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TREED-H.N.N Groups

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Abstract Combinatorial properties of groups were studied via the graphs of groups. This led to substantial development regarding the properties of groups by providing easier proofs. Contributions of many researchers, such as Cohen [1], Chiswell [2,3], Khanfar [4] and many others enhanced this methodology and encouraged the other researchers to apply the concept of graphs of groups, trees and other details in order to develop further applications of groups, free groups, free product of groups, amalgamated free products, H N N groups, etc.
This paper is based on groups acting on graphs using the above terminologies and standard procedures to explain the nature of some specific elements of groups.
On the other hand Length Functions, which was basically established and developed by Lynden [5], has proved its major role in this area.

Keywords Automorphism of a graph, Bipartite Graph, Conjugate Elements, Group actions, Length Functions, Normal Forms

1. Introduction

In this paper we start with some basic definitions and some essential properties of groups, which enable us to deal with groups acting on connected graphs. This will lead us to construct a graph of groups associated with the action of a group G on a graph X. Then the maximal tree T in the graph X is chosen to construct a group which acts on this tree. The latter will be isometric to a subgroup generated by an element associated with a reduced path in the graph.
Axioms of Length Functions, which were introduced by Lyndon [6] and developed by many others, are listed in the basic part of this paper. This is used to study the nature of many groups and analyse their properties.
Applications of Length Functions require the normal form theorem, which is related to closed and reduced path in the graphs.

Basics

Definition 1.1 A Graph X is a pair (V(X), E(X)) of two disjoined sets of elements ; a non-empty set V(X), called vertices and a set E(X) called the edges, with a function t: E(X) →V(X) and a function E(X) →E(X) denoted by: $e \to e^{-1}$ such that $e = (e^{-1})^{-1}$ for all e in E(X), e^{-1} is called the inverse of e.
Definition 1.2 We define $O(e) = t(e^{-1})$ so that $t(e) = O(e^{-1})$. O (e) and t (e) are called the end points of the edge e. O(e) is called the origin of e, and t (e) is called the terminal of e.
Definition 1.3 An edge e with O (e) = t (e) is called a loop. A pair of edges $\{ e, e^{-1} \}$ is called an unoriented edge.
Definition 1.4 An orientation of a graph X is a set consisting of exactly one member of each unoriented $\{ e, e^{-1} \}$ for which $e \neq e^{-1}$, together with every edge $e = e^{-1}$.

Definition 1.5 A graph Y is a subgraph of a graph X if $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$; and if $e \in E(Y)$ then O (e), t (e) and e^{-1} have the same meaning in Y as they have in X. We write $Y \subseteq X$.

Definition 1.6 A path P of length n in a graph X is a finite sequence of edges: $P = e_1 \ldots. e_n$, $n \geq 1$, such that $t(e_i) = O(e_{i+1})$ for i = 1, …, n – 1. We define O (P) = O (e_1) and t (P) = t (e_n) and say that, P is a path from O (e_1) to t (e_n). The path Pis closed if O (e_1) = t (e_n), and reduced if $e_{i+1} \neq e_i^{-1}$ for i = 1, … , n-1. At each vertex c of X, we define an empty path 1_v of length zero (a path without edges) from v to v.

Definition 1.7 If $P_1 = e_1 \ldots e_n$ and $P_2 = e_1' \ldots e_m'$ are paths in X and if t (P_1) = O (P_2), then we define their product: $P_1 P_2 = e_1 \ldots e_n e_1' \ldots e_m'$. P_1P_2 when defined, is a path from O (P_1) to t (P_2). The inverse P^{-1} of the path P is the path $P^{-1} = e_n^{-1} \ldots e_1^{-1}$.

Definition 1.8 A non- empty reduced closed path is a circuit.

Definition 1.9 A graph X is connected if for every pair of vertices u, v in X, there is a path in X from u to v.

Definition 1.10 A tree in X is a connected subgraph T of X which contains no circuits.

A maximal Tree T in a connected graph X is a subgraph in X which is maximal with respect to inclusion.

Definition 10.11 A graph X = ((v (X), E (X)) is bipartite if $V(X) = V_1(X) \cup V_2(X)$ with $V_1(X)$, $V_2(X)$ both edge free.

