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Some applications to $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal in BRK-algebras

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Abstract The purpose of this paper is to introduce the notion of $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal in BRK-algebra and investigate some related properties. The concepts of implication-based interval valued fuzzy quasi-associative ideal and implication operators in Lukasiewicz system of continuous-valued logic in BRK-algebra are introduced.

Keywords BRK-algebra; interval valued fuzzy quasi-associative ideal; $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal; implication-based interval valued fuzzy quasi-associative ideal; implication operators in Lukasiewicz system of continuous-valued logic

1. Introduction

Algebraic structures play a central role in mathematics with wide variety of applications in various disciplines such as control engineering, theoretical physics, artificial intelligence, computer sciences, expert system, management science, operation research, information sciences, coding theory etc. On the other hand, in behavior information concerning various aspects of uncertainty, non-classical logic (a large extension and development of classical logic) is careful to be further powerful system than the classical logic one. The non-classical logic, therefore, has nowadays become a useful tool in computer science. Moreover, non-classical logic deals with the fuzzy information and uncertainty.

The theory of BRK-algebra was first initiated by Bandaru in [1]. The notion of fuzzy sets, proposed by Zadeh [29] in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex or ill defined to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields, for example, artificial intelligence, computer science, control engineering, expert system, management science, operation research and many others. The concept was applied to the theory of groupoids and groups by Rosenfeld [26], where he introduced the fuzzy subgroup of a group.

A new type of fuzzy subgroup, which is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced by Bhakat and Das [4] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [25]. Murali [24] proposed the definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. It was found that the most viable generalization of Rosenfeld's fuzzy subgroup is $(\in, \in \vee q)$-fuzzy subgroup. Bhakat [2-3] initiated the concepts of $(\in \vee q)$-level subsets, $(\in, \in \vee q)$-fuzzy normal, quasi-normal and maximal subgroups. Many researchers utilized these concepts to generalize some concepts of algebra (see [6, 8-11, 15-18, 28, 33-37]). In [7], Davvaz studied $(\in, \in \vee q)$-fuzzy subnearrings and ideals. In [15-17], Jun introduced the concept of (α, β)-fuzzy subalgebras/ideals in BCK/BCI-algebras. The notion of (α, β)-fuzzy positive implicative ideal in BCK-algebras was initiated by Zulfiqar in [33]. In [18], Jun defined $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras. In [34], Zulfiqar introduced the concept of subimplicative (α, β)-fuzzy ideals in BCH-algebras. In [36], Zulfiqar and Shabir defined the concept of positive implicative $(\in, \in \vee q)$-fuzzy ideals $(\overline{\in} \vee \overline{q})$-fuzzy ideals, fuzzy ideals with thresholds) in BCK-algebras.

The theory of interval valued fuzzy sets was proposed forty years ago as a natural extension of fuzzy sets. Interval valued fuzzy set was introduced by Zadeh [30]. The theory was further enriched by many authors [5, 12-14, 19-23, 27, 31-32]. In [5], Biswas defined interval valued fuzzy subgroups of Rosenfeld's nature, and investigated some elementary properties. Jun, initiated the notion of interval valued fuzzy subalgebras/ideals in BCK-algebras [14]. In [19], Latha et al. introduced the idea of interval valued (α, β)-fuzzy subgroups. In [20], Ma et al. defined the theory of interval valued $(\in, \in \vee q)$-fuzzy ideals of pseudo MV-algebras. In [21-22], Ma et al. studied $(\in, \in \vee q)$-interval valued fuzzy ideals in BCI-algebras. Mostafa et al. initiated the notion of interval valued fuzzy KU-ideals in KU-algebras [23]. In [27], Saeid defined the concept of interval valued fuzzy BG-algebras. Zhan et al. [32] initiated the notion of interval valued $(\in, \in \vee q)$-fuzzy filters of pseudo BL-algebras. Zulfiqar and Shabir introduced the notion of $(\overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideals in BCK-algebras in [37].

In the present paper, we define the concept of $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal in BRK-algebra and investigate some related properties. The notions of implication-based interval valued fuzzy quasi-associative ideal and implication operators in Lukasiewicz system of continuous-valued logic in BRK-algebra are introduced.

2. Preliminaries

Throughout this paper X always denotes a BRK-algebra without any specification. We also include some basic aspects that are necessary for this paper.

A BRK-algebra [1, 8-9] is a non-empty set X with a consonant 0 and a binary operation "$*$" satisfying the following axioms:

(BRK-1) $x * 0 = x$

(BRK-2) $(x * y) * x = 0 * y$

for all x, y $\in$ X.

We can define a partial order "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$.

