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Algebraic properties of $(\overline{\alpha}, \overline{\beta})$- interval valued fuzzy fantastic ideals in BRK-algebras

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Abstract In this paper, we define the concept of $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy fantastic ideals in BRK-algebra, where $\overline{\alpha}$, $\overline{\beta}$ are any one of $\overline{\in}$, $\overline{q}$, $\overline{\in} \vee \overline{q}$, $\overline{\in} \wedge \overline{q}$ and investigate some of their related properties.

Keywords fuzzy fantastic, BRK-algebras

1. Introduction

The notion of BRK-algebra was first introduced by Bandaru in [1]. The fuzzy sets, proposed by Zadeh [25] in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex or ill defined to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields, for example, artificial intelligence, computer science, control engineering, expert system, management science, operation research and many others. The concept was applied to the theory of groupoids and groups by Rosenfeld [22], where he introduced the fuzzy subgroup of a group.

A new type of fuzzy subgroup, which is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced by Bhakat and Das [4] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [21]. Murali [20] proposed the definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. It was found that the most viable generalization of Rosenfeld's fuzzy subgroup is $(\in, \in \vee q)$-fuzzy subgroup. Bhakat [2-3] initiated the concepts of $(\in \vee q)$-level subsets, $(\in, \in \vee q)$-fuzzy normal, quasi-normal and maximal subgroups. Many researchers utilized these concepts to generalize some concepts of algebra (see [6, 24, 29-32]). In [7], Davvaz studied $(\in, \in \vee q)$-fuzzy subnearrings and ideals. In [11-13], Jun introduced the concept of (α, β)-fuzzy subalgebras/ideals in BCK/BCI-algebras. The notion of (α, β)-fuzzy positive implicative ideal in BCK-algebras was initiated by Zulfiqar in [29]. In [14], Jun defined $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras. In [30], Zulfiqar introduced the concept of subimplicative (α, β)-fuzzy ideals in BCH-algebras. In [32], Zulfiqar and Shabir defined the concept of positive implicative $(\in, \in \vee q)$-fuzzy ideals $(\overline{\in} \vee \overline{q})$-fuzzy ideals, fuzzy ideals with thresholds) in BCK-algebras.

The theory of interval valued fuzzy sets was proposed forty years ago as a natural extension of fuzzy sets. Interval valued fuzzy set was introduced by Zadeh [26]. The theory was further enriched by many authors [8-9, 27-28, 33]. In [5], Biswas defined interval valued fuzzy subgroups of Rosenfeld's nature, and investigated some elementary properties. Jun, initiated the notion of interval valued fuzzy subalgebras/ideals in BCK-algebras [10]. In [15], Latha et al. introduced the idea of interval valued (α, β)-fuzzy subgroups. In [16], Ma et al.

defined the theory of interval valued $(\in, \in \vee q)$-fuzzy ideals of pseudo MV-algebras. In [17-18], Ma et al. studied $(\in, \in \vee q)$-interval valued fuzzy ideals in BCI-algebras. Mostafa et al. initiated the notion of interval valued fuzzy KU-ideals in KU-algebras [19]. In [23], Saeid defined the concept of interval valued fuzzy BG-algebras. Zhan et al. [28] initiated the notion of interval valued $(\in, \in \vee q)$-fuzzy filters of pseudo BL-algebras. Zulfiqar and Shabir introduced the notion of $(\overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideals in BCK-algebras in [33].

In the present paper, we define the concept of $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy fantastic ideals in BRK-algebra, where $\overline{\alpha}$, $\overline{\beta}$ are any one of $\overline{\in}$, $\overline{q}$, $\overline{\in} \vee \overline{q}$, $\overline{\in} \wedge \overline{q}$ and investigate some of their related properties.

2. Preliminaries

Throughout this paper X always denote a BRK-algebra without any specification. We also include some basic aspects that are necessary for this paper.

Definition 2.1. [1] A BRK-algebra X is a general algebra (X, $*$, 0) of type (2, 0) satisfying the following conditions:

(BRK-1) $l * 0 = l$

(BRK-2) $(l * m) * l = 0 * m$

for all $l, m \in$ X.

We can define a partial order "$\leq$" on X by $l \leq m$ if and only if $l * m = 0$.

Definition 2.2. [31] A non-empty subset I of a BRK-algebra X is called an ideal of X if it satisfies the conditions (I1) and (I2), where

(I1) $0 \in$ I,

(I2) $l * m \in$ I and $m \in$ I imply $l \in$ I,

for all $l, m \in$ X.

Definition 2.3. A non-empty subset I of a BRK-algebra X is called a fantastic ideal of X if it satisfies the conditions (I1) and (I3), where

(I1) $0 \in$ I,

(I3) $(l * m) * n \in$ I and $n \in$ I imply $l * (m * (m * l)) \in$ I,

for all $l, m, n \in$ X.

We now review some fuzzy logic concepts. Recall that the real unit interval [0, 1] with the totally ordered relation "$\leq$" is a complete lattice, with $\wedge$ = min and $\vee$ = max, 0 and 1 being the least element and the greatest element, respectively.

Definition 2.4. An interval valued fuzzy set $\tilde{\lambda}$ of a universe X is a function from X into the unit closed interval [0, 1], that is $\tilde{\lambda}$: X $\rightarrow$ H[0, 1], for each $l \in$ X

$$\tilde{\lambda}(l) = [\lambda^{-}(l), \lambda^{+}(l)] \in \text{H}[0, 1].$$

Definition 2.5. For an interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X and $[0, 0] < \tilde{t} \leq [1, 1]$, the crisp set

$$\tilde{\lambda}_{\tilde{t}} = \{l \in \text{X} \mid \tilde{\lambda}(l) \geq \tilde{u}\}$$

is called the level subset of $\tilde{\lambda}$.