Proposition 1.1 If X and Y are graphs, then a morphism $f : X \to Y$is a mapping which takes vertices to vertices, edges to edges and such that f (O(x)) = O(f(x)**)** and $f(x^{-1}) = (f(x))^{-1}$ for all edges x in X.

f is called an isomorphism if it is one-to-one and onto. An isomorphism $f : X \to X$is called automorphism of X. The automorphism of X form a group, denoted by Aut X.

Definition 1.12 A group G acts on X if G can be represented in Aut X, that is, if there is a group homomorphism $\emptyset$; $G \to$ Aut X.

If X is a vertex or an edge in X, $g \in G$, we write gx for $\emptyset$ (g) (x). If x is an edge, then g O(x) = O (gx), $g x^{-1} = (gx)^{-1}$.

A group G acts without inversions on a graph X, if $gx \neq x^{-1}$ for any $g \in G$ and any $x \in E(X)$. G acts with inversion on X, if $gx = x^{-1}$ for some $g \in G$ and some $x \in E(X)$.

Let V(X)/G denote the set of G-orbits in V(X) and E(X)/G denote the set of G-orbits in E(X), then we can form the graph: X/G = (V(X))/G, E(X)/G).

X/G is the graph whose vertices and edges are the G-orbits in V(X) and E(X), X/G is called “Quotient graph”, with induced inverses and orbits.

The morphism $\rho : X \to X/G$ is called the projection.

Lemma 1.1 Let a group G act ona connected graph X with quotient Y = X/G. Let $\rho : X \to Y$ be a projection, and T a maximal tree of Y. Then there exists a morphism q: $T \to X$ such that ρ q is the identity on T.

Proof See [4].

2. Bass-Serre Theory

Take a graph with vertices A, G_i, $I \in I$ and edges $\emptyset_i: A \to G_i$, construct a graph:

$\mathcal{F} = < A, G_i, t_i, ;rel\ G_i, t^{-1}\ a\ t = \emptyset_i(a),\ \text{all}\ a \in A >$

If we put $t_i = 1$ for all i, then we get $G *_A G_i$.

Now construct a subgroup $\mathcal{F}_o$ of $\mathcal{F}$ which consists of all elements of the form:$a_o\ t_{i_1} g_{i_1} t_{i_1}^{-1} a_1\ t_{i_2} g_{i_2} t_{i_2}^{-1} \ldots . t_{i_n}^{-1}$,

Then define σ: $G \to \mathcal{F}_o$by: $g_i \to t_i\, a_i\, t_i^{-1}$ and $a \to a$

Recall that $G = < G_{i;}\ relG_{i,,} \emptyset_I (a) = \emptyset_j (a), a \in A >$

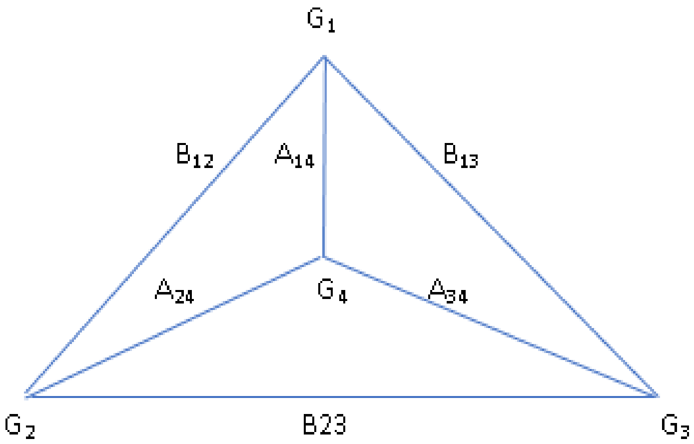
σ: $\emptyset_i (a) \to t_i \emptyset_i (a)\, t_i^{-1} = t_j \emptyset_j (a)\, t_j^{-1} = \sigma\, (\emptyset_j (a)\,)$

Hence σ defines a homomorphism from G to $\mathcal{F}$

Define $\pi\ (\ a_o\ t_{i_1} g_{i_1} t_{i_1}^{-1} a_1\ t_{i_2} g_{i_2} t_{i_2}^{-1} \ldots\ ) = a_o\ t_{i_1} g_{i_1} t_{i_1}^{-1} a_1\ t_{i_2} g_{i_2} t_{i_2}^{-1} \ldots$

Hence $\sigma\pi$ =on $\mathcal{F}_o$, also $\pi\ \sigma$ on G. So $G \approx \mathcal{F}_o$

Example 1 Let G_i be the vertices and A_{ij} be the amalgamated subgroups between the groups G_i and G_j for some i and some j. Take the H N N Extension with generator t_i for each additional edge and pair of isomorphic subgroups, then the following diagram is a connected Graph of groups each vertex is a group , each edge is a group isomorphic to a subgroup of its vertex group.



