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Novel properties of (α, β)-fuzzy strong ideals in BCH-algebras

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Abstract In this paper, we define the concept of (α, β)-fuzzy strong ideals in BCH-algebra, where α, β are any one of $\in$, q, $\in \vee q$, $\in \wedge q$ and investigate some of their related properties. We show that every (α, β)-fuzzy strong ideal of a BCH-algebra X is an (α, β)-fuzzy ideal of Xd. We prove that a fuzzy set λ of a BCH-algebra X is a fuzzy strong ideal of X if and only if λ is an $(\in, \in)$-fuzzy strong ideal of X. We show that when an $(\in, \in \vee q)$-fuzzy strong ideal of a BCH-algebra X is an $(\in, \in)$-fuzzy strong ideal of X. The intersection and union of any family of $(\in, \in \vee q)$-fuzzy strong ideals of a BCH-algebra X are an $(\in, \in \vee q)$-fuzzy strong ideal of X.

Keywords Precise poverty alleviation, Picture fuzzy set, Cosine similarity measure

1. Introduction

The fundamental concept of fuzzy set was given by Zadeh in his pioneering paper [18], of 1965 provides a natural framework for generalizing some of the basic notions of algebra. Extensive applications of fuzzy set theory have been found in various fields, for example, artificial intelligence, computer science, control engineering, expert system, management science, operation research and many others. The concept was applied to the theory of groupoids and groups by Rosenfeld [16], where he introduced the fuzzy subgroup of a group.

A new type of fuzzy subgroup, which is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced by Bhakat and Das [5] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [15]. Murali [14] proposed the definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. It was found that the most viable generalization of Rosenfeld's fuzzy subgroup is $(\in, \in \vee q)$-fuzzy subgroup. Bhakat [2-3] initiated the concepts of $(\in \vee q)$-level subsets, $(\in, \in \vee q)$-fuzzy normal, quasi-normal and maximal subgroups. Many researchers utilized these concepts to generalize some concepts of algebra (see [9-13, 20-25]). In [9-11], Jun defined the notion of (α, β)-fuzzy subalgebras/ideals in BCK/BCI-algebras. The concept of (α, β)-fuzzy strong ideal in BCK-algebras was initiated by Zulfiqar in [20]. In [13], Jun et al. initiated general types of (α, β)-fuzzy ideals of hemirings. Shabir et al. [17], studied $(\in, \in \vee q)$-fuzzy ideals in semigroups and characterized regular semigroups by the properties of these fuzzy ideals. In [12], Jun defined $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras. Dudek et al. [6], defined (α, β)-fuzzy ideals of hemirings. The notion of $(\in, \in \vee q)$-fuzzy ideal in BCI-algebra was discussed in [19]. In [21], Zulfiqar introduced the notion of sub-implicative (α, β)-fuzzy ideals in BCH-algebras. Zulfiqar and Shabir defined the concept of positive implicative

$(\in, \in \vee q)$-fuzzy ideals ($(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals, fuzzy ideals with thresholds) in BCK-algebras in [24]. The concept of $(\in, \in \vee q_k)$-fuzzy fantastic ideals in BCI-algebras was initiated by Zulfiqar in [22]. In [25], Zulfiqar and Shabir introduced the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sub-commutative ideals in BCI-algebras and discussed some of its properties. The notion of $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy fantastic ideals in BCH-algebras was initiated in [23].

In the present paper, we define the notion of (α, β)-fuzzy strong ideals in BCH-algebra, where α, β are any one of $\in$, q, $\in \vee q$, $\in \wedge q$ and investigate some of their related properties.

2. Preliminaries

Throughout this paper X always denote a BCH-algebra without any specification. We also include some basic aspects that are necessary for this paper.

By a BCH-algebra [7-8], we mean an algebra $(X, *, 0)$ of type (2, 0) satisfying the axioms:

(BCH-1) $x * x = 0$

(BCH-2) $x * y = 0$ and $y * x = 0$ imply $x = y$

(BCH-3) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

We can define a partial order "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.1. [1, 21, 23] In any BCH-algebra X, the following are true:

(i) $x * (x * y) \leq y$

(ii) $0 * (x * y) = (0 * x) * (0 * y)$

(iii) $x \leq 0$ implies $x = 0$

(iv) $x * 0 = x$

for all $x, y \in X$.