Definition 2.6. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is called an interval valued fuzzy ideal of X if it satisfies the conditions (F1) and (F2), where

(F1) $\tilde{\lambda}(0) \geq \tilde{\lambda}(l)$,

(F2) $\tilde{\lambda}(l) \geq \tilde{\lambda}(l * m) \wedge \tilde{\lambda}(m)$,

for all $l, m \in$ X.

Definition 2.7. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is called an interval valued fuzzy fantastic ideal of X if it satisfies the conditions (F1) and (F3), where

(F1) $\tilde{\lambda}(0) \geq \tilde{\lambda}(l)$,

(F3) $\tilde{\lambda}(l * (m * (m * l))) \geq \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n)$,

for all $l, m, n \in$ X.

An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X having the form

$$\tilde{\lambda}(m) = \begin{cases} \tilde{u}\ (\neq [0, 0]) & if\ \ m = l \\ [0, 0] & if\ \ m \neq l \end{cases}$$

is said to be an interval valued fuzzy point with support l and value $\tilde{u}$ and is denoted by $l_{\tilde{u}}$. An interval valued fuzzy point $l_{\tilde{u}}$ is said to belong to (resp., quasi-coincident with) an interval valued fuzzy set $\tilde{\lambda}$, written as $l_{\tilde{u}} \in \tilde{\lambda}$ (resp. $l_{\tilde{u}}\ q\ \tilde{\lambda}$) if $\tilde{\lambda}(l) \geq \tilde{u}$ (resp. $\tilde{\lambda}(l) + \tilde{u} > [1, 1]$). By $l_{\tilde{u}} \in \vee\, q\ \tilde{\lambda}$ ($l_{\tilde{u}} \in \wedge\, q\ \tilde{\lambda}$) we mean that $l_{\tilde{u}} \in \tilde{\lambda}$ or $l_{\tilde{u}}\ q\ \tilde{\lambda}$ ($l_{\tilde{u}} \in \tilde{\lambda}$ and $l_{\tilde{u}}\ q\ \tilde{\lambda}$).

In what follows let α and β denote any one of $\in$, q, $\in \vee q$, $\in \wedge q$ and $\alpha \neq \in \wedge q$ unless otherwise specified. To say that $l_{\tilde{u}}\ \overline{\alpha}\ \tilde{\lambda}$ means that $l_{\tilde{u}}\ \alpha\ \tilde{\lambda}$ does not hold.

3. $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy fantastic ideals

In this section, we define the concept of $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy fantastic ideal in a BRK-algebra and investigate some of their properties. Throughout this paper X will denote a BRK-algebra and $\overline{\alpha}$, $\overline{\beta}$ are any one of $\overline{\in}$, $\overline{q}$, $\overline{\in} \vee \overline{q}$, $\overline{\in} \wedge \overline{q}$ unless otherwise specified.

Definition 3.1. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is called an $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy subalgebra of X, where $\overline{\alpha} \neq \overline{\in} \wedge \overline{q}$, if it satisfies the condition

$$(l * m)_{\tilde{u}_1 \wedge \tilde{u}_2}\ \overline{\alpha}\ \tilde{\lambda} \Rightarrow l_{\tilde{u}_1}\ \overline{\beta}\ \tilde{\lambda} \text{ or } m_{\tilde{u}_2}\ \overline{\beta}\ \tilde{\lambda},$$

for all $[0, 0] < \tilde{u}_1, \tilde{u}_2 \leq [1, 1]$ and $l, m \in$ X.

Let $\tilde{\lambda}$ be an interval valued fuzzy set of a BRK-algebra X such that $\tilde{\lambda}(l) \geq [0.5, 0.5]$ for all $l \in$ X. Let $l \in$ X and $[0, 0] < \tilde{u} \leq [1, 1]$ be such that

$$l_{\tilde{u}}\ \overline{\in} \wedge \overline{q}\ \tilde{\lambda}.$$

Then

$$\tilde{\lambda}(l) < \tilde{u} \text{ and } \tilde{\lambda}(l) + \tilde{u} \leq [1, 1].$$

It follows that

$$2\tilde{\lambda}(l) = \tilde{\lambda}(l) + \tilde{\lambda}(l) < \tilde{\lambda}(l) + \tilde{u} \leq [1, 1].$$

This implies that $\tilde{\lambda}(l) < [0.5, 0.5]$. This means that

$$\{ l_{\tilde{u}} \mid l_{\tilde{u}} \ \overline{\in} \wedge \overline{q} \ \tilde{\lambda} \} = \phi .$$

Therefore, the case $\overline{\alpha} = \overline{\in} \wedge \overline{q}$ in the above definition is omitted.

Definition 3.2. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is called an $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy ideal of X, where $\overline{\alpha} \neq \overline{\in} \wedge \overline{q}$, if it satisfies the conditions (A) and (B), where

(A) $0_{\tilde{u}} \ \overline{\alpha} \ \tilde{\lambda} \Rightarrow l_{\tilde{u}} \ \overline{\beta} \ \tilde{\lambda}$,

(B) $l_{\tilde{u}_1 \wedge \tilde{u}_2} \ \overline{\alpha} \ \tilde{\lambda} \Rightarrow (l * m)_{\tilde{u}_1} \ \overline{\beta} \ \tilde{\lambda}$ or $m_{\tilde{u}_2} \ \overline{\beta} \ \tilde{\lambda}$,

for all $[0, 0] < \tilde{u}, \tilde{u}_1, \tilde{u}_2 \leq [1, 1]$ and $l, m \in$ X.