Then H N N extension can be constructed by:
G = < G_1, G_2, t, ; rel G_1, rel G_2, $\phi_1(a) = \phi_2(a)$, $t\,\psi_1(b)t^{-1} = \psi_2(b), a \in A, b \in B$>
G' = < G_1, G_2, t, ; rel G_1, rel G_2, $\psi_1(b) = \psi_2(b)$, $t\,\phi_1(a)t^{-1} = \phi_2(a), a \in A, b \in B$>

Definition 1.13 A graph of groups is a connected bipartite graph with vertices E and V and edges joining E to V. Each vertex is a group and each edge is an isomorphism ϕ of a group G_e in E into a group $G_v$in V.

Let $\mathcal{F}$= < E, V, $t_i$for each edge in the graph; relE, rel V, t^{-1} a t =$\emptyset_i$ (a), all a ∈ A >

Definition 1.14 The tree product of groups is given by:
G= < E, V ; rel E, rel V, $\phi_{i_j}(g_{e_i}) = g_{e_i}$, $wehere\ g_{e_i} \in G_{e_i} in\ E\ and\ \phi_{i_j}\ is\ on\ the\ tree$ >

Choose a maximal tree T in the graph, then $G_T = \mathcal{F}/\ t_i = 1$ all t_i in T and
G_T = < E, V, t_i, ; rel E, rel V, $t_i^{-1} g\ t_i = \phi_i(g)$, g ∈ G_e, $t_j = 1\ for\ all\ t_j\ in\ a\ max.tree\ T$ > is called Treed H N N group.

$\mathcal{F}_{e_o}$ is a subgroup of $\mathcal{F}$ **of all elements which can be expressed in the form**
$g_{e_o} t_o\, g_{v_1} t_1^{-1} g_{e_2}\, t_2\, g_{v_3}\, t_3^{-1}\, g_{e_4}\, t_4\ \ldots.t_n^{-1}\, g'_{e_o}$ corresponding to a path from e_o to e_o in the graph if:

$$t_o\, t_1^{-1}\, t_2\, t_3^{-1}\ \ldots.t_n^{-1}$$

t_1, t_2 both edges starting at e_2….etc.
t_2, t_3 both edges ending at e_2 ….

Theorem 1.1 $G_T \approx \mathcal{F}_{e_o}$

Proof See [1].

Corollary 1.1 G_T is independent of maximal tree T.

Definition 1.15 An element $x = g_{e_o}\, t_1\, g_{v_1}\, t_2^{-1}\, g_{e_3}\, t_3\ \ldots\ t_n^{-1}\, g'_{e_o}, n \geq 0\ of\ \mathcal{F}_{e_o}$ is said to be in a reduced form if it does not contain $t_i^{-1}\, g_{e_i}\, t_i$, with $g_{e_i} \in G_e$ or $t_i\, g_{v_i}\, t_i^{-1}$ with $g_{v_i} \in \phi_i(G_{e_i})$.

Reduced forms are not unique.

Definition 1.16 Given a collection of coset representatives of $\phi_i(G_e)\ in\ G_v$ with $\phi_i : G_e \to G_v$, a ward $w = g_{e_o}\, t_1\, g_{v_1}\, t_2^{-1}\, g_{e_3}\, t_3\ \ldots\ t_n^{-1}\, g'_{e_o}, n \geq 0$, representing an element of $\mathcal{F}_{e_o}$is in normal form if:

(i) g_{e_o} is an arbitrary element in a group $G_e \in E$

(ii) all g_{v_i} are in the collection of coset representatives of $\phi_i(G_{e_{i-1}})\ in\ G_v$

(iii) g_{e_i} = 1, i = 1,2, ……n

(iv) There is no subward$t_i^{-1}\, g_{e_i}\, t_i$ where $g_{e_i} \in G_e$ $,nor\ t_i\, g_{v_i}\, t_i^{-1}$, $g_{v_i} \in \phi_i(G_{e_i})$, i = 1, …, n.