Definition 2.2. [21] A nonempty subset S of a BCH-algebra X is called a subalgebra of X if it satisfies

$$x * y \in S, \text{ for all } x, y \in S.$$

Definition 2.3. [21] A non-empty subset I of a BCH-algebra X is called an ideal of X if it satisfies the conditions (I1) and (I2), where

(I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$,

for all $x, y \in X$.

Definition 2.4. A non-empty subset I of a BCH-algebra X is called a strong ideal of X if it satisfies the conditions (I1) and (I3), where

(I1) $0 \in I$,

(I3) $(x * y) * z \in I$ and $y \in I$ imply $x * z \in I$,

for all $x, y, z \in X$.

We now review some fuzzy logic concepts. Recall that the real unit interval [0, 1] with the totally ordered relation "$\leq$" is a complete lattice, with $\wedge$ = min and $\vee$ = max, 0 and 1 being the least element and the greatest element, respectively.

A fuzzy set λ of a universe X is a function from X into the unit closed interval [0, 1], that is $\lambda : X \rightarrow [0, 1]$. For a fuzzy set λ of a BCH-algebra X and $t \in (0, 1]$, the crisp set

$$\lambda_t = \{x \in X \mid \lambda(x) \geq t\}$$

is called the level subset of λ [4].

Definition 2.5. [21] A fuzzy set λ of a BCH-algebra X is called a fuzzy ideal of X if it satisfies the conditions (F1) and (F2), where

(F1) $\lambda(0) \geq \lambda(x)$,

(F2) $\lambda(x) \geq \lambda(x * y) \wedge \lambda(y)$,

for all $x, y \in X$.

Definition 2.6. A fuzzy set λ of a BCH-algebra X is called a fuzzy strong ideal of X if it satisfies the conditions (F1) and (F3), where

(F1) $\lambda(0) \geq \lambda(x)$,

(F3) $\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y)$,

for all $x, y, z \in X$.

Theorem 2.7. Every fuzzy strong ideal of a BCH-algebra X is a fuzzy ideal of X.
Proof. Straightforward.

Theorem 2.8. A fuzzy set λ of a BCH-algebra X is a fuzzy strong ideal of X if and only if, for every $t \in (0, 1]$, λ_t is either empty or a strong ideal of X.
Proof. Straightforward.
A fuzzy set λ of a BCH-algebra X having the form

$$\lambda(y) = \begin{cases} t \in (0, 1] & if \ \ y = x, \\ 0 & if \ \ y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t [21].
For a fuzzy point x_t and a fuzzy set λ in a set X, Pu and Liu [15] gave meaning to the symbol $x_t\alpha\lambda$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp., quasi-coincident with) a fuzzy set λ, written as $x_t \in \lambda$ (resp. $x_t q \lambda$) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$). By $x_t \in \vee q \lambda$ ($x_t \in \wedge q \lambda$) we mean that $x_t \in \lambda$ or $x_t q \lambda$ ($x_t \in \lambda$ and $x_t q \lambda$). For all $t_1, t_2 \in [0, 1]$, $\min\{t_1, t_2\}$ and $\max\{t_1, t_2\}$ will be denoted by $t_1 \wedge t_2$ and $t_1 \vee t_2$, respectively.
In what follows let α and β denote any one of $\in$, q, $\in \vee q$, $\in \wedge q$ and $\alpha \neq \in \wedge q$ unless otherwise specified. To say that $x_t \overline{\alpha} \lambda$ means that $x_t \alpha \lambda$ does not hold.

3. (α, β)-fuzzy strong ideals

In this section, we define the concepts of (α, β)-fuzzy ideal and (α, β)-fuzzy strong ideal in a BCH-algebra and investigate some of their properties. Throughout this paper X will denote a BCH-algebra and α, β are any one of $\in$, q, $\in \vee q$, $\in \wedge q$ unless otherwise specified.

Definition 3.1. A fuzzy set λ of a BCH-algebra X is called an (α, β)-fuzzy subalgebra of X, where $\alpha \neq \in \wedge q$, if it satisfies the condition

$$x_{t_1} \alpha \lambda, \ y_{t_2} \alpha \lambda, \Rightarrow (x * y)_{t_1 \wedge t_2} \beta \lambda$$

for all $t_1, t_2 \in (0, 1]$ and $x, y \in X$.
Let λ be a fuzzy set of a BCH-algebra X such that $\lambda(x) \leq 0.5$ for all $x \in X$. Let $x \in X$ and $t \in (0, 1]$ be such that

$$x_t \in \wedge q \lambda.$$

Then

$$\lambda(x) \geq t \text{ and } \lambda(x) + t > 1.$$

It follows that

$$2\lambda(x) = \lambda(x) + \lambda(x) \geq \lambda(x) + t > 1.$$

This implies that $\lambda(x) > 0.5$. This means that

$$\{x_t \mid x_t \in \wedge q\lambda\} = \phi.$$

Therefore, the case $\alpha = \in \wedge q$ in the above definition is omitted.

