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On the Upper Bound Estimates for the Coefficients of Certain Subclass Bi-univalent Functions of Complex Order

Nizami Mustafa, Taner Turaç

Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, TURKEY nizamimustafa@gmail.com Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey
nizamimustafa@gmail.com

Abstract In this paper, we introduce and investigate a subclass of analytic and bi-univalent functions of complex order in the open unit disk in complex plane. We obtain upper bound estimates for the initial coefficients $|a_2|, |a_3|$ and $|a_4|$ of the functions belonging to this class.

Keywords Analytic functions, Bi-univalent functions, Coefficient bounds
Subject Classification Primary 30C45, 30C50, Secondary 30C80

1. Introduction

Let A be the class of analytic functions in the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots = z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathbb{C}, \quad (1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by S the subclass of A consisting of functions which are also univalent in. Some of the important subclass of S is the class $\Re(\alpha, \beta)$ defined by

$$\Re(\alpha, \beta) = \left\{f \in S : Re\left(f^{'}(z) + \beta z f^{''}(z)\right) > \alpha, z \in U\right\}, \alpha[0,1), \beta \geq 0.$$

Gao and Zhou [9] investigated and showed some mapping properties of this class $\Re(\alpha, \beta)$.

In the special case, we have subclass $\Re(\beta)$ defined by

$$\Re(\beta) = \left\{f \in S : Re\left(f^{'}(z) + \beta f^{''}(z)\right) > 0, z \in U\right\}, \beta \geq 0$$

for $\alpha = 0$.

Early, by Alintaş et al. [1] were investigated a subclass $\Re(\alpha, \beta, \tau)$ of analytic and bi-univalent functions consisting of function $f \in T$ which satisfied the condition

$$\left|\frac{1}{\tau}\left[f^{'}(z) + \beta z f^{''}(z) - 1\right]\right| \leq \alpha, \beta \in [0,1], \alpha \in (0,1], \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}, z \in U.$$

Here T is the subclass of A consisting of the functions f in the form

$$f(z) = z - a_2 z^2 - a_3 z^3 - \ldots - a_n z^n - \ldots = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0.$$

Altintaş et al. [1] found necessary and sufficient conditions for the functions belonging to this class.

It is well-known that (see, for example, [6]) every function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \ldots,$$

$$w \in D = \{w : |w| < r_0(f)\}, r_0(f) \geq \frac{1}{4}.$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent. We denote by Σ the subclass of bi-univalent functions in U given by (1).

In 1967, Lewin[14] showed that for every function of the form (1) the second coefficient satisfies the estimate $|a_2| < 1.51$. In 1967, Brannan and Clunie [2] conjectured that $|a_2| < \sqrt{2}$ for $f \in \Sigma$. In 1984, Tan [21] obtained the bound for $|a_2|$, namely, taht $|a_2| < 1.485,$ which is the best known estimate for functions in the class Σ. In 1985, Kedzierawski [12] proved the Brannan-Clunie conjecture for bi-starlike functions. Brannan and Taha [3] obtained estimates on the initial two coefficients $|a_2|$ and $|a_3|$ for the functions in the classes of bi-starlike and bi-convex functions of order α, $\alpha \in [0,1)$.

The study of bi-univalent functions was revived, in recently years by Srivastava et al. [19] and a considerably large number of sequels to the work of Srivastava et al. [18] have appeared in the literature. In particular, several results on coefficient estimates for the initial three coefficients $|a_2|, |a_3|$ and $|a_4|$ were proved for various subclasses of Σ (see, for example, [4, 7, 11, 16, 19, 20, 22, 23]).

Recently, Deniz [5] and Kumar et al. [13] both extended and improved the results of Brannan and Taha [3] by generalizing their classes by means of the principle of subordination between analytic functions.

Despite the numerous studies mentioned above, the problem of estimating the coefficients $|a_n|(n = 2,3,\ldots)$ for the general class functions Σ is still open (see also [20] in this connection).