Theorem 1.2 (Normal Form Theorem)Every element of$\mathcal{F}_{e_o}$ has a unique normal form.

Proof See [1].

3. Length Functions

Definition 3.1: A length function | | on a group G, is a function giving each element x of G a real number $|x|$, such that for all $x , y , z \in G$, the following axioms are satisfied.

$A1' |e| = 0$, e is the identity elements of G .

$A2$ $|x^{-1}| = |x|$

$A4$ $d(x,y) < d(y,z) \Longrightarrow d(x,y) = d(x,z)$, where $d(x,y) = \frac{1}{2}\ (|x| + |y| - |xy^{-1}|$

Lyndon in [6] showed that $A4$ is equivalent to $d(x,y) \geq \min\{ d(y,z), d(x,z)\}$ and to $d(y,z), d(x,z) \geq m \Longrightarrow d(x,z) \geq m$.

$A1'$, $A2$ and $A4$ imply that: $|x| \geq d(x,y) = d(y,x) \geq 0$.

Assuming, A2 and A4 only, it is easy to show that:

i. $d(x,y) \geq |e|$
ii. $|x| \geq |e|$
iii. $d(x,y) \leq |x| - \frac{1}{2}|e|$, see [7]

The axiom A3 states that: $d(x,y) \geq 0$ is deducible from $A1'$, $A2$. Also, $A1'$ is a weaker versionof the following axiom: $A1$: $|x| = 0$ if and only if $x = 1$ in G.

Lyndon [6] showed that if G is any group with length function and x, y and z are elements in G, then the following properties will hold.

Proposition 3.2 $d(xy,y) + d(x,y^{-1}) = |y|$

Proposition 3.3 $d(x,y^{-1}) + d(y,z^{-1}) \leq |y|$implies that $|x\ y\ z| \leq |x| - |y| + |z|$

Proposition 3.4 $d(x,y^{-1}) + d(y,z^{-1}) \leq |y|$implies that$d(xy,z^{-1}) = d(y,z^{-1})$

Proposition 3.5 $d(x,y) + d(x^{-1},y^{-1}) \geq |x| = |y|$implies that $|(xy^{-1})^2| \leq |xy^{-1}|)$

It follows from proposition 3.2 that for any $x,y \in G$, $d\ (x,y) = |y| - d\ (x\ y^{-1}, y^{-1}) \leq |y|$ by A3.

Since $d(x,y) = d\ (y\ ,x)$, we get: $d\ (x\ ,y) \leq \min\{|x|\ , |y|\}$.

A5 states that:$d\ (x\ ,y) +\ d\ (x^{-1}\ ,y^{-1}) > |x| = y \Rightarrow x = y$

Definition 3.1: A non-trivial element g of a group G is called Non-Archimedean if $|g^2| \leq |g|$

Definition 3.2: Let G be a group with length function. An element $x \neq 1$ in G is called Archimedean if $|x| \leq |x^2|$.

The following Axioms and results are added by Lyndon and others

$A0$ $x \neq 1 \Longrightarrow |x| < |x^2|$

$C0$ $d(x\ ,y\ )$is always an integer

$C1$ $x \neq 1$, $|x^2| \leq |x|\ implies$ $|x|$ is odd.

$C2\ \ For\ \ no\ \ x\ is\ |x^2| = |x| + 1$

$C3\ \ if\ |x|\ is\ odd\ then\ |x^2| \geq |x|$

$C1'\ if\ |x|\ is\ even\ and\ |x| \neq 0\ , then\ |x^2| > |x|$

$N0$ $|x^2| \leq |x|\ implies\ x^2 = 1\ \ is\ x = x^{-1}$

$N1^*\ G\ is\ general\ by\ \{x \in G :\ |x| \leq 1\}$

Definition 3.3: The set of all Non-Archimedean elements of G will be denoted by N and is given by: $N = \{x\ \in G :\ |x^2| \leq |x|\}$

Lyndon also introduced the following set in [1]: $M = \{xy\ \in G : |xy| +\ |yx| < 2|x| = 2\ |y|\}$, and showed that $M \subseteq N$. The nature of the elements of M and N is investigated in the next section.