Definition 3.2. A fuzzy set λ of a BCH-algebra X is called an (α, β)-fuzzy ideal of X, where $\alpha \neq \in \wedge q$, if it satisfies the conditions (A) and (B), where

(A) $x_t \alpha \lambda \Rightarrow 0_t \beta \lambda$,

(B) $(x * y)_{t_1} \alpha \lambda,\ y_{t_2} \alpha \lambda \Rightarrow x_{t_1 \wedge t_2} \beta \lambda$,

for all t, t_1, $t_2 \in (0, 1]$ and x, y $\in$ X.

Definition 3.3. A fuzzy set λ of a BCH-algebra X is called an (α, β)-fuzzy strong ideal of X, where $\alpha \neq \in \wedge q$, if it satisfies the conditions (A) and (C), where

(A) $x_t \alpha \lambda \Rightarrow 0_t \beta \lambda$,

(C) $((x * y) * z)_{t_1} \alpha \lambda, y_{t_2} \alpha \lambda \Rightarrow (x * z)_{t_1 \wedge t_2} \beta \lambda$,

for all t, t_1, $t_2 \in (0, 1]$ and x, y, z $\in$ X.

Theorem 3.4. Every (α, β)-fuzzy strong ideal of a BCH-algebra X is an (α, β)-fuzzy ideal of Xd.

Proof. Let λ be an (α, β)-fuzzy strong ideal of X. Then for all t_1, $t_2 \in (0, 1]$ and x, y, z $\in$ X, we have

$$((x * y) * z)_{t_1} \alpha \lambda,\ y_{t_2} \alpha \lambda \Rightarrow (x * z)_{t_1 \wedge t_2} \beta \lambda.$$

Putting z = 0 in above, we get

$$((x * y) * 0)_{t_1} \alpha \lambda, (y * 0)_{t_2} \alpha \lambda \Rightarrow (x * 0)_{t_1 \wedge t_2} \beta \lambda.$$

This implies

$$(x * y)_{t_1} \alpha \lambda, y_{t_2} \alpha \lambda \Rightarrow x_{t_1 \wedge t_2} \beta \lambda \quad \text{(by Proposition 2.1(iv))}$$

This means that λ satisfies the condition (B). Combining with (A) implies that λ is an (α, β)-fuzzy ideal of X.

Theorem 3.5. For any fuzzy set λ of a BCH-algebra X, the condition (F1) and (F3) are equivalent to the conditions:

(D) $x_t \in \lambda \Rightarrow 0_t \in \lambda$,

(E) $((x * y) * z)_{t_1} \in \lambda,\ y_{t_2} \in \lambda \Rightarrow (x * z)_{t_1 \wedge t_2} \in \lambda$,

for all t, t_1, $t_2 \in$ (0, 1 and x, y, z $\in$ X.

Proof. (F1) $\Rightarrow$ (D)

Let x $\in$ X and t $\in$ (0, 1] be such that $x_t \in \lambda$, that is $\lambda(x) \geq t$.

Then by (F1), we have

$$\lambda(0) \geq \lambda(x) \geq t, \text{ and so } 0_t \in \lambda.$$

(D) $\Rightarrow$ (F1)

Since $x_{\lambda(x)} \in \lambda$, for x $\in$ X. Thus by hypothesis $0_{\lambda(x)} \in \lambda$, so we have

$$\lambda(0) \geq \lambda(x).$$

(F3) $\Rightarrow$ (E)

Let x, y, z $\in$ X and t_1, $t_2 \in (0, 1]$ be such that

$$((x * y) * z)_{t_1} \in \lambda \text{ and } y_{t_2} \in \lambda.$$

Then

$$\lambda((x * y) * z) \geq t_1 \text{ and } \lambda(y) \geq t_2.$$

By (F3), we have

$$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y)$$
$$\geq t_1 \wedge t_2.$$

Thus

$$(x * z)_{t_1 \wedge t_2} \in \lambda.$$

(E) $\Rightarrow$ (F3)

Note that for every x, y, z $\in$ X,

$$((x * y) * z)_{\lambda((x * y) * z)} \in \lambda \text{ and } y_{\lambda(y)} \in \lambda.$$

Hence by hypothesis

$$(x * z)_{\lambda((x * y) * z) \wedge \lambda(y)} \in \lambda.$$

This implies that

$$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y).$$

Theorem 3.6. Every $(\in \vee q, \in \vee q)$-fuzzy strong ideal of a BCH-algebra X is an $(\in, \in \vee q)$-fuzzy strong ideal of X.