Motivated by the aforementioned works, we define a new subclass of bi-univalent functions Σ as follows.

Definition 1 *A function $f \in \Sigma$ given by (1) is said to be in the class $\Im_\Sigma(\alpha, \beta, \tau)$ if the following conditions are satisfied*

$$Re\left\{1 + \frac{1}{\tau}\left[f'(z) + \beta z f''(z) - 1\right]\right\} > \alpha, \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}, \alpha \in [0,1), \beta \geq 0, z \in U$$

and

$$Re\left\{1 + \frac{1}{\tau}\left[g'(w) + \beta w g''(w) - 1\right]\right\} > \alpha, \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}, \alpha \in [0,1), \beta \geq 0, w \in D$$

where the function $g = f^{-1}$.

Remark 1 *Choose $\tau = 1$ in the Definition 1, we have function class*

$\Im_\Sigma(\alpha, \beta, 1) = H_\Sigma(\alpha, \beta), \alpha \in [0,1), \beta \geq 0$ *; that is,*

$$f \in H_\Sigma(\alpha, \beta) \Leftrightarrow Re\left(f'(z) + \beta z f''(z)\right) > \alpha, z \in U$$

and

$$Re\left(g'(w) + \beta w g''(w)\right) > \alpha, w \in D$$

where $g = f^{-1}$.

Remark 2 *Choose* $\beta = 0$ *in the Definition 1, we have function class* $\Im_{\Sigma}(\alpha,0,\tau), \alpha \in [0,1), \tau \in \mathbb{C}^*$ *; that is,*

$$f \in \Im_{\Sigma}(\alpha,0,\tau) \Leftrightarrow Re\left\{1+\frac{1}{\tau}\left[f^{'}(z)-1\right]\right\} > \alpha, z \in U$$

and

$$Re\left\{1+\frac{1}{\tau}\left[g^{'}(w)-1\right]\right\} > \alpha, w \in D$$

where $g = f^{-1}$.

Remark 3 *Choose* $\beta = 0, \tau = 1$ *in the Definition 1, we have function class* $\Im_{\Sigma}(\alpha,0,1) = \Re_{\Sigma}(\alpha,0), \alpha \in [0,1)$ *; that is,*

$$f \in \Re_{\Sigma}(\alpha,0) \Leftrightarrow Re\left(f^{'}(z)\right) > \alpha, z \in U$$

and

$$Re\left(g^{'}(w)\right) > \alpha, w \in D$$

where $g = f^{-1}$.

Remark 4 *Choose* $\beta = 1$ *in the Definition 1, we have function class* $\Im_{\Sigma}(\alpha,1,\tau), \alpha \in [0,1), \tau \in \mathbb{C}^*$ *; that is*

$$f \in \Im_{\Sigma}(\alpha,1,\tau) \Leftrightarrow Re\left\{1+\frac{1}{\tau}\left[f^{'}(z)+zf^{''}(z)-1\right]\right\} > \alpha, z \in U$$

and

$$Re\left\{1+\frac{1}{\tau}\left[g^{'}(w)+wg^{''}(w)-1\right]\right\} > \alpha, w \in D,$$

where $g = f^{-1}$.

Remark 5 *Choose* $\beta = 1, \tau = 1$ *in the Definition 1, we have function class* $\Im_{\Sigma}(\alpha,1,1) = \Re_{\Sigma}(\alpha,1), \alpha \in [0,1)$ *; that is,*

$$f \in \Re_{\Sigma}(\alpha,1) \Leftrightarrow Re\left(f^{'}(z)+zf^{''}(z)\right) > \alpha, z \in U$$

and

$$Re\left(g^{'}(w)+\beta wg^{''}(w)\right) > \alpha, w \in D$$

where $g = f^{-1}$.