Definition 3.3. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is called an $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy fantastic ideal of X, where $\overline{\alpha} \neq \overline{\in} \wedge \overline{q}$, if it satisfies the conditions (A) and (C), where

(A) $0_{\tilde{u}} \ \overline{\alpha} \ \tilde{\lambda} \Rightarrow l_{\tilde{u}} \ \overline{\beta} \ \tilde{\lambda}$,

(C) $(l * (m * (m * l)))_{\tilde{u}_{11} \wedge \tilde{u}_2} \ \overline{\alpha} \ \tilde{\lambda} \Rightarrow ((l * m) * n)_{\tilde{u}_1} \ \overline{\beta} \ \tilde{\lambda}$ or $n_{\tilde{u}_2} \ \overline{\beta} \ \tilde{\lambda}$,

for all $[0, 0] < \tilde{u}, \tilde{u}_1, \tilde{u}_2 \leq [1, 1]$ and $l, m, n \in$ X.

Theorem 3.4. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is an interval valued fuzzy fantastic ideal of X if and only if $\tilde{\lambda}$ is an $(\overline{\in}, \overline{\in})$-interval valued fuzzy fantastic ideal of X.

Proof. Suppose $\tilde{\lambda}$ is an interval valued fuzzy fantastic ideal of X. Let $0_{\tilde{u}} \ \overline{\in} \ \tilde{\lambda}$ for $[0, 0] < \tilde{u} \leq [1, 1]$. Then $\tilde{\lambda}(0) < \tilde{u}$. By (F1), we have

$$\tilde{u} > \tilde{\lambda}(0) \geq \tilde{\lambda}(l),$$

this implies that

$$\tilde{u} > \tilde{\lambda}(l),$$

that is,

$$l_{\tilde{u}} \ \overline{\in} \ \tilde{\lambda} .$$

Let $l, m \in$ X and $[0, 0] < \tilde{u}, \tilde{v} \leq [1, 1]$ be such that

$$(l * (m * (m * l)))_{\tilde{u} \wedge \tilde{v}} \ \overline{\in} \ \tilde{\lambda} .$$

Then

$$\tilde{\lambda}(l * (m * (m * l))) < \tilde{u} \wedge \tilde{v} .$$

Since $\tilde{\lambda}$ is an interval valued fuzzy fantastic ideal of X. So

$$\tilde{u} \wedge \tilde{v} > \tilde{\lambda}(l * (m * (m * l))) \geq \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n).$$

This implies that

$$\tilde{u} > \tilde{\lambda}((l * m) * n) \ \text{ or } \ \tilde{v} > \tilde{\lambda}(n),$$

that is,

$$((l * m) * n)_{\tilde{u}} \ \overline{\in} \ \tilde{\lambda} \ \text{ or } \ n_{\tilde{v}} \ \overline{\in} \ \tilde{\lambda} .$$

This shows that $\tilde{\lambda}$ is an $(\overline{\in}, \overline{\in})$-interval valued fuzzy fantastic ideal of X.

Conversely, assume that $\widetilde{\lambda}$ is an $(\overline{\in}, \overline{\in})$-interval valued fuzzy fantastic ideal of X. To show $\widetilde{\lambda}$ is an interval valued fuzzy fantastic ideal of X. Suppose there exists $l \in$ X such that

$$\widetilde{\lambda}(0) < \widetilde{\lambda}(l).$$

Select $[0, 0] < \widetilde{u} \leq [1, 1]$ such that

$$\widetilde{\lambda}(0) < \widetilde{u} \leq \widetilde{\lambda}(l).$$

Then $0_{\widetilde{u}} \overline{\in} \widetilde{\lambda}$ but $l_{\widetilde{u}} \in \widetilde{\lambda}$, which is a contradiction. Hence

$$\widetilde{\lambda}(0) \geq \widetilde{\lambda}(l), \text{ for all } l \in X.$$

Now suppose there exist $l, m, n \in$ X such that

$$\widetilde{\lambda}(l * (m * (m * l))) < \widetilde{\lambda}((l * m) * n) \wedge \widetilde{\lambda}(n).$$

Select $[0, 0] < \widetilde{u} \leq [1, 1]$ such that

$$\widetilde{\lambda}(l * (m * (m * l))) < \widetilde{u} \leq \widetilde{\lambda}((l * m) * n) \wedge \widetilde{\lambda}(n).$$

Then $(l * (m * (m * l)))_{\widetilde{u}} \overline{\in} \widetilde{\lambda}$ but $((l * m) * n)_{\widetilde{u}} \in \widetilde{\lambda}$ and $n_{\widetilde{u}} \in \widetilde{\lambda}$, which is a contradiction.
Hence

$$\widetilde{\lambda}(l * (m * (m * l))) \geq \widetilde{\lambda}((l * m) * n) \wedge \widetilde{\lambda}(n).$$

This shows that $\widetilde{\lambda}$ is an interval valued fuzzy fantastic ideal of X.

4. $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideals

In this section, we define the notion of $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantatstic ideal in a BRK-algebra and investigate some of their related properties.

Definition 4.1. Let $\widetilde{\lambda}$ be an interval valued fuzzy set of a BRK-algebra X. Then $\widetilde{\lambda}$ is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X if it satisfies the conditions (D) and (E), where

(D) $0_{\widetilde{u}} \overline{\in} \widetilde{\lambda} \Rightarrow l_{\widetilde{u}} \overline{\in} \vee \overline{q} \widetilde{\lambda}$,

(E) $(l * (m * (m * l)))_{\widetilde{u} \wedge \widetilde{v}} \overline{\in} \widetilde{\lambda} \Rightarrow ((l * m) * n)_{\widetilde{u}} \overline{\in} \vee \overline{q} \widetilde{\lambda}$ or $n_{\widetilde{v}} \overline{\in} \vee \overline{q} \widetilde{\lambda}$,

for all $l, m, n \in$ X and $[0, 0] < \widetilde{u}, \widetilde{v} \leq [1, 1]$.