Proof. Let λ be an $(\in \vee q, \in \vee q)$-fuzzy strong ideal of X. Let x $\in$ X and t $\in$ (0, 1] be such that $x_t \in \lambda$. Then

$$x_t \in \vee q \lambda$$

and so

$$0_t \in \vee q \lambda.$$

Let x, y, z $\in$ X and $t_1, t_2 \in (0, 1]$ be such that

$$((x * y) * z)_{t_1} \in \lambda \text{ and } y_{t_2} \in \lambda.$$

Then

$$((x * y) * z)_{t_1} \in \vee q \lambda \text{ and } y_{t_2} \in \vee q \lambda.$$

This implies that

$$(x * z)_{t_1 \wedge t_2} \in \vee q \lambda.$$

Therefore λ is an $(\in, \in \vee q)$-fuzzy strong ideal of X.

Theorem 3.7. A fuzzy set λ of a BCH-algebra X is an $(\in, \in \vee q)$-fuzzy strong ideal of X if and only if it satisfies the conditions (F) and (G), where

(F) $\lambda(0) \geq \lambda(x) \wedge 0.5$,

(G) $\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y) \wedge 0.5$,

for all x, y, z $\in$ X.

Proof. Suppose λ is an $(\in, \in \vee q)$-fuzzy strong ideal of X. Let x $\in$ X be such that

$$\lambda(0) < \lambda(x) \wedge 0.5.$$

If $\lambda(x) < 0.5$, then $\lambda(0) < \lambda(x)$. Select t $\in$ (0, 0.5) such that

$$\lambda(0) < t \leq \lambda(x).$$

Then

$$x_t \in \lambda \text{ but } 0_t \overline{\in \vee q} \lambda,$$

which is a contradiction. If $\lambda(x) \geq 0.5$, then $\lambda(0) < 0.5$. This implies that

$$x_{0.5} \in \lambda \text{ but } 0_{0.5} \overline{\in \vee q} \lambda.$$

Again a contradiction. Hence

$$\lambda(0) \geq \lambda(x) \wedge 0.5,$$

for all $x \in X$.

Now we show that condition (G) holds. On the contrary assume that there exist $x, y, z \in X$ such that

$$\lambda(x * z) < \lambda((x * y) * z) \wedge \lambda(y) \wedge 0.5.$$

If $\lambda((x * y) * z) \wedge \lambda(y) < 0.5$, then

$$\lambda(x * z) < \lambda((x * y) * z) \wedge \lambda(y).$$

Select $t \in (0, 0.5)$ such that

$$\lambda(x * z) < t \leq \lambda((x * y) * z) \wedge \lambda(y).$$

Then

$$((x * y) * z)_t \in \lambda \text{ and } y_t \in \lambda \text{ but } (x * z)_t \overline{\in \vee q} \lambda,$$

which is a contradiction. If $\lambda((x * y) * z) \wedge \lambda(y) \geq 0.5$, then

$$\lambda(x * z) < 0.5.$$

This implies

$$((x * y) * z)_{0.5} \in \lambda \text{ and } y_{0.5} \in \lambda \text{ but } (x * z)_{0.5} \overline{\in \vee q} \lambda.$$

Again a contradiction. Hence

$$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y) \wedge 0.5.$$

Conversely, assume that λ satisfies (F) and (G). Let $x \in X$ and $t \in (0, 1]$ be such that $x_t \in \lambda$. Then $\lambda(x) \geq t$. By condition (F), we have

$$\lambda(0) \geq \lambda(x) \wedge 0.5$$
$$\geq t \wedge 0.5.$$

If $t \leq 0.5$, then $\lambda(0) \geq t$. This implies $0_t \in \lambda$.