Definition 2.1. [1] A non-empty subset S of a BRK-algebra X is called a subalgebra of X if it satisfies

$$x * y \in S, \text{ for all } x, y \in S.$$

Definition 2.2. [9] A non-empty subset I of a BRK-algebra X is called an ideal of X if it satisfies the conditions (I1) and (I2), where

(I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$,

for all x, y $\in$ X.

Definition 2.3. A non-empty subset I of a BRK-algebra X is called a quasi-associative ideal of X if it satisfies the conditions (I1) and (I3), where

(I1) $0 \in I$,

(I3) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$,

for all x, y, z $\in$ X.

We now review some interval valued fuzzy logic concepts. First, we denote by $\bar{c} = [c^-, c^+]$ a closed interval of [0, 1], where $0 \leq c^- \leq c^+ \leq 1$ and denote by H[0, 1] the set of all such closed intervals of [0, 1].
Define on H[0, 1] an order relation "≤" by

(1) $\bar{c}_1 \leq \bar{c}_2 \iff c_1^- \leq c_2^-$ and $c_1^+ \leq c_2^+$

(2) $\bar{c}_1 = \bar{c}_2 \iff c_1^- = c_2^-$ and $c_1^+ = c_2^+$

(3) $\bar{c}_1 < \bar{c}_2 \iff c_1^- \leq c_2^-$ and $c_1^+ \neq c_2^+$

(4) $p\bar{c} = [pc^-, pc^+]$, whenever $0 \leq p \leq 1$

(5) $\text{rmax}\{\bar{c}_i, \bar{d}_i\} = [\max\{c_i^-, d_i^-\}, \max\{c_i^+, d_i^+\}]$

(6) $\text{rmin}\{\bar{c}_i, \bar{d}_i\} = [\min\{c_i^-, d_i^-\}, \min\{c_i^+, d_i^+\}]$

(7) $\text{rinf}\,\bar{c}_i = [\Lambda_{i\in I}\, c_i^-, \Lambda_{i\in I}\, c_i^+]$

(8) $\text{rsup}\,\bar{c}_i = [\vee_{i\in I}\, c_i^-, \vee_{i\in I}\, c_i^+]$

Where

$$\bar{c}_i = [c_i^-, c_i^+],\ \bar{d}_i = [d_i^-, d_i^+] \in H[0, 1],\ i \in I.$$

Then, H[0, 1] with ≤ is a complete lattice, with ∧ = rmin, ∨ = rmax, $\bar{0} = [0, 0]$ and $\bar{1} = [1, 1]$ being its least element and the greatest element, respectively.

An interval valued fuzzy set $\tilde{\lambda}$ of a universe X is a function from X into the unit closed interval [0, 1], that is $\tilde{\lambda} : X \to H[0, 1]$, $\tilde{\lambda}(x) \in H[0, 1]$, where for each $x \in X$

$$\tilde{\lambda}(x) = [\lambda^-(x), \lambda^+(x)] \in H[0, 1].$$

For an interval valued fuzzy set $\tilde{\lambda}$ in a BRK-algebra X and $[0, 0] < \tilde{t} \leq [1, 1]$, the crisp set

$$\tilde{\lambda}_{\tilde{t}} = \{x \in X \mid \tilde{\lambda}(x) \geq \tilde{t}\}$$

is called the level subset of $\tilde{\lambda}$.

We also note that, since every $c \in [0, 1]$ is in correspondence with the interval $[c, c] \in H[0, 1]$, it follows that a fuzzy set is a particular case of interval valued fuzzy set. First we note that an interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is a pair of fuzzy sets (λ^-, λ^+) of X such that $\lambda^-(x) \leq \lambda^+(x)$, for all $x \in X$.

If $\bar{C}$, $\bar{D}$ are two interval valued fuzzy sets of a BRK-algebra X, then we define

$$\bar{C} \leq \bar{D} \text{ if and only if for all } x \in X,\ C^-(x) \leq D^-(x) \text{ and } C^+(x) \leq D^+(x),$$

$$\bar{C} = \bar{D} \text{ if and only if for all } x \in X,\ C^-(x) = D^-(x) \text{ and } C^+(x) = D^+(x).$$

Also, the union and intersection are defined as follows:

If $\bar{C}$ and $\bar{D}$ are two interval valued fuzzy sets of a BRK-algebra X, where

$$\bar{C}(x) = [C^-(x), C^+(x)],\quad \bar{D}(x) = [D^-(x), D^+(x)], \text{ for all } x \in X,$$

then

$(\bar{C} \cup \bar{D})(x) = \bar{C}(x) \vee \bar{D}(x) = [\max\{C^-(x), D^-(x)\}, \max\{C^+(x), D^+(x)\}]$

$(\bar{C} \cap \bar{D})(x) = \bar{C}(x) \wedge \bar{D}(x) = [\min\{C^-(x), D^-(x)\}, \min\{C^+(x), D^+(x)\}]$

$\overline{C^c}(x) = [1 - C^+(x), 1 - C^-(x)]$

where the operation "c" is the complement of interval valued fuzzy set of X.