Recently, the class $\Im(\alpha,\beta,\tau)$ were investigated by Mustafa et al. [15]. They give the sufficient, and sufficient and necessary conditions for the functions belong to this class. Also, they find the upper bound estimates for the some initial coefficients of the functions belonging to this class and its special cases.

The class $\Im_{\Sigma}(\alpha,0,1) = \Re_{\Sigma}(\alpha,0) = N_{\Sigma}(\alpha)$ were investigated by Srivastava et al. [19].

Recently, by Frasin [8] investigated subclass $\Im_{\Sigma}(\alpha,\beta,1) = H_{\Sigma}(\alpha,\beta), \alpha \in [0,1), \quad \beta > 0$ with condition $2(1-\alpha)\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{\beta n+1} \le 1$. He found estimates on two first coefficients for the functions in this class.

The object of the present paper is to find the upper bound estimates for three initial coefficients $|a_2|, |a_3|$ and $|a_4|$ of the functions belonging to the class $\Im_\Sigma(\alpha,\beta,\tau)$ and its special cases.

To prove our main results, we need require the following lemmas.

Lemma 1 *(See, for example, [17]) If $p \in P$, then the estimates $|p_n| \le 2, n = 1,2,3,...$ are sharp, where P is the family of all functions p, analytic in U for which $p(0)=1$ and $Re(p(z)) > 0, z \in U$ and*

$$p(z) = 1 + p_1 z + p_2 z^2 + ..., z \in U. \tag{2}$$

Lemma 2 *(See, for example,[10]) If the function $p \in P$ is given by the series (2), then*

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)w$$

for some x and z with $|x| \le 1$ and $|z| \le 1$.

2. Coefficient bound estimates for the function class $\Im_\Sigma(\alpha,\beta,\tau)$

In this section, we prove the following theorem on upper bound estimates for the initial three coefficients of the function belonging to the class $\Im_\Sigma(\alpha,\beta,\tau)$.

Theorem 1 *Let the function f given by (1) be in the class $\Im_\Sigma(\alpha,\beta,\tau)$. Then,*

$$|a_2| \le \frac{(1-\alpha)|\tau|}{1+\beta}, |a_3| \le \begin{cases} \dfrac{2(1-\alpha)|\tau|}{3(1+2\beta)} & \text{if } |\tau| \in (0,\tau_0), \\ \dfrac{(1-\alpha)^2|\tau|^2}{(1+\beta)^2} & \text{if } |\tau| \ge \tau_0, \end{cases}$$

where $\tau_0 = \dfrac{2(1+\beta)^2}{3(1-\alpha)(1+2\beta)}$.

Also,

$$|a_4| \le h_1(\alpha,\beta,\tau)t_0^3 + 2h_2(\alpha,\beta,\tau)t_0^2 + h_3(\alpha,\beta,\tau),$$

where

$$h_1(\alpha,\beta,\tau) = h_2(\alpha,\beta,\tau) + \frac{(1-\alpha)|\tau|}{8(1+3\beta)}, h_2(\alpha,\beta,\tau) = \frac{5(1-\alpha)^2|\tau|^2}{24(1+\beta)(1+2\beta)t_0},$$

$$h_3(\alpha,\beta,\tau) = \frac{(1-\alpha)|\tau|}{2(1+3\beta)}, t_0 = \frac{6(1+\beta)(1+2\beta)}{5(1-\alpha)|\tau|(1+3\beta) + 6(1+\beta)(1+2\beta)}.$$

Proof. Let $f \in \Im_\Sigma(\alpha,\beta,\tau), \alpha \in [0,1), \beta \ge 0, \tau \in \mathbb{C}^*$ and $g = f^{-1}$.