If $t > 0.5$, then $\lambda(0) \geq 0.5$. This implies

$$\lambda(0) + t > 0.5 + 0.5 = 1,$$

that is, $0_t q \lambda$. Hence

$$0_t \in \vee q \lambda.$$

Let $x, y, z \in X$ and $t_1, t_2 \in (0, 1]$ be such that

$$((x * y) * z)_{t_1} \in \lambda \text{ and } y_{t_2} \in \lambda.$$

Then

$$\lambda((x * y) * z) \geq t_1 \text{ and } \lambda(y) \geq t_2.$$

By condition (G), we have

$$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y) \wedge 0.5$$
$$\geq t_1 \wedge t_2 \wedge 0.5.$$

If $t_1 \wedge t_2 \leq 0.5$, then

$$\lambda(x * z) \geq t_1 \wedge t_2.$$

This implies

$$(x * z)_{t_1 \wedge t_2} \in \lambda.$$

If $t_1 \wedge t_2 > 0.5$, then

$$\lambda(x * z) \geq 0.5.$$

This implies

$$\lambda(x * z) + t_1 \wedge t_2 > 0.5 + 0.5 = 1,$$

i.e.,

$$(x * z)_{t_1 \wedge t_2} q \lambda.$$

Hence

$$(x * z)_{t_1 \wedge t_2} \in \vee q \lambda.$$

This shows that λ is an $(\in, \in \vee q)$-fuzzy strong ideal of X.

Theorem 3.8. A fuzzy set λ of a BCH-algebra X is a fuzzy strong ideal of X if and only if λ is an $(\in, \in)$-fuzzy strong ideal of X.

Proof. Suppose λ is a fuzzy strong ideal of X and $x_t \in \lambda$ for $x \in X$ and $t \in (0, 1]$. Then $\lambda(x) \geq t$. By Definition 2.6, $\lambda(0) \geq \lambda(x)$, we have $\lambda(0) \geq t$, that is $0_t \in \lambda$. Let $x, y, z \in X$ and $t, r \in (0, 1]$ be such that

$$((x * y) * z)_t \in \lambda \text{ and } y_r \in \lambda.$$

Then

$$\lambda((x * y) * z) \geq t \text{ and } \lambda(y) \geq r.$$

By Definition 2.6, we have

$$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y) \geq t \wedge r.$$

This implies that

$$(x * z)_{t \wedge r} \in \lambda.$$

This shows that λ is an $(\in, \in)$-fuzzy strong ideal of X.

Conversely, assume that λ is an $(\in, \in)$-fuzzy strong ideal of X. Suppose there exists $x \in X$ such that

$$\lambda(0) < \lambda(x).$$

Select $t \in (0, 1]$ such that

$$\lambda(0) < t \leq \lambda(x).$$

Then $x_t \in \lambda$ but $0_t \overline{\in} \lambda$, which is a contradiction. Hence

$$\lambda(0) \geq \lambda(x), \text{ for all } x \in X.$$

Now suppose there exist $x, y, z \in X$ such that

$$\lambda(x * z) < \lambda((x * y) * z) \wedge \lambda(y).$$

Select $t \in (0, 1]$ such that

$$\lambda(x * z) < t \leq \lambda((x * y) * z) \wedge \lambda(y).$$

Then $((x * y) * z)_t \in \lambda$ and $y_t \in \lambda$ but $(x * z)_t \overline{\in} \lambda$, which is a contradiction. Hence

$$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y).$$

This shows that λ is a fuzzy strong ideal of X.

Theorem 3.9. Every $(\in, \in)$-fuzzy strong ideal of a BCH-algebra X is an $(\in, \in \vee q)$-fuzzy strong ideal of X.

Proof. Straightforward.

Corollary 3.10. Every fuzzy strong ideal of a BCH-algebra X is an $(\in, \in \vee q)$-fuzzy strong ideal of X.

Proof. By Theorem 3.8, every fuzzy strong ideal of a BCH-algebra X is an $(\in, \in)$-fuzzy strong ideal of X. Hence by above Theorem 3.9, every fuzzy strong ideal of X is an $(\in, \in \vee q)$-fuzzy strong ideal of X.

Next we show that when an $(\in, \in \vee q)$-fuzzy strong ideal of a BCH-algebra X is an $(\in, \in)$-fuzzy strong ideal of X.

Theorem 3.11. Let λ be an $(\in, \in \vee q)$-fuzzy strong ideal of a BCH-algebra X such that $\lambda(x) < 0.5$ for all $x \in X$. Then λ is an $(\in, \in)$-fuzzy strong ideal of X.