By the joint of two interval valued fuzzy sets, we know

$$\bar{C} \cup \overline{C^c}(x) = [\max\{C^-(x), 1 - C^+(x)\}, \max\{C^+(x), 1 - C^-(x)\}].$$

Definition 2.4. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is called an interval valued fuzzy ideal of X if it satisfies the conditions (F1) and (F2), where

(F1) $\tilde{\lambda}(0) \geq \tilde{\lambda}(x)$,

(F2) $\tilde{\lambda}(x) \geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y)$,

for all x, y $\in$ X.

Definition 2.5. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is called an interval valued fuzzy quasi-associative ideal of X if it satisfies the conditions (F1) and (F3), where

(F1) $\tilde{\lambda}(0) \geq \tilde{\lambda}(x)$,

(F3) $\tilde{\lambda}(x * z) \geq \tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y)$,

for all x, y, z $\in$ X.

Example 2.6. Let X = {0, 1, 2, 3} be a BRK-algebra with the following Cayley table [1]:

∗	0	1	2	3
0	0	1	0	1
1	1	0	1	0
2	2	1	0	1
3	3	2	3	0

We define an interval valued fuzzy set $\tilde{\lambda}$ by $\tilde{\lambda}(0)$ = [0.9, 0.9], $\tilde{\lambda}(1)$ = [0.7, 0.7], $\tilde{\lambda}(2)$ = [0.6, 0.6] and $\tilde{\lambda}(3)$ = [0.4, 0.4]. Simple calculations show that $\tilde{\lambda}$ is an interval valued fuzzy quasi-associative ideal of X.

Theorem 2.7. The intersection of any family of an interval valued fuzzy quasi-associative ideal of a BRK-algebra X is an interval valued fuzzy quasi-associative ideal of X.

Proof. Let $\{\tilde{\lambda}_i\}_{i \in I}$ be a family of an interval valued fuzzy quasi-associative ideal of a BRK-algebra X and x $\in$ X. So

$$\tilde{\lambda}_i(0) \geq \tilde{\lambda}_i(x)$$

for all i $\in$ I. Thus

$$\begin{aligned}(\wedge_{i \in I} \tilde{\lambda}_i)(0) &= \wedge_{i \in I} (\tilde{\lambda}_i(0)) \\ &\geq \wedge_{i \in I} (\tilde{\lambda}_i(x)) \\ &= (\wedge_{i \in I} \tilde{\lambda}_i)(x).\end{aligned}$$

Thus

$$(\wedge_{i \in I} \tilde{\lambda}_i)(0) \geq (\wedge_{i \in I} \tilde{\lambda}_i)(x).$$

Let x, y, z $\in$ X. Since each $\tilde{\lambda}_i$ is an interval valued fuzzy quasi-associative ideal of X. So

$$\tilde{\lambda}_i(x * z) \geq \tilde{\lambda}_i(x * (y * z)) \wedge \tilde{\lambda}_i(y)$$

for all i $\in$ I. Thus

$$\begin{aligned}(\wedge_{i \in I} \tilde{\lambda}_i)(x * z) &= \wedge_{i \in I} (\tilde{\lambda}_i(x * z)) \\ &\geq \wedge_{i \in I} \tilde{\lambda}_i(x * (y * z)) \wedge \tilde{\lambda}_i(y)\end{aligned}$$

$$= (\wedge_{i \in I} \tilde{\lambda}_i)(x * (y * z)) \wedge (\wedge_{i \in I} \tilde{\lambda}_i)(y).$$

Therefore

$$(\wedge_{i \in I} \tilde{\lambda}_i)(x * z) \geq (\wedge_{i \in I} \tilde{\lambda}_i)(x * (y * z)) \wedge (\wedge_{i \in I} \tilde{\lambda}_i)(y).$$

Hence, $\wedge_{i \in I} \tilde{\lambda}_i$ is an interval valued fuzzy quasi-associative ideal of X.

Theorem 2.8. The union of any family of an interval valued fuzzy quasi-associative ideal of a BRK-algebra X is an interval valued fuzzy quasi-associative ideal of X.