Theorem 4.2. Let $\widetilde{\lambda}$ be an interval valued fuzzy set of a BRK-algebra X. Then the conditions (D) equivalent to (F) and (E) equivalent to (G), where

(F) $\widetilde{\lambda}(0) \vee [0.5, 0.5] \geq \widetilde{\lambda}(l)$,

(G) $\widetilde{\lambda}(l * (m * (m * l))) \vee [0.5, 0.5] \geq \widetilde{\lambda}((l * m) * n) \wedge \widetilde{\lambda}(n)$,

for all $l, m, n \in$ X.

Proof. (D) $\Rightarrow$ (F)

Let $l \in$ X be such that

$$\widetilde{\lambda}(l) > \widetilde{\lambda}(0) \vee [0.5, 0.5].$$

Select $\widetilde{u}$ such that

$$\widetilde{\lambda}(l) \geq \widetilde{u} > \widetilde{\lambda}(0) \vee [0.5, 0.5].$$

Then

$$0_{\widetilde{u}} \overline{\in} \widetilde{\lambda}.$$

But

$$\tilde{\lambda}(l) \geq \tilde{u} \text{ and } \tilde{\lambda}(l) + \tilde{u} > [1, 1],$$

that is,

$$l_{\tilde{u}} \in \tilde{\lambda} \text{ and } l_{\tilde{u}}\, q\, \tilde{\lambda},$$

which is a contradiction.
Hence

$$\tilde{\lambda}(0) \vee [0.5, 0.5] \geq \tilde{\lambda}(l).$$

(F) $\Rightarrow$ (D)

Let $0_{\tilde{u}} \,\overline{\in}\, \tilde{\lambda}$. Then $\tilde{\lambda}(0) < \tilde{u}$.

If $\tilde{\lambda}(0) \geq [0.5, 0.5]$, then by condition (F), we have

$$\tilde{u} > \tilde{\lambda}(0) \geq \tilde{\lambda}(l)$$

and so

$$\tilde{\lambda}(l) < \tilde{u},$$

that is,

$$l_{\tilde{u}} \,\overline{\in}\, \tilde{\lambda}.$$

If $\tilde{\lambda}(0) < [0.5, 0.5]$, then by condition (F), we have

$$[0.5, 0.5] \geq \tilde{\lambda}(l).$$

Suppose $l_{\tilde{u}} \in \tilde{\lambda}$. Then $\tilde{\lambda}(l) \geq \tilde{u}$. Thus $[0.5, 0.5] \geq \tilde{u}$.
Hence

$$\tilde{\lambda}(l) + \tilde{u} \leq [0.5, 0.5] + [0.5, 0.5] = [1, 1]$$

that is,

$$l_{\tilde{u}}\, \overline{q}\, \tilde{\lambda}.$$

This implies that

$$l_{\tilde{u}} \,\overline{\in} \vee \overline{q}\, \tilde{\lambda}.$$

(E) $\Rightarrow$ (G)
Suppose there exist $l, m, n \in$ X such that

$$\tilde{\lambda}(l * (m * (m * l))) \vee [0.5, 0.5] < \tilde{u} = \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n).$$

Then

$$[0, 0] < \tilde{u} \leq [1, 1],\ (l*(m*(m*l)))_{\tilde{u}} \,\overline{\in}\, \tilde{\lambda} \text{ and } ((l*m)*n)_{\tilde{u}} \in \tilde{\lambda},\ n_{\tilde{u}} \in \tilde{\lambda}.$$

It follows that

$$((l*m)*n)_{\tilde{u}}\, \overline{q}\, \tilde{\lambda} \text{ or } n_{\tilde{u}}\, \overline{q}\, \tilde{\lambda}.$$

Then

$$\tilde{\lambda}((l * m) * n) + \tilde{u} \leq [1, 1]$$

or

$$\tilde{\lambda}(n) + \tilde{u} \leq [1, 1].$$

As $\tilde{u} \leq \tilde{\lambda}((l * m) * n)$ and $\tilde{u} \leq \tilde{\lambda}(n)$, it follows that

$$\tilde{u} \leq [0.5, 0.5].$$

This is a contradiction. So

$$\tilde{\lambda}(l * (m * (m * l))) \vee [0.5, 0.5] \geq \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(l).$$

(G) $\Rightarrow$ (E)

Let $l, m, n \in$ X and $[0, 0] < \tilde{u}, \tilde{v} \leq [1, 1]$ be such that

$$(l*(m*(m*l)))_{\tilde{u}\wedge\tilde{v}} \,\overline{\in}\, \tilde{\lambda}.$$

Then

$$\tilde{\lambda}(l * (m * (m * l))) < \tilde{u} \wedge \tilde{v}.$$

(*a*) If $\tilde{\lambda}(l * (m * (m * l))) \geq [0.5, 0.5]$, then by condition (G), we have

$$\tilde{\lambda}(l * (m * (m * l))) \geq \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n).$$

Thus

$$\tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n) < \tilde{u} \wedge \tilde{v},$$

and consequently

$$\tilde{\lambda}((l * m) * n) < \tilde{u} \text{ or } \tilde{\lambda}(n) < \tilde{v}.$$