Proof. Let $x \in X$ and $t \in (0, 1]$ be such that $x_t \in \lambda$. Then $\lambda(x) \geq t$. Since

$$\lambda(0) \geq \lambda(x) \wedge 0.5 = \lambda(x)$$

$\geq t$

we have $0_t \in \lambda$. Now let x, y, z $\in$ X and $t_1, t_2 \in (0, 1]$ be such that

$$((x * y) * z)_{t_1} \in \lambda \text{ and } y_{t_2} \in \lambda.$$

Then

$$\lambda((x * y) * z) \geq t_1 \text{ and } \lambda(y) \geq t_2.$$

It follows from Theorem 3.7(G) that

$$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y) \wedge 0.5$$
$$= \lambda((x * y) * z) \wedge \lambda(y)$$
$$\geq t_1 \wedge t_2.$$

Thus

$$(x * z)_{t_1 \wedge t_2} \in \lambda.$$

Therefore λ is an $(\in, \in)$-fuzzy strong ideal of X.

Corollary 3.12. Let λ be an $(\in, \in \vee q)$-fuzzy strong ideal of a BCH-algebra X such that $\lambda(x) < 0.5$ for all x $\in$ X. Then λ is a fuzzy strong ideal of X.

Theorem 3.13. Let I be a strong ideal of a BCH-algebra X. Then the fuzzy set λ of X defined by

$$\lambda(x) = \begin{cases} \geq 0.5 & \text{if } x \in I \\ 0 & \text{otherwise}, \end{cases}$$

is an $(\in, \in \vee q)$-fuzzy strong ideal of X.

Proof. Let I be a strong ideal of X. Let x $\in$ X and t $\in$ (0, 1] be such that $x_t \in \lambda$. Then

$$\lambda(x) \geq t > 0.$$

Thus

$$\lambda(x) \geq 0.5.$$

This implies that x $\in$ I. Since I is a strong ideal of X. So 0 $\in$ I. Hence

$$\lambda(0) \geq 0.5.$$

If $t \leq 0.5$, then $\lambda(0) \geq 0.5 \geq t$. This implies that $0_t \in \lambda$.

If $t > 0.5$, then

$$\lambda(0) + t \geq 0.5 + 0.5 > 1$$

and so $0_t\, q\, \lambda$. This implies that

$$0_t \in \vee q \lambda.$$

Let x, y, z $\in$ X and t, r $\in$ (0, 1] be such that

$$((x * y) * z)_t \in \lambda \text{ and } y_r \in \lambda.$$

Then

$$\lambda((x * y) * z) \geq t > 0 \text{ and } \lambda(y) \geq r > 0.$$

Thus

$$\lambda((x * y) * z) \geq 0.5 \text{ and } \lambda(y) \geq 0.5.$$

This implies that

$$(x * y) * z \in I \text{ and } y \in I.$$

Since I is a strong ideal of X. So

$$x * z \in I.$$

Hence

$$\lambda(x * z) \geq 0.5.$$

If $t \wedge r \leq 0.5$, then

$$\lambda(x * z) \geq 0.5 \geq t \wedge r.$$

This implies that

$$(x * z)_{t \wedge r} \in \lambda.$$

If $t \wedge r > 0.5$, then

$$\lambda(x * z) + t \wedge r \geq 0.5 + 0.5 > 1$$

and so

$$(x * z)_{t \wedge r}\, q\, \lambda.$$

Thus

$$(x * z)_{t \wedge r} \in \vee\, q\, \lambda.$$

Hence λ is an $(\in, \in \vee\, q)$-fuzzy strong ideal of X.

Theorem 3.14. Let I be a non-empty subset of a BCH-algebra X. Then I is a strong ideal of X if the fuzzy set λ of X defined by

$$\lambda(x) = \begin{cases} \geq 0.5 & \text{if } x \in I \\ 0 & \text{otherwise}, \end{cases}$$

is a $(q, \in \vee\, q)$-fuzzy strong ideal of X.

Proof. Let I be a strong ideal of X. Let $x \in X$ and $t \in (0, 1]$ be such that $x_t\, q\, \lambda$. Then

$$\lambda(x) + t > 1$$

So $x \in I$. Since I is a strong ideal of X. So $0 \in I$. Hence

$$\lambda(0) \geq 0.5.$$

If $t \leq 0.5$, then

$$\lambda(0) \geq 0.5 \geq t.$$

This implies that $0_t \in \lambda$.