Then,

$$1 + \frac{1}{\tau}\left[f'(z) + \beta z f''(z) - 1\right] = \alpha + (1-\alpha)p(z) \tag{3}$$

and

$$1 + \frac{1}{\tau}\left[g'(w) + \beta w g''(w) - 1\right] = \alpha + (1-\alpha)q(w) \tag{4}$$

where functions $p(z)=1+p_1z+p_2z^2+\ldots$ and $q(w)=1+q_1w+q_2w^2+\ldots$ are in the class P.
Comparing the coefficients in (3) and (4), we have

$$a_2=\frac{\tau(1-\alpha)}{2(1+\beta)}p_1, a_3=\frac{\tau(1-\alpha)}{3(1+2\beta)}p_2, a_4=\frac{\tau(1-\alpha)}{4(1+3\beta)}p_3 \tag{5}$$

and

$$-a_2=\frac{\tau(1-\alpha)}{2(1+\beta)}q_1, 2a_2^2-a_3=\frac{\tau(1-\alpha)}{3(1+2\beta)}q_2, -5a_2^3+5a_2a_3-a_4=\frac{\tau(1-\alpha)}{4(1+3\beta)}q_3. \tag{6}$$

From the first equality of (5) and (6), we find that

$$\frac{\tau(1-\alpha)}{2(1+\beta)}p_1=a_2=-\frac{\tau(1-\alpha)}{2(1+\beta)}q_1\left(p_1=-q_1\right). \tag{7}$$

Also, from the second equality of (5) and (6), considering (7), we get

$$a_3=\frac{\tau^2(1-\alpha)^2}{4(1+\beta)^2}p_1^2+\frac{\tau(1-\alpha)}{6(1+2\beta)}\left(p_2-q_2\right). \tag{8}$$

Subtracting the third equality of (6) from the third equality of (5) and considering (7), we can easily obtain that

$$a_4=\frac{5\tau^2(1-\alpha)^2}{24(1+\beta)(1+2\beta)}p_1\left(p_2-q_2\right)+\frac{\tau(1-\alpha)}{8(1+3\beta)}\left(p_3-q_3\right). \tag{9}$$

In view of Lemma 2, since (see (7)) $p_1=-q_1$, we can write

$$p_2-q_2=\frac{4-p_1^2}{2}(x-y) \tag{10}$$

and

$$p_3-q_3=\frac{p_1^3}{2}+\frac{p_1(4-p_1^2)}{2}(x+y)-\frac{p_1(4-p_1^2)}{4}\left(x^2+y^2\right) \tag{11}$$
$$+\frac{4-p_1^2}{2}\left[(1-|x|^2)z-(1-|y|^2)w\right]$$

for some x,y and z,w with $|x|\le 1,|y|\le 1,|z|\le 1$ and $|w|\le 1$.

Since $|p_1|\le 2$, we may assume without any restriction that $t\in[0,2]$, where $t=|p_1|$.
From (7), we easily see that

$$|a_2|\le\frac{(1-\alpha)|\tau|}{2(1+\beta)}t, t\in[0,2];$$

that is,

$$|a_2|\le\frac{(1-\alpha)|\tau|}{1+\beta}. \tag{12}$$

Substituting the expression (10) in (8) and using triangle inequality, taking $|x|=\xi,|y|=\eta$, we can easily see that

$$|a_3|\le C_1(t)(\xi+\eta)+C_2(t)=F(\xi,\eta), \tag{13}$$

where

$$C_1(t)=\frac{(1-\alpha)|\tau|(4-t^2)}{12(1+2\beta)}\ge 0, C_2(t)=\frac{(1-\alpha)^2|\tau|^2}{(1+\beta)^2}t^2\ge 0, t\in[0,2].$$

It is clear that the maximum of the function $F(\xi,\eta)$ occurs at $(\xi,\eta)=(1,1)$. Therefore,

$$F(\xi,\eta)\le \max\{F(\xi,\eta):\xi,\eta\in[0,1]\}=F(1,1)=2C_1(t)+C_2(t). \tag{14}$$

Define the function $G:[0,2]\to \mathbb{R}$ as follows

$$G(t)=2C_1(t)+C_2(t) \tag{15}$$

for fixed value of $\tau\in\mathbb{C}^*$.