Proof. Let $\{\tilde{\lambda}_i\}_{i \in I}$ be a family of an interval valued fuzzy quasi-associative ideal of a BRK-algebra X and x $\in$ X. So

$$\tilde{\lambda}_i(0) \geq \tilde{\lambda}_i(x)$$

for all i $\in$ I. Thus

$$(\vee_{i \in I} \tilde{\lambda}_i)(0) = \vee_{i \in I} (\tilde{\lambda}_i(0))$$

$$\geq \vee_{i \in I} (\tilde{\lambda}_i(x))$$

$$= (\vee_{i \in I} \tilde{\lambda}_i)(x).$$

Thus

$$(\vee_{i \in I} \tilde{\lambda}_i)(0) \geq (\vee_{i \in I} \tilde{\lambda}_i)(x).$$

Let x, y, z $\in$ X. Since each $\tilde{\lambda}_i$ is an interval valued fuzzy quasi-associative ideal of X. So

$$\tilde{\lambda}_i(x * z) \geq \tilde{\lambda}_i(x * (y * z)) \wedge \tilde{\lambda}_i(y)$$

for all i $\in$ I. Thus

$$(\vee_{i \in I} \tilde{\lambda}_i)(x * z) = \vee_{i \in I} (\tilde{\lambda}_i(x * z))$$

$$\geq \vee_{i \in I} (\tilde{\lambda}_i(x * (y * z)) \wedge \tilde{\lambda}_i(y)$$

$$= (\vee_{i \in I} \tilde{\lambda}_i)(x * (y * z)) \wedge (\vee_{i \in I} \tilde{\lambda}_i)(y).$$

Therefore

$$(\vee_{i \in I} \tilde{\lambda}_i)(x * z) \geq (\vee_{i \in I} \tilde{\lambda}_i)(x * (y * z)) \wedge (\vee_{i \in I} \tilde{\lambda}_i)(y).$$

Hence, $\vee_{i \in I} \tilde{\lambda}_i$ is an interval valued fuzzy quasi-associative ideal of X.

3. $(\in, \in \vee q)$-interval valued fuzzy ideals

In this section, we discuss $(\in, \in \vee q)$-interval valued fuzzy ideal in a BRK-algebra and study their related properties.

An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X having the form

$$\tilde{\lambda}(y) = \begin{cases} \tilde{t} \ (\neq [0, 0]) & if \ \ y = x \\ [0, 0] & if \ \ y \neq x \end{cases}$$

is said to be an interval valued fuzzy point with support x and value $\tilde{t}$ and is denoted by $x_{\tilde{t}}$. An interval valued fuzzy point $x_{\tilde{t}}$ is said to belong to (resp., quasi-coincident with) an interval valued fuzzy set $\tilde{\lambda}$, written as $x_{\tilde{t}} \in \tilde{\lambda}$ (resp. $x_{\tilde{t}}\, q\, \tilde{\lambda}$) if $\tilde{\lambda}$(x) ≥ $\tilde{t}$ (resp. $\tilde{\lambda}$(x) + $\tilde{t}$ > [1, 1]). By $x_{\tilde{t}} \in \vee q\, \tilde{\lambda}$ ($x_{\tilde{t}} \in \wedge q\, \tilde{\lambda}$) we mean that $x_{\tilde{t}} \in \tilde{\lambda}$ or $x_{\tilde{t}}\, q\, \tilde{\lambda}$ ($x_{\tilde{t}} \in \tilde{\lambda}$ and $x_{\tilde{t}}\, q\, \tilde{\lambda}$).

Definition 3.1. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is called an $(\in, \in \vee q)$-interval valued fuzzy ideal of X if it satisfies the conditions (A) and (B), where

(A) $x_{\tilde{t}} \in \tilde{\lambda} \Rightarrow 0_{\tilde{t}} \in \vee q \tilde{\lambda}$,

(B) $(x * y)_{\tilde{t}} \in \tilde{\lambda},\ y_{\tilde{r}} \in \tilde{\lambda} \Rightarrow x_{\tilde{t} \wedge \tilde{r}} \in \vee q \tilde{\lambda}$,

for all [0, 0] < $\tilde{t}, \tilde{r}$ ≤ [1, 1], and x, y ∈ X.

Lemma 3.2. The conditions (A) and (B) in Definition 3.1, are equivalent to the following conditions, respectively:

(C) $\tilde{\lambda}(0) \geq \tilde{\lambda}(x) \wedge [0.5, 0.5]$,

(D) $\tilde{\lambda}(x) \geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5]$,

for all x, y ∈ X.

Proof. Straightforward.