If $t > 0.5$, then

$$\lambda(0) + t \geq 0.5 + 0.5 > 1$$

and so $0_t\, q\, \lambda$. This implies that

$$0_t \in \vee\, q\, \lambda.$$

Let $x, y, z \in X$ and $t, r \in (0, 1]$ be such that

$$((x * y) * z)_t\, q\, \lambda \text{ and } y_r\, q\, \lambda.$$

Then

$$\lambda((x * y) * z) + t > 1 \text{ and } \lambda(y) + r > 1.$$

So

$$(x * y) * z \in I \text{ and } y \in I.$$

Since I is a strong ideal of X. So

$$x * z \in I.$$

Thus

$$\lambda(x * z) \geq 0.5.$$

If $t \wedge r \leq 0.5$, then

$$\lambda(x * z) \geq 0.5 \geq t \wedge r.$$

So

$$(x * z)_{t \wedge r} \in \lambda.$$

If $t \wedge r > 0.5$, then

$$\lambda(x * z) + t \wedge r \geq 0.5 + 0.5 > 1$$

and so

$$(x * z)_{t \wedge r}\, q\, \lambda.$$

Thus

$$(x * z)_{t \wedge r} \in \vee\, q\, \lambda.$$

Hence λ is an $(q, \in \vee\, q)$-fuzzy strong ideal of X.

Theorem 3.15. Let I be a non-empty subset of a BCH-algebra X. Then I is a strong ideal of X if the fuzzy set λ of X defined by

$$\lambda(x) = \begin{cases} \geq 0.5 & \text{if } x \in I \\ 0 & \text{otherwise}, \end{cases}$$

is an $(\in \vee\, q, \in \vee\, q)$-fuzzy strong ideal of X.

Proof. Let I be a strong ideal of X. Let $x \in X$ and $t \in (0, 1]$ be such that

$$x_t \in \vee\, q\, \lambda.$$

This implies that

$$x_t \in \lambda \text{ or } x_t\, q\, \lambda.$$

If $x_t\, q\, \lambda$. This implies that

$$\lambda(x) + t > 1.$$

This implies that $x \in I$. Since I is a strong ideal of X. So $0 \in I$. Hence

$$\lambda(0) \geq 0.5.$$

If $t \leq 0.5$, then

$$\lambda(0) \geq 0.5 \geq t.$$

This implies that $0_t \in \lambda$.

If $t > 0.5$, then

$$\lambda(0) + t \geq 0.5 + 0.5 > 1$$

and so $0_t\, q\, \lambda$. This implies that

$$0_t \in \vee\, q\, \lambda.$$

Let $x, y, z \in X$ and $t, r \in (0, 1]$ be such that

$$((x * y) * z)_t \in \vee\, q\, \lambda \text{ and } y_r \in \vee\, q\, \lambda.$$

This implies that

$$((x * y) * z)_t \in \lambda \text{ or } ((x * y) * z)_t\, q\, \lambda$$

and

$$y_r \in \lambda \text{ or } y_r\, q\, \lambda.$$

If $((x * y) * z)_t\, q\, \lambda$ and $y_r\, q\, \lambda$. This implies that

$$\lambda((x * y) * z) + t > 1 \text{ and } \lambda(y) + r > 1.$$

So

$$(x * y) * z \in I \text{ and } y \in I.$$

Since I is a strong ideal of X. So

$$x * z \in I.$$

Thus

$$\lambda(x * z) \geq 0.5.$$

If $t \wedge r \leq 0.5$, then

$$\lambda(x * z) \geq 0.5 \geq t \wedge r.$$

So

$$(x * z)_{t \wedge r} \in \lambda.$$

If $t \wedge r > 0.5$, then

$$\lambda(x * z) + t \wedge r \geq 0.5 + 0.5 > 1$$

and so

$$(x * z)_{t \wedge r}\, q\, \lambda.$$

Thus

$$(x * z)_{t \wedge r} \in \vee\, q\, \lambda.$$

Hence λ is an $(\in \vee q, \in \vee q)$-fuzzy strong ideal of X.

Theorem 3.16. The intersection of any family of $(\in, \in \vee q)$-fuzzy strong ideals of a BCH-algebra X is an $(\in, \in \vee q)$-fuzzy strong ideal of X.

