau| - 1]\gamma\}a_n$$
$$\le {}_{n=2}^{\infty} \{n + [(1-\alpha)|\tau| - 1]\gamma\}[1 + (n-1)\beta]a_n \le (1-\alpha)|\tau|;$$

that is,

$$(1+\beta) {}_{n=2}^{\infty} n|a_n| \le (1-\alpha)|\tau| + [1 - (1-\alpha)|\tau|]\gamma(1+\beta) {}_{n=2}^{\infty} |a_n|.$$

Using the inequality (15) to the last inequality, we obtain

$$(1+\beta) {}_{n=2}^{\infty} n|a_n| \le \frac{2(1-\alpha)|\tau|}{\{2 + [(1-\alpha)|\tau| - 1]\gamma\}},$$

which, immediately yields the inequality (16).
Thus, the proof of Theorem3 is completed.
Setting $\tau = 1$ in Theorem 3, we obtain the following corollary.

Corollary 13 *(see [[7], Lemma 2.1]) Let the function* f *given by (2) belongs to the class* $T(\alpha, \beta; \gamma)$. *Then,*

$$ {}_{n=2}^{\infty} a_n \le \frac{1-\alpha}{(1+\beta)(2-\alpha\gamma)}$$

and

$$ {}_{n=2}^{\infty} na_n \le \frac{2(1-\alpha)}{(1+\beta)(2-\alpha\gamma)}$$

Remark 17 *The result obtained in the Corollary 13 verifies to Lemma 2 (with* $n = p = 1$ *) in [2].*

From Theorem 2, for the coefficient bound estimates, we arrive at the following result.

Corollary 14 *Let* $f \in TS^*CK(\alpha, \beta, \gamma, \tau), \tau \in \mathrm{R}^* = \mathrm{R} - \{0\}$. *Then,*

$$|a_n| \le \frac{(1-\alpha)|\tau|}{\{n + [(1-\alpha)|\tau| - 1]\gamma\}[1 + (n-1)\beta]}, n = 2,3,\dots..$$

Remark 18 *Numerous consequences of the Corollary 14 can indeed be deduced by specializing the various parameters involved. Many of these consequences were proved by earlier workers on the subject (see, for example, [1, 8, 10]).*

Now, we give following properties of the class $T(\alpha,\beta,\gamma;\tau)$.

Theorem 4 *The subclass* $TS^{*}CK(\alpha,\beta,\gamma,\tau),\tau\in\mathbb{R}^{*}=\mathbb{R}-\{0\}$ *is convex set.*

Proof. Assume that each of the functions $f,g\in TS^{*}CK(\alpha,\beta,\gamma,\tau),\tau\in\mathbb{R}^{*}=\mathbb{R}-\{0\},\alpha\in[0,1),\beta,\gamma\in[0,1]$, with $g(z)=z-\sum_{n=2}^{\infty}b_n z^n$. Then, for $\lambda\in[0,1]$, we write

$$\varphi(z)=\lambda f(z)+(1-\lambda)g(z)=z-\sum_{n=2}^{\infty}c_n z^n$$

where $c_n=\lambda a_n+(1-\lambda)b_n, n=2,3,...$.

Using necessary part of Theorem2, we write

$$\begin{aligned}&\sum_{n=2}^{\infty}\{n+[(1-\alpha)|\tau|-1]\gamma\}[1+(n-1)\beta]c_n\\&=\lambda\sum_{n=2}^{\infty}\{n+[(1-\alpha)|\tau|-1]\gamma\}[1+(n-1)\beta]a_n\\&+(1-\lambda)\sum_{n=2}^{\infty}\{n+[(1-\alpha)|\tau|-1]\gamma\}[1+(n-1)\beta]b_n\\&\le\lambda(1-\alpha)|\tau|+(1-\lambda)(1-\alpha)|\tau|=(1-\alpha)|\tau|;\end{aligned}$$

that is,

$$\sum_{n=2}^{\infty}\{n+[(1-\alpha)|\tau|-1]\gamma\}[1+(n-1)\beta]c_n\le(1-\alpha)|\tau|.$$

Next, using the sufficiently part of the Theorem 2, we have $\varphi\in T(\alpha,\beta;\gamma)$. This completes the proof of Theorem 4.

From Theorem 4, we obtain the following result.

Corollary 15 *(see [7], Lemma 2) The subclass* $T(\alpha,\beta;\gamma)$ *of the analytic functions in the open unit disk is convex set.*

3. Results and Discussion

In this paper, was introduced two new subclasses $S^{*}CK(\alpha,\beta,\gamma,\tau)$ and $TS^{*}CK(\alpha,\beta,\gamma,\tau)$ of the analytic functions on the open unit disk in the complex plane. The various geometric properties of the functions belonging to these classes have examined. Also, sharp inequalities for the coefficient bound estimates for the functions belonging to these classes are given. Taking advantage of the results obtained in this study can be given the distortion and growth theorems for the class $TS^{*}CK(\alpha,\beta,\gamma,\tau)$ with $\tau\in\mathbb{R}^{*}=\mathbb{R}-\{0\}$ defined in the study. In addition, the radii of starlikeness and convexity can be examined for this class.

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