Example 3.3. Let X = {0, 1, 2, 3} be a BRK-algebra with the following Cayley table [1]:

*	0	1	2	3
0	0	1	0	1
1	1	0	1	0
2	2	1	0	1
3	3	2	3	0

We define an interval valued fuzzy set $\tilde{\lambda}$ by $\tilde{\lambda}$(0) = [0.89, 0.89], $\tilde{\lambda}$(1) = [0.78, 0.78], $\tilde{\lambda}$(2) = [0.67, 0.67] and $\tilde{\lambda}$(3) = [0.56, 0.56]. Then $\tilde{\lambda}$ is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

4. $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal

In this section, we define the concept of $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal in a BRK-algebra and investigate some of their related properties.

Definition 4.1. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is called an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X if it satisfies the conditions (A) and (E), where

(A) $x_{\tilde{t}} \in \tilde{\lambda} \Rightarrow 0_{\tilde{t}} \in \vee q \tilde{\lambda}$,

(E) $(x * (y * z))_{\tilde{t}} \in \tilde{\lambda},\ y_{\tilde{r}} \in \tilde{\lambda} \Rightarrow (x * z)_{\tilde{t} \wedge \tilde{r}} \in \vee q \tilde{\lambda}$,

for all [0, 0] < $\tilde{t}, \tilde{r}$ ≤ [1, 1] and x, y, z ∈ X.

Lemma 4.2. The condition (E) is equivalent to the condition (F), where

(F) $\tilde{\lambda}(x * z) \geq \tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5]$

for all x, y, z ∈ X.

Proof. Straightforward.

Corollary 4.3. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X if it satisfies the conditions (C) and (F).

Example 4.4. Let X = {0, 1, 2} be a BRK-algebra with the following Cayley table [1]:

$*$	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

We define an interval valued fuzzy set $\tilde{\lambda}$ by $\tilde{\lambda}$ (0) = [0.88, 0.88], $\tilde{\lambda}$ (1) = [0.74, 0.74] and $\tilde{\lambda}$ (2) = [0.53, 0.53]. Simple calculations show that $\tilde{\lambda}$ is an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X.

Theorem 4.5. Every $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideals of a BRK-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X.

Proof. Let $\tilde{\lambda}$ be $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X. Then for all x, y, z $\in$ X, we have

$$(x * (y * z))_{\tilde{t}} \in \tilde{\lambda},\ y_{\tilde{r}} \in \tilde{\lambda} \quad \Rightarrow \quad (x * z)_{\tilde{t} \wedge \tilde{r}} \in \vee \text{q} \tilde{\lambda}.$$

Putting z = 0 in above, we get

$$(x * (y * 0))_{\tilde{t}} \in \tilde{\lambda},\ y_{\tilde{r}} \in \tilde{\lambda} \quad \Rightarrow \quad (x * 0)_{\tilde{t} \wedge \tilde{r}} \in \vee \text{q} \tilde{\lambda}.$$

This implies that

$$(x * y)_{\tilde{t}} \in \tilde{\lambda},\ y_{\tilde{r}} \in \tilde{\lambda} \quad \Rightarrow \quad x_{\tilde{t} \wedge \tilde{r}} \in \vee \text{q} \tilde{\lambda} \qquad \text{(by BRK-1)}$$

This means that $\tilde{\lambda}$ satisfies the condition (B). Combining with (A) implies that $\tilde{\lambda}$ be an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

Theorem 4.6. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X if and only if the set $\tilde{\lambda}_{\tilde{t}}$ $(\neq \phi)$ is a quasi-associative ideal of X for all [0, 0] < $\tilde{t}$ $\leq$ [0.5, 0.5].

Proof. Let $\tilde{\lambda}$ be an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X and [0, 0] < $\tilde{t}$ $\leq$ [0.5, 0.5]. If $\tilde{\lambda}_{\tilde{t}} \neq \phi$, then x $\in$ $\tilde{\lambda}_{\tilde{t}}$. This implies that $\tilde{\lambda}$ (x) $\geq$ $\tilde{t}$. By condition (C), we have

$$\begin{aligned} \tilde{\lambda}(0) &\geq \tilde{\lambda}(x) \wedge [0.5, 0.5] \\ &\geq \tilde{t} \wedge [0.5, 0.5] \\ &= \tilde{t}. \end{aligned}$$

Thus $\tilde{\lambda}$ (0) $\geq$ $\tilde{t}$. Hence 0 $\in$ $\tilde{\lambda}_{\tilde{t}}$.