It follows that

$$((l*m)*n)_{\tilde{u}} \,\overline{\in}\, \tilde{\lambda} \text{ or } n_{\tilde{v}} \,\overline{\in}\, \tilde{\lambda},$$

and hence

$$((l*m)*n)_{\tilde{u}} \,\overline{\in} \vee \overline{q}\, \tilde{\lambda} \text{ or } n_{\tilde{v}} \,\overline{\in} \vee \overline{q}\, \tilde{\lambda}.$$

(b) If $\tilde{\lambda}(l * (m * (m * l))) < [0.5, 0.5]$, then by condition (G), we have

$$[0.5, 0.5] \geq \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n).$$

Suppose $((l*m)*n)_{\tilde{u}} \in \tilde{\lambda}$ and $n_{\tilde{v}} \in \tilde{\lambda}$. Then

$$\tilde{\lambda}((l * m) * n) \geq \tilde{u} \text{ and } \tilde{\lambda}(n) \geq \tilde{v}.$$

Thus

$$[0.5, 0.5] \geq \tilde{u} \wedge \tilde{v}.$$

Hence

$$\tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n) + \tilde{u} \wedge \tilde{v} \leq [0.5, 0.5] + [0.5, 0.5] = [1, 1]$$

that is

$$((l*m)*n)_{\tilde{u}} \,\overline{q}\, \tilde{\lambda} \text{ or } n_{\tilde{v}} \,\overline{q}\, \tilde{\lambda}.$$

This implies that

$$((l*m)*n)_{\tilde{u}} \,\overline{\in} \vee \overline{q}\, \tilde{\lambda} \text{ or } n_{\tilde{v}} \,\overline{\in} \vee \overline{q}\, \tilde{\lambda}.$$

Corollary 4.3. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X if and only if it satisfies the conditions (F) and (G).

Example 4.4. Let X = {0, *a*, *b*} be a BRK-algebra with the following Cayley table:

$*$	0	a	b
0	0	b	b
a	a	0	0
b	b	0	0

We define an interval valued fuzzy set $\tilde{\lambda}$ by $\tilde{\lambda}$ (0) = [0.4, 0.4], $\tilde{\lambda}$ (a) = [0.2, 0.2] and $\tilde{\lambda}$ (b) = [0.3, 0.3]. Simple calculations show that $\tilde{\lambda}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X.

Example 4.5. Let X = {0, 1, 2, 3} be a BRK-algebra with the following Cayley table:

∗	0	1	2	3
0	0	1	0	1
1	1	0	1	0
2	2	1	0	1
3	3	2	3	0

We define an interval valued fuzzy set $\tilde{\lambda}$ by $\tilde{\lambda}$ (0) = [1, 1], $\tilde{\lambda}$ (1) = [0.4, 0.4], $\tilde{\lambda}$ (2) = [0.3, 0.3] and $\tilde{\lambda}$ (3) = [0.7, 0.7]. Simple calculations show that $\tilde{\lambda}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy ideal of X but not an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X.
Putting l = 3, m = 0 and n = 0 in (G), we get

$$\tilde{\lambda}(l * (m * (m * l))) \vee [0.5, 0.5] \geq \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n)$$
$$\tilde{\lambda}(3 * (0 * (0 * 3))) \vee [0.5, 0.5] \geq \tilde{\lambda}((3 * 0) * 0) \wedge \tilde{\lambda}(0)$$
$$\tilde{\lambda}(3 * (0 * 1) \vee [0.5, 0.5] \geq \tilde{\lambda}(3 * 0) \wedge \tilde{\lambda}(0)$$
$$\tilde{\lambda}(3 * 1) \vee [0.5, 0.5] \geq \tilde{\lambda}(3) \wedge \tilde{\lambda}(0)$$
$$\tilde{\lambda}(2) \vee [0.5, 0.5] \geq \tilde{\lambda}(3) \wedge \tilde{\lambda}(0)$$
$$[0.3, 0.3] \vee [0.5, 0.5] \geq [0.7, 0.7] \wedge [1, 1]$$
$$[0.5, 0.5] \ngeq [0.7, 0.7].$$

Theorem 4.6. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X if and only if for any [0, 0] < $\tilde{u}$ ≤ [1, 1], $\tilde{\lambda}_{\tilde{u}}$ = {l ∈ X | $\tilde{\lambda}$ (l) ≥ $\tilde{u}$ } is a fantastic ideal of X.

Proof. Let $\tilde{\lambda}$ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X and [0.5, 0.5] < $\tilde{u}$ ≤ [1, 1]. If $\tilde{\lambda}_{\tilde{u}} \neq \phi$, then $l \in \tilde{\lambda}_{\tilde{u}}$. This implies that $\tilde{\lambda}$ (l) ≥ $\tilde{u}$. By condition (F), we have

$$\tilde{\lambda}(0) \vee [0.5, 0.5] \geq \tilde{\lambda}(l) \geq \tilde{u}.$$

Thus $\tilde{\lambda}$ (0) ≥ $\tilde{u}$. Hence $0 \in \tilde{\lambda}_{\tilde{u}}$.

Let $(l * m) * n \in \tilde{\lambda}_{\tilde{u}}$ and $n \in \tilde{\lambda}_{\tilde{u}}$. Then

$$\tilde{\lambda}((l * m) * n) \geq \tilde{u} \quad \text{and} \quad \tilde{\lambda}(n) \geq \tilde{u}.$$

By condition (G), we have

$$\tilde{\lambda}(l * (m * (m * l))) \vee [0.5, 0.5] \geq \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n)$$
$$\geq \tilde{u} \wedge \tilde{u}$$
$$= \tilde{u}.$$

Thus $\tilde{\lambda}$ (l ∗ (m ∗ (m ∗ l))) ≥ $\tilde{u}$, that is $l * (m * (m * l)) \in \tilde{\lambda}_{\tilde{u}}$. Therefore $\tilde{\lambda}_{\tilde{u}}$ is a fantastic ideal of X.