Substituting the value $C_1(t)$ and $C_2(t)$ in (15), we obtain

$$G(t)=A(\alpha,\beta,\tau)t^2+B(\alpha,\beta,\tau),$$

where

$$A(\alpha,\beta,\tau)=\frac{(1-\alpha)^2|\tau|^2}{(1+\beta)^2}\left[|\tau|-\frac{2(1+\beta)^2}{3(1-\alpha)(1+2\beta)}\right], B(\alpha,\beta,\tau)=\frac{2(1-\alpha)|\tau|}{3(1+2\beta)}$$

Now, we must investigate the maximum of the function $G(t)$ in the interval $[0,2]$.

By simple computation, we can easily show that

$$G'(t)=A(\alpha,\beta,\tau)t.$$

It is clear that $G'(t)<0$ if $A(\alpha,\beta,\tau)<0$; that is, the function $G(t)$ is a decreasing function if $|\tau|\in(0,\tau_0)$, where $\tau_0=\dfrac{2(1+\beta)^2}{3(1-\alpha)(1+2\beta)}$.

Therefore,

$$G(t)\le \max\{G(t):t\in[0,2]\}=G(0)=2C_1(0)=\frac{2(1-\alpha)|\tau|}{3(1+2\beta)}. \tag{16}$$

Also, $G'(t)\ge 0$ if $|\tau|\ge\tau_0$; that is, the function $G(t)$ is an increasing function for $|\tau|\ge\tau_0$.

Therefore,

$$G(t)\le \max\{G(t):t\in[0,2]\}=G(2)=2C_2(2)=\frac{(1-\alpha)^2|\tau|^2}{(1+\beta)^2}. \tag{17}$$

Substituting the expressions (10) and (11) in (9) and using triangle inequality, putting $|x|=\zeta, |y|=\varsigma$, we can easily see that

$$|a_4|\le c_1(t)(\zeta^2+\varsigma^2)+c_2(t)(\zeta+\varsigma)+c_3(t)=\phi(\zeta,\varsigma), \tag{18}$$

where

$$c_1(t)=\frac{(1-\alpha)(4-t^2)(t-2)|\tau|}{32(1+3\beta)}\le 0,$$

$$c_2(t)=\frac{(1-\alpha)(4-t^2)|\tau|t[5|\tau|(1-\alpha)(1+3\beta)+3(1+\beta)(1+2\beta)]}{48(1+\beta)(1+2\beta)(1+3\beta)}\ge 0,$$

$$c_3(t)=\frac{(1-\alpha)|\tau|}{16(1+3\beta)}(t^3-2t^2+8)\ge 0, t\in[0,2].$$

Now, we need to maximize the function $\phi(\zeta,\varsigma)$ on the closed square $\Omega=\{(\zeta,\varsigma):\zeta,\varsigma\in[0,1]\}$.

Since the coefficients $c_1(t)$, $c_2(t)$ and $c_3(t)$ of the function $\phi(\zeta,\varsigma)$ is dependent to variable t, we must investigate the maximum of $\phi(\zeta,\varsigma)$ respect to t taking into account these cases $t=0, t=2$ and $t\in(0,2)$.

Let us $t=0$. Then, we write

$$\phi(\zeta,\varsigma)=-\frac{(1-\alpha)|\tau|}{4(1+3\beta)}(\zeta^2+\varsigma^2)+\frac{(1-\alpha)|\tau|}{2(1+3\beta)}.$$

In this case, we will examine the maximum of the function $\phi(\zeta,\varsigma)$ taking into account the sing of $\Delta(\zeta,\varsigma)=\phi_{\zeta\zeta}(\zeta,\varsigma)\phi_{\varsigma\varsigma}(\zeta,\varsigma)-[\phi_{\zeta\varsigma}(\zeta,\varsigma)]^2$.