4. Applications of Length Functions

The normal form theorem 1.2 for treed H N N groups allow us to assign a well-defined length to each element of the group.

Definition 4.1. Let G_T be a treed H N N group, G_T = < E, V, t_i, ; rel E, rel V, $t_i^{-1} g_{e_i} t_i = \phi_i(g_{e_i}), g_{e_i} \in G_e \; in \; E, t_j = 1 \; for \; all \; t_j \; in \; a \; max.tree \; T$ >.

Define a length function on the elements of G_T by:

$|x| = n, if \; x = g_{e_o} t_1 g_{v_1} t_2^{-1} g_{e_3} t_3 \ldots t_n^{-1} g'_{e_o}$ is in a reduced form and $n \geq 0$.

$x \to |x|$ satisfies the following axioms:
$A1'$ $|e| = 0$, e is the identity elements of G_T .
$A2$ $|x^{-1}| = |x|$
$A4$ $d(x,y) < d(y,z) \Longrightarrow d(x,y) = d(x,z)$, where $d(x,y) = \frac{1}{2}$ ($|x| + |y| - |xy^{-1}|$

Corollary 4.1. If G_T is a treed H N N group, then the set N consists of conjugates of elements of a group G_V in V or G_e in E.

Proof Suppose $x \in N$, i e, $|x^2| \leq |x|$ and let

$x^2 = g_{e_o} t_1 g_{v_1} t_2^{-1} g_{e_3} t_3 \ldots t_n^{-1} g'_{e_o} g_{e_o} t_1 g_{v_1} t_2^{-1} g_{e_3} t_3 \ldots t_n^{-1} g'_{e_o}$

g'_{e_o} and g_{e_o} are in G_e and $|x^2| \leq |x|$, so a subward of length 2r in the middle will be consolidated.

$x^2 = g_{e_o} t_1 g_{v_1} t_2^{-1} \ldots t_{n-r}^{\varepsilon} (g_{s_{n-r}} t_{n-r+1}^{-\varepsilon} g_{s_{n-r+1}} \ldots t_n^{-1} \quad g_{e_o} \; g_{e_o} t_1^{-1} \ldots t_r^{\varepsilon} g_{s_r} \quad) t_{r+1}^{-\varepsilon} g_{s_{r+1}} \ldots$
$t_n^{-1} g'_{e_o}$

$= g_{e_o} t_1 g_{v_1} t_2^{-1} \ldots t_{n-r}^{\varepsilon} (k_r) t_{r+1}^{-\varepsilon} g_{s_{r+1}} \ldots t_n^{-1} g'_{e_o}$, in a reduced form.

If $\varepsilon = +1$, then $k_r \in G_v \; and \; if \; \varepsilon = -1, then \; k_r \in G_e$

Suppose $\varepsilon = +1$, then $g_{s_{n-r-1}} = g_{e_{n-r}} \in G_e$

$|x| = n$, So, $2n - 2r \leq n$, and $r > \frac{n}{2}$, So $r \geq n - r$.

Take $x = g_{e_o} t_1 g_{v_1} t_2^{-1} \ldots t_r^{\varepsilon} g_{s_r} t_{r+1}^{-\varepsilon} \ldots t_n^{-1} g'_{e_o}$

Suppose $\varepsilon = 1$, then $g_{s_r} = g_{v_r} \in G_v$, and $x = (g_{e_o} t_1 g_{v_1} t_2^{-1} \ldots t_r) g_{v_r} (t_{r+1}^{-1} \ldots t_n^{-1} g'_{e_o})$

$$x = (g_{e_o} t_1 g_{v_1} t_2^{-1} \ldots t_r)(g_{e_o} t_1 g_{v_1} t_2^{-1} \ldots t_r)^{-1}$$

$$x = (g_{e_o} t_1 g_{v_1} t_2^{-1} \ldots t_r) g_{v_r} t_{r+1}^{-1} k_{n-r} t_r (g_{e_o} t_1 g_{v_1} t_2^{-1} \ldots t_r)^{-1}$$

$r \geq n - r, so \; k_{n-r}$ is a part of k_r which has consolidated.