Conversely, assume that $\tilde{\lambda}$ is an interval valued fuzzy set of X such that $\tilde{\lambda}_{\tilde{u}}$ ($\neq \phi$) is a fantastic ideal of X for all [0.5, 0.5] < $\tilde{u}$ ≤ [1, 1]. Let l ∈ X be such that

$$\tilde{\lambda}(0) \vee [0.5, 0.5] < \tilde{\lambda}(l).$$

Select [0.5, 0.5] < $\tilde{u}$ ≤ [1, 1] such that

$$\tilde{\lambda}(0) \vee [0.5, 0.5] < \tilde{u} \leq \tilde{\lambda}(l).$$

Then $l \in \tilde{\lambda}_{\tilde{u}}$ but $0 \notin \tilde{\lambda}_{\tilde{u}}$, a contradiction. Hence

$$\tilde{\lambda}(0) \vee [0.5, 0.5] \geq \tilde{\lambda}(l).$$

Now assume that $l, m, n \in$ X such that

$$\tilde{\lambda}(l * (m * (m * l))) \vee [0.5, 0.5] < \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n).$$

Select [0.5, 0.5] < $\tilde{u}$ ≤ [1, 1] such that

$$\tilde{\lambda}(l * (m * (m * l))) \vee [0.5, 0.5] < \tilde{u} \leq \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n).$$

Then $(l * m) * n$ and n are in $\tilde{\lambda}_{\tilde{u}}$ but $l * (m * (m * l)) \notin \tilde{\lambda}_{\tilde{u}}$, a contradiction. Hence

$$\tilde{\lambda}(l * (m * (m * l))) \vee [0.5, 0.5] \geq \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n).$$

This shows that $\tilde{\lambda}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X.

Corollary 4.7. Every interval valued fuzzy fantastic ideal of a BRK-algebra X is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X.

Theorem 4.8. Let I be a non-empty subset of a BRK-algebra X. Then I is a fantastic ideal of X if and only if the interval valued fuzzy set $\tilde{\lambda}$ of X defined by

$$\tilde{\lambda}(l) = \begin{cases} \leq [0.5, 0.5] & if\ \ l \in X - I \\ [1, 1] & if\ \ l \in I, \end{cases}$$

is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X.

Proof. Let I be a fantastic ideal of X. Then $0 \in$ I. This implies that $\tilde{\lambda}(0)$ = [1, 1]. Thus

$$\tilde{\lambda}(0) \vee [0.5, 0.5] = [1, 1] \geq \tilde{\lambda}(l).$$

It means that $\tilde{\lambda}$ satisfies the condition (F).

Now let $l, m, n \in$ X. If $(l * m) * n$ and n are in I, then $l * (m * (m * l)) \in$ I. This implies that

$$\tilde{\lambda}(l * (m * (m * l))) \vee [0.5, 0.5] = [1, 1] = \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n).$$

If one of $(l * m) * n$ and n is not in I, then

$$\tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n) \leq [0.5, 0.5] \leq \tilde{\lambda}(l * (m * (m * l))) \vee [0.5, 0.5].$$

Thus $\tilde{\lambda}$ satisfies the condition (G). Hence $\tilde{\lambda}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X.

Conversely, assume that $\tilde{\lambda}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X. Let $l \in$ I. Then by condition (F), we have

$$\tilde{\lambda}(0) \vee [0.5, 0.5] \geq \tilde{\lambda}(l) = [1, 1].$$

This implies that $0 \in$ I. Let $l, m, n \in$ X be such that $(l * m) * n$ and n are in I. Then by condition (G), we have

$$\tilde{\lambda}(l * (m * (m * l))) \vee [0.5, 0.5] \geq \tilde{\lambda}((l * m) * n) \wedge \tilde{\lambda}(n) = [1, 1].$$

This implies that

$$\tilde{\lambda}(l * (m * (m * l))) = [1, 1],$$

that is

$$l * (m * (m * l)) \in \text{I}.$$

Hence I is a fantastic ideal of X.

Theorem 4.9. Let I be a non-empty subset of a BRK-algebra X. Then I is a fantastic ideal of X if and only if the interval valued fuzzy set $\tilde{\lambda}$ of X defined by

$$\tilde{\lambda}(l) = \begin{cases} \leq [0.5,\ 0.5] & if\ \ l \in X - I \\ [1,\ 1] & if\ \ l \in I, \end{cases}$$

is a $(\overline{q},\ \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X.

Proof. Let I be a fantastic ideal of X. Let $[0, 0] < \tilde{u} \leq [1, 1]$ be such that $0_{\tilde{u}}\ \overline{q}\ \tilde{\lambda}$. Then

$$\tilde{\lambda}(0) + \tilde{u} \leq [1, 1],$$

so $0 \notin$ I. This implies that I = ϕ. Thus, if $\tilde{u} > [0.5, 0.5]$, then

$$\tilde{\lambda}(l) \leq [0.5, 0.5] < \tilde{u},$$

so $l_{\tilde{u}}\ \overline{\in}\ \tilde{\lambda}$. If $\tilde{u} \leq [0.5, 0.5]$, then

$$\tilde{\lambda}(l) + \tilde{u} \leq [0.5, 0.5] + [0.5, 0.5] = [1, 1].$$

This implies that $l_{\tilde{u}}\ \overline{q}\ \tilde{\lambda}$. Hence

$$l_{\tilde{u}}\ \overline{\in} \vee \overline{q}\ \tilde{\lambda}.$$

Now let $l, m, n \in$ X and $[0, 0] < \tilde{u}, \tilde{v} \leq [1, 1]$ be such that