By simple computation, we can easily see that

$$\phi_{\zeta}^{'}(\zeta,\varsigma)=-\frac{(1-\alpha)|\tau|}{2(1+3\beta)}\zeta, \phi_{\varsigma}^{'}(\zeta,\varsigma)=-\frac{(1-\alpha)|\tau|}{2(1+3\beta)}\varsigma$$

and

$$\phi_{\varsigma\zeta}^{'}(\zeta,\varsigma)=\phi_{\zeta\varsigma}(\zeta,\varsigma)=0, \phi_{\zeta\zeta}^{''}(\zeta,\varsigma)=\phi_{\varsigma\varsigma}^{''}(\zeta,\varsigma)=-\frac{(1-\alpha)|\tau|}{2(1+3\beta)}, (\zeta,\varsigma)\in\Omega.$$

Thus,

$$\Delta(\zeta_0,\varsigma_0)=\left(\frac{(1-\alpha)|\tau|}{2(1+3\beta)}\right)^2>0 \text{ and } \phi_{\varsigma\varsigma}^{''}(\zeta_0,\varsigma_0)<0;$$

that is, (ζ_0,ς_0) is a maximum point for the function $\phi(\zeta,\varsigma)$, where $(\zeta_0,\varsigma_0)=(0,0)$. Therefore, in the case $t=0$

$$\phi(\zeta,\varsigma)\le \max\{(\zeta,\varsigma):\zeta,\varsigma\in[0,1]\}=\phi(0,0)=\frac{(1-\alpha)|\tau|}{2(1+3\beta)}. \tag{19}$$

For $t=2$, the function $\phi(\zeta,\varsigma)$ is a constant function as follows

$$\phi(\zeta,\varsigma)=c_3(2)=\frac{(1-\alpha)|\tau|}{2(1+3\beta)}. \tag{20}$$

In the case $t\in(0,2)$, we will examine the maximum of the function $\phi(\zeta,\varsigma)$ taking into account the sign of $\Lambda(\zeta,\varsigma)=\phi_{\zeta\zeta}(\zeta,\varsigma)\phi_{\varsigma\varsigma}(\zeta,\varsigma)-[\phi_{\zeta\varsigma}(\zeta,\varsigma)]^2$.

By simple computation, we can easily see that

$$\phi_{\zeta}^{'}(\zeta,\varsigma)=2c_1(t)\zeta+c_2(t), \phi_{\zeta}^{'}(\zeta,\varsigma)=2c_1(t)\varsigma+c_2(t)$$

and

$$\phi_{\zeta\varsigma}^{''}(\zeta,\varsigma)=\phi_{\varsigma\zeta}^{''}(\zeta,\varsigma)=0$$

$$\phi_{\zeta\zeta}^{''}(\zeta,\varsigma)=\phi_{\varsigma\varsigma}^{''}(\zeta,\varsigma)=2c_1(t), (\zeta,\varsigma)\in\Omega.$$

Thus (ζ_0,ς_0), where $\zeta_0=\varsigma_0=\frac{-c_2(t)}{2c_1(t)}$, is likely a critical point of the function $\phi(\zeta,\varsigma)$. We can easily show that $(\zeta_0,\varsigma_0)\in\Omega$; that is, $\frac{-c_2(t)}{2c_1(t)}\le 1$ for $t\le t_0$, where $t_0=\frac{6(1+\beta)(1+2\beta)}{5(1-\alpha)|\tau|(1+3\beta)+6(1+\beta)(1+2\beta)}$.