Let x $*$ (y $*$ z) $\in$ $\tilde{\lambda}_{\tilde{t}}$ and y $\in$ $\tilde{\lambda}_{\tilde{t}}$. Then

$$\tilde{\lambda}(x * (y * z)) \geq \tilde{t} \text{ and } \tilde{\lambda}(y) \geq \tilde{t}.$$

By condition (F), we have

$$\begin{aligned} \tilde{\lambda}(x * z) &\geq \tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5] \\ &\geq \tilde{t} \wedge \tilde{t} \wedge [0.5, 0.5] \\ &= \tilde{t} \wedge [0.5, 0.5] \\ &= \tilde{t}. \end{aligned}$$

Thus

$$\tilde{\lambda}(x * z) \geq \tilde{t},$$

that is,

$$x * z \in \tilde{\lambda}_{\tilde{t}}.$$

Therefore $\tilde{\lambda}_{\tilde{t}}$ is a quasi-associative ideal of X.

Conversely, assume that $\tilde{\lambda}$ is an interval valued fuzzy set of X such that $\tilde{\lambda}_{\tilde{t}}$ $(\neq \phi)$ is a quasi-associative ideal of X for all [0, 0] < $\tilde{t}$ $\leq$ [0.5, 0.5]. Let x $\in$ X be such that

$$\tilde{\lambda}(0) < \tilde{\lambda}(x) \wedge [0.5, 0.5].$$

Select [0, 0] < $\tilde{t}$ $\leq$ [0.5, 0.5] such that

$$\tilde{\lambda}(0) < \tilde{t} \leq \tilde{\lambda}(x) \wedge [0.5, 0.5].$$

Then x $\in$ $\tilde{\lambda}_{\tilde{t}}$ but 0 $\notin$ $\tilde{\lambda}_{\tilde{t}}$, a contradiction. Hence

$$\tilde{\lambda}(0) \geq \tilde{\lambda}(x) \wedge [0.5, 0.5].$$

Now assume that x, y, z $\in$ X such that

$$\tilde{\lambda}(x * z) < \tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5].$$

Select [0, 0] < $\tilde{t}$ $\leq$ [0.5, 0.5] such that

$$\tilde{\lambda}(x * z) < \tilde{t} \leq \tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5].$$

Then x * (y * z) and y are in $\tilde{\lambda}_{\tilde{t}}$ but x * z $\notin$ $\tilde{\lambda}_{\tilde{t}}$, a contradiction. Hence

$$\tilde{\lambda}(x * z) \geq \tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5].$$

This shows that $\tilde{\lambda}$ is an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X.

Theorem 4.7. The intersection of any family of $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of a BRK-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X.

Proof. Let $\{\tilde{\lambda}_i\}_{i \in I}$ be a family of $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of a BRK-algebra X and x $\in$ X. So

$$\tilde{\lambda}_i(0) \geq \tilde{\lambda}_i(x) \wedge [0.5, 0.5]$$

for all i $\in$ I. Thus

$$\begin{aligned}(\wedge_{i \in I} \tilde{\lambda}_i)(0) &= \wedge_{i \in I} (\tilde{\lambda}_i(0)) \\ &\geq \wedge_{i \in I} (\tilde{\lambda}_i(x) \wedge [0.5, 0.5]) \\ &= (\wedge_{i \in I} \tilde{\lambda}_i)(x) \wedge [0.5, 0.5].\end{aligned}$$

Thus

$$(\wedge_{i \in I} \tilde{\lambda}_i)(0) \geq (\wedge_{i \in I} \tilde{\lambda}_i)(x) \wedge [0.5, 0.5].$$

Let x, y, z $\in$ X. Since each $\tilde{\lambda}_i$ is an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X. So

$$\tilde{\lambda}_i(x * z) \geq \tilde{\lambda}_i(x * (y * z)) \wedge \tilde{\lambda}_i(y) \wedge [0.5, 0.5]$$

for all i $\in$ I. Thus

$$(\wedge_{i \in I} \tilde{\lambda}_i)(x * z) = \wedge_{i \in I} (\tilde{\lambda}_i(x * z))$$

$$\geq \bigwedge_{i \in I} (\tilde{\lambda}_i (x * (y * z)) \wedge \tilde{\lambda}_i (y) \wedge [0.5, 0.5])$$

$$= (\bigwedge_{i \in I} \tilde{\lambda}_i)(x * (y * z)) \wedge (\bigwedge_{i \in I} \tilde{\lambda}_i)(y) \wedge [0.5, 0.5].$$

Therefore

$$(\bigwedge_{i \in I} \tilde{\lambda}_i)(x * z) \geq (\bigwedge_{i \in I} \tilde{\lambda}_i)(x * (y * z)) \wedge (\bigwedge_{i \in I} \tilde{\lambda}_i)(y) \wedge [0.5, 0.5].$$

Hence, $\bigwedge_{i \in I} \tilde{\lambda}_i$ is an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X.