$$(l*(m*(m*l)))_{\tilde{u}\wedge\tilde{v}}\ \overline{q}\ \tilde{\lambda}.$$

Then

$$\tilde{\lambda}(l*(m*(m*l))) + \tilde{u} \wedge \tilde{v} \leq [1, 1],$$

so $l*(m*(m*l)) \notin$ I. This implies that either

$$(l*m)*n \notin \text{ I or } n \notin \text{ I}.$$

Suppose $(l*m)*n \notin$ I. Thus, if $\tilde{u} > [0.5, 0.5]$, then

$$\tilde{\lambda}((l*m)*n) \leq [0.5, 0.5] < \tilde{u}$$

and so $\tilde{\lambda}((l*m)*n) \leq \tilde{u}$. This implies that

$$((l*m)*n)_{\tilde{u}}\ \overline{\in}\ \tilde{\lambda}.$$

If $\tilde{u} < [0.5, 0.5]$ and $((l*m)*n)_{\tilde{u}} \in \tilde{\lambda}$, then

$$\tilde{\lambda}((l*m)*n) \geq \tilde{u}.$$

As

$$[0.5, 0.5] \geq \tilde{\lambda}((l*m)*n),$$

so $[0.5, 0.5] \geq \tilde{u}$. Thus

$$\tilde{\lambda}((l*m)*n) + \tilde{u} \leq [0.5, 0.5] + [0.5, 0.5] = [1, 1],$$

that is

$$((l*m)*n)_{\tilde{u}}\ \overline{q}\ \tilde{\lambda}.$$

Hence

$$((l*m)*n)_{\tilde{u}}\ \overline{\in} \vee \overline{q}\ \tilde{\lambda}.$$

Similarly, if $n \notin$ I, then

$$n_{\tilde{v}}\ \overline{\in} \vee \overline{q}\ \tilde{\lambda}.$$

This shows that $\tilde{\lambda}$ is a $(\overline{q},\ \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X.

Conversely, assume that $\tilde{\lambda}$ is a $(\overline{q},\ \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X. Let $l \in$ I. If $0 \notin$ I, then $\tilde{\lambda}(0) \leq [0.5, 0.5]$. Now for any $[0, 0] < \tilde{u} \leq [0.5, 0.5]$.

$$\tilde{\lambda}(0) + \tilde{u} \leq [0.5, 0.5] + [0.5, 0.5] = [1, 1],$$

this implies that $0_{\tilde{u}}\ \overline{q}\ \tilde{\lambda}$. Thus

$$l_{\tilde{u}}\ \overline{\in} \vee \overline{q}\ \tilde{\lambda}.$$

But

$$\tilde{\lambda}(l) = [1, 1] > \tilde{u} \text{ and } \tilde{\lambda}(l) + \tilde{u} > [1, 1]$$

implies that

$$l_{\tilde{u}} \in \wedge\, q\, \tilde{\lambda}.$$

This is a contradiction. Hence $0 \in$ I.
Now suppose $l, m, n \in$ X such that $(l * m) * n$ and $n \in$ I. We have to show that $l * (m * (m * l)) \in$ I. On contrary assume that $l * (m * (m * l)) \notin$ I. Then

$$\tilde{\lambda}(l * (m * (m * l))) \leq [0.5, 0.5].$$

Now for $[0, 0] < \tilde{u} \leq [1, 1]$

$$\tilde{\lambda}(l * (m * (m * l))) + \tilde{u} \leq [0.5, 0.5] + [0.5, 0.5] = [1, 1],$$

this is

$$(l*(m*(m*l)))_{\tilde{u}}\ \overline{q}\ \tilde{\lambda}.$$

Thus

$$((l*m)*n)_{\tilde{u}}\ \overline{\in} \vee \overline{q}\ \tilde{\lambda} \text{ or } n_{\tilde{u}}\ \overline{\in} \vee \overline{q}\ \tilde{\lambda}.$$

But $(l * m) * n$ and $n \in$ I implies

$$\tilde{\lambda}((l * m) * n) = \tilde{\lambda}(n) = [1, 1].$$

This implies that

$$((l*m)*n)_{\tilde{u}} \in \wedge\, q\, \tilde{\lambda} \text{ and } n_{\tilde{u}} \in \wedge\, q\, \tilde{\lambda},$$

which is a contradiction. Hence $l * (m * (m * l)) \in$ I.

Theorem 4.10. Let I be a non-empty subset of a BRK-algebra X. Then I is a fantastic ideal of X if and only if the interval valued fuzzy set $\tilde{\lambda}$ of X defined by

$$\tilde{\lambda}(l) = \begin{cases} \leq [0.5, 0.5] & \text{if } l \in X - I \\ [1, 1] & \text{if } l \in I, \end{cases}$$

is an $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X.
Proof. The proof follows from the proof of Theorem 4.8 and Theorem 4.9.

Theorem 4.11. The intersection of any family of $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideals of a BRK-algebra X is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X.