$k_{n-r} \in G_e$,so $t_{r+1}^{-1} k_{n-r} t_r = g_{v_{n-r}} \in G_{v}$.

Suppose $g_{e_o} t_1 g_{v_1} t_2^{-1} \ldots t_r = g \in G$, then

$$x = g \, g_{v_r} g_{v_{n-r}} \, g^{-1} = g g_v \, g^{-1}, \qquad where \; g_v \in x = (g_{e_o} t_1 g_{v_1} t_2^{-1} \ldots t_r$$

Hence x is a conjugate of an element of a group G_v in V.

If we suppose that $\varepsilon = -1$, then x will be a conjugate of an element of a group G_e in E.

Corollary 4.2 If G_T is a treed H N N group then the equivalent elements of N lie in the same conjugate of the group G_e in E or a group G_v in V.

i.e $x \sim y \; if \; and \; only \; if \; x = g g_v \, g^{-1}, y = g g'_v \, g^{-1}, where \; g \in G \; and \; g_v, g'_v \in G_v$ in V.

or $x = g g_e \; g^{-1} \; and \; y = g g'_e \; g^{-1}, where \; g_e \in C$.

Proof Similar to Corollary 4.1.

Theorem 4.1 If G_T is a treed H N N group, then M consists of conjugates of elements of a group $G_e \in E$.i.e$x \in M \; if \; and \; only \; if \; x = g g_e \, g^{-1}, where \; g_e \in G_e \; in \; E, and \; g \in G$

Proof Let xy be an element of M and suppose

$x = g_{e_o} t_1 g_{v_1} t_2^{-1} g_{e_3} t_3 \ldots t_n^{-1} g'_{e_o}$ and $y = h_{e_o} c_1 h_{u_1} c_2^{-1} h_{e_3} c_3 \ldots c_n^{-1} h'_{e_o}$ and >0.

$$xy = g_{e_o}\, t_1\, g_{v_1}\, t_1^{-1}\, g_{e_3}\, t_3\, \dots\, t_n^{-1}\, g'_{e_o}\, h_{e_o} c_1 h_{u_1} c_2^{-1}\, h_{e_3}\, c_3\, \dots\, c_n^{-1}\, h'_{e_o}$$

$|x| = |y| = n, \quad |xy| + |yx| < 2n, \quad |xy| < 2n$

So the least form of xy is not reduced, and suppose 2r terms consolidate in the middle, then

$xy = g_{e_o}\, t_1\, g_{v_1}\, t_1^{-1}\, g_{e_3}\, t_3\, \dots\, t_{n-r}^{\varepsilon} k_r\, c_{r+1}^{-\varepsilon} \dots\, c_n^{-1}\, h'_{e_o}$ and

$yx = h_{e_o} c_1 h_{u_1} c_2^{-1}\, h_{e_3}\, c_3\, \dots\, c_{n-1}^{-\delta} c_s\, t_{s+1}^{-\delta}\, \dots t_n^{-1} g'_{e_o}$ are in reduced forms, for r, s >0, k_s , c_s are elements of $G_e\ or\ G_v$, and $\varepsilon, \delta = \pm 1$.

$$|xy| + |yx| < 2|x| = 2\,|y|$$

2n – 2r + 2n – 2s < 2n

r + s > n implies r > n – s

s > n – r

Take $xy = g_{e_o}\, t_1\, g_{v_1}\, t_1^{-1}\, g_{e_3}\, t_3\, \dots\, t_s^{\varepsilon}\, k_{n-s}\, c_{n-s+1}^{-\varepsilon}\, \dots\, c_n^{-1}\, h'_{e_o}$, where k_{n-s} is a part of k_r, for r > n – s.