Theorem 4.8. The union of any family of $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of a BRK-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X.

Proof. Let $\{\tilde{\lambda}_i\}_{i \in I}$ be a family of $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of a BRK-algebra X and x ∈ X. So

$$\tilde{\lambda}_i (0) \geq \tilde{\lambda}_i (x) \wedge [0.5, 0.5]$$

for all i ∈ I. Thus

$$(\bigvee_{i \in I} \tilde{\lambda}_i)(0) = \bigvee_{i \in I} (\tilde{\lambda}_i (0))$$

$$\geq \bigvee_{i \in I} (\tilde{\lambda}_i (x) \wedge [0.5, 0.5])$$

$$= (\bigvee_{i \in I} \tilde{\lambda}_i)(x) \wedge [0.5, 0.5].$$

Thus

$$(\bigvee_{i \in I} \tilde{\lambda}_i)(0) \geq (\bigvee_{i \in I} \tilde{\lambda}_i)(x) \wedge [0.5, 0.5].$$

Let x, y, z ∈ X. Since each $\tilde{\lambda}_i$ is an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X. So

$$\tilde{\lambda}_i (x * z) \geq \tilde{\lambda}_i (x * (y * z)) \wedge \tilde{\lambda}_i (y) \wedge [0.5, 0.5]$$

for all i ∈ I. Thus

$$(\bigvee_{i \in I} \tilde{\lambda}_i) (x * z) = \bigvee_{i \in I} (\tilde{\lambda}_i (x * z))$$

$$\geq \bigvee_{i \in I} (\tilde{\lambda}_i (x * (y * z)) \wedge \tilde{\lambda}_i (y) \wedge [0.5, 0.5])$$

$$= (\bigvee_{i \in I} \tilde{\lambda}_i)(x * (y * z)) \wedge (\bigvee_{i \in I} \tilde{\lambda}_i)(y) \wedge [0.5, 0.5]$$

Therefore

$$(\bigvee_{i \in I} \tilde{\lambda}_i)(x * z) \geq (\bigvee_{i \in I} \tilde{\lambda}_i)(x * (y * z)) \wedge (\bigvee_{i \in I} \tilde{\lambda}_i)(y) \wedge [0.5, 0.5].$$

Hence, $\bigvee_{i \in I} \tilde{\lambda}_i$ is an $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal of X.

Definition 4.9. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is called an interval valued fuzzy quasi-associative ideal with thresholds $\tilde{\varepsilon}$ and $\tilde{\delta}$ of X, $[0, 0] < \tilde{\varepsilon}, \tilde{\delta} \leq [1, 1]$, if it satisfies (G) and (H), where

(G) $\tilde{\lambda}(0) \vee \tilde{\varepsilon} \geq \tilde{\lambda}(x) \wedge \tilde{\delta}$

(H) $\tilde{\lambda}(x * z) \vee \tilde{\varepsilon} \geq \tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y) \wedge \tilde{\delta}$

for all x, y, z ∈ X.

Example 4.10. Suppose X = {*0*, *a*, *b*, *c*} be a BRK-algebra with the following Cayley table:

$*$	*0*	*a*	*b*	*c*
0	*0*	*b*	*b*	*0*
a	*a*	*0*	*0*	*b*
b	*b*	*0*	*0*	*B*
c	*c*	*a*	*a*	*0*

We define an interval valued fuzzy set $\tilde{\lambda}$ by $\tilde{\lambda}$ (0) = [0.89, 0.89], $\tilde{\lambda}$ (*a*) = [0.38, 0.38], $\tilde{\lambda}$ (*b*) = [0.49, 0.49] and $\tilde{\lambda}$ (*c*) = [0.64, 0.64]. Simple calculations show that $\tilde{\lambda}$ is an interval valued fuzzy quasi-associative ideal of X with thresholds $\tilde{\varepsilon}$ = [0.98, 0.98] and $\tilde{\delta}$ = [0.50, 0.50].

5. Implication-based interval valued fuzzy quasi-associative ideal

In this section, we define the concept of implication-based interval valued fuzzy quasi-associative ideal in a BRK-algebra and investigate some related properties.

Fuzzy propositional calculus is an extension of the Aristotelean propositional calculus. In fuzzy propositional calculus the truth set is taken [0, 1] instead of {0, 1}, which is the truth set in Aristotelean propositional calculus. In fuzzy logic some of the operators, like $\wedge$, $\vee$, $\neg$, $\to$ can be defined by using truth tables. One can also use the extension principle to obtain the definitions of these operators.