Proof. Let $\{\tilde{\lambda}_i\}_{i \in I}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideals of a BRK-algebra X and $l \in$ X. So

$$\tilde{\lambda}_i(0) \vee [0.5, 0.5] \geq \tilde{\lambda}_i(l)$$

for all i $\in$ I. Thus

$$(\wedge_{i \in I} \tilde{\lambda}_i)(0) \vee [0.5, 0.5] = \wedge_{i \in I} (\tilde{\lambda}_i(0) \vee [0.5, 0.5])$$

$$\geq \wedge_{i \in I} (\tilde{\lambda}_i(l))$$

$$= (\wedge_{i \in I} \tilde{\lambda}_i)(l).$$

Thus

$$(\underset{i\in I}{\wedge}\tilde{\lambda}_i)(0)\vee[0.5, 0.5]\geq(\underset{i\in I}{\wedge}\tilde{\lambda}_i)(l).$$

Let $l, m, n \in$ X. Since each $\tilde{\lambda}_i$ is an $(\overline{\in}, \overline{\in}\vee\overline{q})$-interval valued fuzzy fantastic ideal of X. So

$$\tilde{\lambda}_i(l*(m*(m*l)))\vee[0.5, 0.5]\geq\tilde{\lambda}_i((l*m)*n)\wedge\tilde{\lambda}_i(n)$$

for all i $\in$ I. Thus

$$(\underset{i\in I}{\wedge}\tilde{\lambda}_i)(l*(m*(m*l)))\vee[0.5, 0.5]=\underset{i\in I}{\wedge}(\tilde{\lambda}_i(l*(m*(m*l)))\vee[0.5, 0.5])$$

$$\geq\underset{i\in I}{\wedge}(\tilde{\lambda}_i((l*m)*n)\wedge\tilde{\lambda}_i(n))$$

$$=(\underset{i\in I}{\wedge}\tilde{\lambda}_i)((l*m)*n)\wedge(\underset{i\in I}{\wedge}\tilde{\lambda}_i)(n)$$

Thus

$$(\underset{i\in I}{\wedge}\tilde{\lambda}_i)(l*(m*(m*l)))\vee[0.5, 0.5]\geq(\underset{i\in I}{\wedge}\tilde{\lambda}_i)((l*m)*n)\wedge(\underset{i\in I}{\wedge}\tilde{\lambda}_i)(n).$$

Hence, $\underset{i\in I}{\wedge}\tilde{\lambda}_i$ is an $(\in, \in\vee q)$-interval valued fuzzy fantastic ideal of X.

Theorem 4.12. The union of any family of $(\overline{\in}, \overline{\in}\vee\overline{q})$-interval valued fuzzy fantastic ideals of a BRK-algebra X is an $(\overline{\in}, \overline{\in}\vee\overline{q})$-interval valued fuzzy fantastic ideal of X.

Proof. Let $\{\tilde{\lambda}_i\}_{i\in I}$ be a family of $(\overline{\in}, \overline{\in}\vee\overline{q})$-interval valued fuzzy fantastic ideals of a BRK-algebra X and $l \in$ X. So

$$\tilde{\lambda}_i(0)\vee[0.5, 0.5]\geq\tilde{\lambda}_i(l)$$

for all i $\in$ I. Thus

$$(\underset{i\in I}{\vee}\tilde{\lambda}_i)(0)\vee[0.5, 0.5]=\underset{i\in I}{\vee}(\tilde{\lambda}_i(0)\vee[0.5, 0.5])$$

$$\geq\underset{i\in I}{\vee}(\tilde{\lambda}_i(l))$$

$$=(\underset{i\in I}{\vee}\tilde{\lambda}_i)(l).$$

Thus

$$(\underset{i\in I}{\vee}\tilde{\lambda}_i)(0)\vee[0.5, 0.5]\geq(\underset{i\in I}{\vee}\tilde{\lambda}_i)(l).$$

Let $l, m, n \in$ X. Since each $\tilde{\lambda}_i$ is an $(\overline{\in}, \overline{\in}\vee\overline{q})$-interval valued fuzzy fantastic ideal of X. So

$$\tilde{\lambda}_i(l*(m*(m*l)))\vee[0.5, 0.5]\geq\tilde{\lambda}_i((l*m)*n)\wedge\tilde{\lambda}_i(n)$$

for all i $\in$ I. Thus

$$(\underset{i\in I}{\vee}\tilde{\lambda}_i)(l*(m*(m*l)))\vee[0.5, 0.5]=\underset{i\in I}{\vee}(\tilde{\lambda}_i(l*(m*(m*l)))\vee[0.5, 0.5])$$

$$\geq\underset{i\in I}{\vee}(\tilde{\lambda}_i((l*m)*n)\wedge\tilde{\lambda}_i(n))$$

$$=(\underset{i\in I}{\vee}\tilde{\lambda}_i)((l*m)*n)\wedge(\underset{i\in I}{\vee}\tilde{\lambda}_i)(n)$$

Thus

$$(\underset{i\in I}{\vee}\tilde{\lambda}_i)(l*(m*(m*l)))\vee[0.5, 0.5]\geq(\underset{i\in I}{\vee}\tilde{\lambda}_i)((l*m)*n)\wedge(\underset{i\in I}{\vee}\tilde{\lambda}_i)(n).$$

Hence, $\underset{i \in I}{\vee} \tilde{\lambda}_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy fantastic ideal of X.

5. Conclusion

To investigate the structure of an algebraic system, we see that the interval valued fuzzy fantastic ideal with special properties always play an important role.

The purpose of this paper is to initiated the concept of $(\overline{\alpha}, \overline{\beta})$- interval valued fuzzy fantastic ideals in BRK-algebra, where $\overline{\alpha}$, $\overline{\beta}$ are any one of $\overline{\in}$, $\overline{q}$, $\overline{\in} \vee \overline{q}$, $\overline{\in} \wedge \overline{q}$ and investigate some of their related properties.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of interval valued fuzzy BRK-algebras and their applications in other branches of algebra. In the future study of interval valued fuzzy BRK-algebras, perhaps the following topics are worth to be considered:

(1) To apply this notion to some other algebraic structures;

(2) To consider these results to some possible applications in computer sciences and information systems in the future.

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