It can easily be see that $t_0\le 1$. Therefore, the function $\phi(\zeta,\varsigma)$ cannot have a critical point for $t\in(t_0,2)$. Hence, we must investigate the maximum of the function $\phi(\zeta,\varsigma)$ for $t\in(0,t_0]$ not for $t\in(0,2)$. Since

$$\Delta(\zeta_0,\varsigma_0)=4c_1^2(t)>0 \text{and} \phi''_{\zeta\varsigma}(\zeta,\varsigma)=2c_1(t)<0,$$

(ζ_0,ς_0) is a maximum point for the function $\phi(\zeta,\varsigma)$. Therefore,

$$\phi(\zeta,\varsigma)\le \max\{(\zeta,\varsigma):\zeta,\varsigma\in[0,1]\}=\phi(\zeta_0,\varsigma_0)=c_3(t)-\frac{c_2^3(t)}{2c_1(t)}.$$

Let the function $H:(0,t_0]\to \mathrm{R}$ defined as follows

$$H(t)=c_3(t)-\frac{c_2^3(t)}{2c_1(t)}. \tag{21}$$

Substituting the value $c_1(t)$, $c_2(t)$ and $c_3(t)$ and in (21), we write

$$H(t)=h_1(\alpha,\beta,\tau)t^3+2h_2(\alpha,\beta,\tau)t^2+h_3(\alpha,\beta,\tau),$$

where

$$h_1(\alpha,\beta,\tau)=h_2(\alpha,\beta,\tau)+\frac{(1-\alpha)|\tau|}{8(1+3\beta)},$$

$$h_2(\alpha,\beta,\tau)=\frac{5(1-\alpha)^2|\tau|^2}{24(1+\beta)(1+2\beta)t_0},h_3(\alpha,\beta,\tau)=\frac{(1-\alpha)|\tau|}{2(1+3\beta)}.$$

By simple computation, we have

$$H'(t)=[3h_1(\alpha,\beta,\tau)t+4h_2(\alpha,\beta,\tau)]t$$

Since $h_1(\alpha,\beta,\tau)>0$ and $h_2(\alpha,\beta,\tau)>0$ for each $\alpha\in(0,1),\beta\ge 0,|\tau|>0,$ then $H'(t)>0$. So, the function $H(t)$ is an increasing function on $(0,t_0]$.

Therefore,

$$\max\{H(t):t\in(0,t_0]\}=H(t_0)=h_1(\alpha,\beta,\tau)t_0^3+h_2(\alpha,\beta,\tau)t_0^2+h_3(\alpha,\beta,\tau) \tag{22}$$

Thus, in the case $t\in(0,2)$, we have

$$|a_4|\le h_1(\alpha,\beta,\tau)t_0^3+h_2(\alpha,\beta,\tau)t_0^2+h_3(\alpha,\beta,\tau). \tag{23}$$

It is clear that

$$[h_1(\alpha,\beta,\tau)t_0+2h_2(\alpha,\beta,\tau)]t_0^2+h_3(\alpha,\beta,\tau)>h_3(\alpha,\beta,\tau)=\frac{(1-\alpha)|\tau|}{2(1+3\beta)}.$$

Consequently, inequality (23) is satisfied for all $t\in[0,2]$.

Thus, from (12), (16), (17) and (23) the proof of Theorem 1 is completed.

In the special cases from Theorem 1, we arrive at the following results.

Corollary 1 *Let the function f given by (1) be in the class $\Im_\Sigma(\alpha,\beta,1)=H_\Sigma(\alpha,\beta)$. Then,*

$$|a_2|\le\frac{1-\alpha}{1+\beta},|a_3|\le\begin{cases}\dfrac{(1-\alpha)^2}{(1+\beta)^2} & \text{if } \alpha\in[0,\alpha_0],\\ \dfrac{2(1-\alpha)}{3(1+2\beta)} & \text{if } \alpha\in(\alpha_0,1),\end{cases}$$

where $\alpha_0=1-\dfrac{2(1+\beta)^2}{3(1+2\beta)}$.