Proof. Let $\{\lambda_i\}_{i \in I}$ be a family of $(\in, \in \vee q)$-fuzzy strong ideals of a BCH-algebra X and $x \in X$. So

$$\lambda_i(0) \geq \lambda_i(x) \wedge 0.5$$

for all $i \in I$. Thus

$$(\bigwedge_{i \in I} \lambda_i)(0) = \bigwedge_{i \in I} \lambda_i(0)$$

$$\geq \bigwedge_{i \in I} (\lambda_i(x) \wedge 0.5)$$

$$= (\bigwedge_{i \in I} \lambda_i)(x) \wedge 0.5.$$

Thus

$$(\bigwedge_{i \in I} \lambda_i)(0) \geq (\bigwedge_{i \in I} \lambda_i)(x) \wedge 0.5.$$

Let $x, y, z \in X$. Since each λ_i is an $(\in, \in \vee q)$-fuzzy strong ideal of X. So

$$\lambda_i(x * z) \geq \lambda_i((x * y) * z) \wedge \lambda_i(y) \wedge 0.5$$

for all $i \in I$. Thus

$$(\bigwedge_{i \in I} \lambda_i)(x * z) = \bigwedge_{i \in I} \lambda_i(x * z)$$

$$\geq \bigwedge_{i \in I} (\lambda_i((x * y) * z) \wedge \lambda_i(y) \wedge 0.5)$$

$$= (\bigwedge_{i \in I} \lambda_i)((x * y) * z) \wedge (\bigwedge_{i \in I} \lambda_i)(y) \wedge 0.5.$$

Therefore

$$(\bigwedge_{i \in I} \lambda_i)(x * z) \geq (\bigwedge_{i \in I} \lambda_i)((x * y) * z) \wedge (\bigwedge_{i \in I} \lambda_i)(y) \wedge 0.5.$$

Hence, $\bigwedge_{i \in I} \lambda_i$ is an $(\in, \in \vee q)$-fuzzy strong ideal of X.

Theorem 3.17. The union of any family of $(\in, \in \vee q)$-fuzzy strong ideals of a BCH-algebra X is an $(\in, \in \vee q)$-fuzzy strong ideal of X.

Proof. Let $\{\lambda_i\}_{i \in I}$ be a family of $(\in, \in \vee q)$-fuzzy strong ideals of a BCH-algebra X and $x \in X$. So

$$\lambda_i(0) \geq \lambda_i(x) \wedge 0.5$$

for all $i \in I$. Thus

$$(\bigvee_{i \in I} \lambda_i)(0) = \bigvee_{i \in I} \lambda_i(0)$$

$$\geq \bigvee_{i \in I} (\lambda_i(x) \wedge 0.5)$$

$$= (\bigvee_{i \in I} \lambda_i)(x) \wedge 0.5.$$

Thus

$$(\vee_{i \in I} \lambda_i)(0) \geq (\vee_{i \in I} \lambda_i)(x) \wedge 0.5.$$

Let x, y, z $\in$ X. Since each λ_i is an $(\in, \in \vee q)$-fuzzy strong ideal of X. So

$$\lambda_i(x * z) \geq \lambda_i((x * y) * z) \wedge \lambda_i(y) \wedge 0.5$$

for all i $\in$ I. Thus

$$(\vee_{i \in I} \lambda_i)(x * z) = \vee_{i \in I} \lambda_i(x * z)$$

$$\geq \vee_{i \in I} (\lambda_i((x * y) * z) \wedge \lambda_i(y) \wedge 0.5)$$

$$= (\vee_{i \in I} \lambda_i)((x * y) * z) \wedge (\vee_{i \in I} \lambda_i)(y) \wedge 0.5$$

Therefore

$$(\vee_{i \in I} \lambda_i)(x * z) \geq (\vee_{i \in I} \lambda_i)((x * y) * z) \wedge (\vee_{i \in I} \lambda_i)(y) \wedge 0.5.$$

Hence, $\vee_{i \in I} \lambda_i$ is an $(\in, \in \vee q)$-fuzzy strong ideal of X.

4. Conclusion

To investigate the structure of an algebraic system, we see that the fuzzy strong ideals with special properties always play an important role.

The purpose of this paper is to introduce the concept of (α, β)-fuzzy strong ideals in BCH-algebra, where α, β are any one of $\in$, q, $\in \vee q$, $\in \wedge q$ and investigate some of their related properties.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of fuzzy BCH-algebras and their applications in other branches of algebra. In the future study of fuzzy BCH-algebras, perhaps the following topics are worth to be considered:

- To characterize other classes of BCH-algebras by using this notion;
- To apply this notion to some other algebraic structures;
- To consider these results to some possible applications in computer sciences and information systems in the future.

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