If $\varepsilon = +1, then\ k_{n-s} \in G_v\ in\ V,\ t_s\, k_{n-s}\, c_{n-s+1}^{-\varepsilon} \in G_e\ in\ E$

$$xy = g_{e_o}\, t_1\, g_{v_1}\, t_1^{-1} \dots g_{e_s}\ t_s\, k_{n-s}\, c_{n-s+1}^{-1}\ (h'_{e_{n-s+1}} \dots\, c_n^{-1}\, h'_{e_o})$$

$$xy = (h'_{e_{n-s+1}} \dots c_n^{-1} h'_{e_o})^{-1} (\, h'_{e_{n-s+1}} \dots c_n^{-1} h'_{e_o}\, g_{e_o}\, t_1\, g_{v_1}\, t_2^{-1} \dots g_{e_s}) t_s k_{n-s}\, c_{n-s+1}^{-1}\, (h'_{e_{n-s+1}} \dots c_n^{-1}\, h'_{e_o})$$

Let $g_e = t_s\, k_{n-s}\, c_{n-s+1}^{-1} \in G_e$, $h'_{e_{n-s+1}} \dots c_n^{-1} h'_{e_o}\, g_{e_o}\, t_1\, g_{v_1}\, t_2^{-1} \dots g_{e_s} = g'_e \in G_e$,

and let $g'_e\, g_e = g''_e \in G_e$.

Hence $xy = \left((h'_{e_{n-s+1}} \dots c_n^{-1} h'_{e_o})^{-1}\right)\ g_e (h'_{e_{n-s+1}} \dots c_n^{-1}\, h'_{e_o})$

Therefore, xy is a conjugate of an element $g_e\ in\ G_e \in E$.

If $\varepsilon = -1, then\ k_{n-s} \in G_e\ in\ E.$

Take $xy = (g_{e_o}\, t_1\, g_{v_1}\, t_1^{-1}\, g_{e_3}\, t_3\, \dots\, t_s^{-1}\, g_{e_s}) t_{s+1}\, k_{n-s-1}\, c_{n-s}^{-1}\, (h'_{e_{n-s}} \dots\, c_n^{-1}\, h'_{e_o}$

Let $g_e = t_{s+1}\, k_{n-s-1}\, c_{n-s}^{-1} \in G_e\ in\ E$, then

$$xy = (h'_{e_{n-s}} \dots c_n^{-1} h'_{e_o})\ \ (h'_{e_{n-s}} \dots c_n^{-1} h'_{e_o})\, (g_{e_o}\, t_1\, g_{v_1}\, t_2^{-1} \dots t_s^{-1} g_{e_s}) g_e\, (h'_{e_{n-s}} \dots c_n^{-1} h'_{e_o})$$

Let $(h'_{e_{n-s}} \dots c_n^{-1} h'_{e_o})\, (g_{e_o}\, t_1\, g_{v_1}\, t_2^{-1} \dots t_s^{-1} g_{e_s}) = g'_e$ which is part of k_s which consolidated, then

$$xy = (h'_{e_{n-s}} \dots c_n^{-1} h'_{e_o})^{-1} (g_o g_e) (h'_{e_{n-s}} \dots c_n^{-1} h'_{e_o})$$

$g_o , g_e \in G_e$, then put $\bar{g}_e = g'_e\, g_e$

Hence $xy = (h'_{e_{n-s}} \dots c_n^{-1} h'_{e_o})^{-1} (\bar{g}_e) (h'_{e_{n-s}} \dots c_n^{-1} h'_{e_o})$

i.e. xy is a conjugate of an element of G_e in E.

The same procedure applies to yx

Corollaries 4.2, 4.3 and Theorem 4.1 show that $M \subseteq N$.

References

[1]. Cohen, D. E. (1978). Combinatorial group Theory, A Topological Approach, London Queen Mary University,

[2]. Chiswell, I.M.(1976). Abstract Length Function groups: Maths. Proc. Camb. Phil. Soc. 451-463.

[3]. Chiswell, I.M.(1971). On groups acting on Trees: Ph D Thesis, University of Michigan.

[4]. Khanfar, M.I. (1978). Combinatorial Properties of Groups with Length Function ; Ph D. Thesis, and University of Birmingham .

[5]. Lyndon and Schupp. (1977). Combinatorial Group Theory, Springer-Verlag, Berlin, Heidelberg, New York.

[6]. Lyndon, R.C.(1963). Length Functions in Groups: Math. Scand., 12, 209-234.

[7]. Wilkens, D.L.(1976). On Non-Archimedian Lengths in Groups: Mathematika, 23, 57-61.