Also,

$$|a_4| \le h_1(\alpha,\beta)t_0^3 + 2h_2(\alpha,\beta)t_0^2 + h_3(\alpha,\beta),$$

where

$$h_1(\alpha,\beta) = h_2(\alpha,\beta) + \frac{1-\alpha}{8(1+3\beta)}, h_2(\alpha,\beta) = \frac{5(1-\alpha)^2}{24(1+\beta)(1+2\beta)t_0},$$
$$h_3(\alpha,\beta) = \frac{1-\alpha}{2(1+3\beta)}, t_0 = \frac{6(1+\beta)(1+2\beta)}{5(1-\alpha)(1+3\beta)+6(1+\beta)(1+2\beta)}.$$

Corollary 2 *Let the function* f *given by (1 be in the class* $\Im_\Sigma(\alpha,0,\tau)$. *Then,*

$$|a_2| \le (1-\alpha)|\tau|, |a_3| \le \begin{cases} \frac{2(1-\alpha)|\tau|}{3} & \text{if } |\tau| \in (0,\tau_0), \\ (1-\alpha)^2|\tau|^2 & \text{if } |\tau| \ge \tau_0, \end{cases}$$

where $\tau_0 = \frac{2}{3(1-\alpha)}$.

Also,

$$|a_4| \le \frac{(1-\alpha)|\tau|}{2}\left\{1 + \frac{[5(1-\alpha)|\tau|+3][25(1-\alpha)^2|\tau|^2 + 60(1-\alpha)|\tau| + 18]}{[5(1-\alpha)|\tau|+6]^3}\right\}.$$

Corollary 3 *Let the function* f *given by (1) be in the class* $\Im_\Sigma(\alpha,0,1) = \Re_\Sigma(\alpha,0) = N_\Sigma(\alpha)$. *Then,*

$$|a_2| \le 1-\alpha, |a_3| \le \begin{cases} (1-\alpha)^2 & \text{if } \alpha \in \left[0,\frac{1}{3}\right], \\ \frac{2(1-\alpha)}{3} & \text{if } \alpha \in \left(\frac{1}{3},1\right), \end{cases}$$

and

$$|a_4| \le \frac{1-\alpha}{2}\left\{1 + \frac{(8-5\alpha)(25\alpha^2 - 110\alpha + 103)}{(11-5\alpha)^3}\right\}.$$

Corollary 4 *Let the function* f *given by (1) be in the class* $\Im_\Sigma(\alpha,1,\tau)$. *Then,*

$$|a_2| \le \frac{(1-\alpha)|\tau|}{2}, |a_3| \le \begin{cases} \frac{2(1-\alpha)|\tau|}{9} & \text{if } |\tau| \in (0,\tau_0), \\ \frac{(1-\alpha)^2|\tau|^2}{4} & \text{if } |\tau| \ge \tau_0, \end{cases}$$

where $\tau_0 = \frac{8}{9(1-\alpha)}$

Also,

$$|a_4| \le \frac{(1-\alpha)|\tau|}{8}\left\{1+\frac{[10(1-\alpha)|\tau|+9][50(1-\alpha)^2|\tau|^2+180(1-\alpha)|\tau|+81]}{4[5(1-\alpha)|\tau|+9]^3}\right\}.$$

Corollary 5 *Let the function* f *given by (1) be in the class* $\Im_\Sigma(\alpha,1,1)=\Re_\Sigma(\alpha,1)$.
Then,

$$|a_2| \le \frac{1-\alpha}{2}, |a_3| \le \begin{cases} \frac{(1-\alpha)^2}{4} & \text{if} \quad \alpha \in \left[0,\frac{1}{9}\right], \\ \frac{2(1-\alpha)}{9} & \text{if} \quad \alpha \in \left(\frac{1}{9},1\right), \end{cases}$$

and

$$|a_4| \le \frac{1-\alpha}{8}\left\{1+\frac{(19-10\alpha)(50\alpha^2-280\alpha+311)}{4(14-5\alpha)^3}\right\}.$$

3. Results and Discussion

In this paper, was introduced a new subclass $\Im_\Sigma(\alpha,\beta,\tau)$ of the analytic functions on the open unit disk in the complex plane. The various geometric properties of the functions belonging to these classes have examined. Also, sharp inequalities for the coefficient bound estimates for the functions belonging to these classes are given.

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