In fuzzy logic the truth value of a fuzzy proposition P is denoted by [P]: In the following, we give fuzzy logic and its corresponding set theoretical notations, which we will use in the paper hereafter.

$[x \in \tilde{\lambda}] = \tilde{\lambda}(x)$

$[x \notin \tilde{\lambda}] = 1 - \tilde{\lambda}(x)$

$[P \wedge Q] = [P] \wedge [Q]$

$[P \vee Q] = [P] \vee [Q]$

$[P \to Q] = 1 \wedge (1 - [P] + [Q])$

$[\forall x P(x)] = \wedge [P(x)]$

$\models P$ if and only if $[P] = 1$ for all valuations.

Of course, various implication operators can be similarly defined. We consider in the following some important implication operators:

Name	Definition of Implication Operators
Early Zadeh	$I_m(x, y) = \max\{1 - x, \min\{x, y\}\}$
Lukasiewicz	$I_a(x, y) = \min\{1, 1 - x + y\}$
Standard Star (Godel)	$I_g(x, y) = \begin{cases} 1 & if\ x \le y, \\ y & if\ x > y, \end{cases}$
Contraposition of Godel	$I_{cg}(x, y) = \begin{cases} 1 & if\ x \le y, \\ 1 - x & if\ x > y, \end{cases}$
Gaines-Rescher	$I_{gr}(x, y) = \begin{cases} 1 & if\ x \le y, \\ 0 & if\ x > y, \end{cases}$
Kleene-Dienes	$I_b(x, y) = \max\{1 - x, y\}$

Where x is the degree of truth (or degree of membership) of the premise and y is the respective value for the consequence and I is the resulting degree of truth for the implication. The "quality" of these implication operators could be evaluated either by empirically or by axiomatically methods.

In the following definition, we consider the implication operators in the Lukasiewicz system of continuous-valued logic.

Definition 5.1. An interval valued fuzzy set $\tilde{\lambda}$ of a BRK-algebra X is called an interval valued fuzzifying quasi-associative ideal of X if it satisfies the conditions (I) and (J), where

(I) $\models [x \in \tilde{\lambda}] \to [0 \in \tilde{\lambda}]$,

(J) $\models [x * (y * z) \in \tilde{\lambda}] \wedge [y \in \tilde{\lambda}] \to [x * z \in \tilde{\lambda}]$,

for all x, y, z $\in$ X.

Clearly, Definition 5.1 is equivalent to Definition 2.5. Therefore an interval valued fuzzifying quasi-associative ideal is an ordinary interval valued fuzzy quasi-associative ideal.

Now we define implication-based interval valued fuzzy quasi-associative ideal in a BRK-algebra.

Definition 5.2. Let $\tilde{\lambda}$ be an interval valued fuzzy set in a BRK-algebra X and $[0, 0] < \tilde{t} \leq [1, 1]$ be a fixed number. Then $\tilde{\lambda}$ is called a $\tilde{t}$ -implication-based interval valued fuzzy quasi-associative ideal of X if it satisfies the conditions (K) and (L), where

(K) $\models_{\tilde{t}} [x \in \tilde{\lambda}] \to [0 \in \tilde{\lambda}]$,

(L) $\models_{\tilde{t}} [x * (y * z) \in \tilde{\lambda}] \wedge [y \in \tilde{\lambda}] \to [x * z \in \tilde{\lambda}]$,

for all x, y, z $\in$ X.

Corollary 5.3. Let I be an implication operator, $[0, 0] < \tilde{t} \leq [1, 1]$ be a fixed interval and $\tilde{\lambda}$ be an interval valued fuzzy set of X. Then $\tilde{\lambda}$ is a $\tilde{t}$ -implication-based interval valued fuzzy quasi-associative ideal of a BRK-algebra X if and only if the following conditions hold:

(M) $I(\tilde{\lambda}(x), \tilde{\lambda}(0)) \geq \tilde{t}$,

(N) $I(\tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y), \tilde{\lambda}(x * z)) \geq \tilde{t}$,

for all x, y, z $\in$ X.

Let $\tilde{\lambda}$ be an interval valued fuzzy set of a BRK-algebra X. Then we have the following results:

Theorem 5.4.

(O) Let $I = I_{gr}$. Then $\tilde{\lambda}$ is a [0.5, 0.5]-implication-based interval valued fuzzy quasi-associative ideal of a BRK-algebra X if and only if $\tilde{\lambda}$ is an interval valued fuzzy quasi-associative ideal with thresholds ($\tilde{\varepsilon}$ = [0, 0], $\tilde{\delta}$ = [0.5, 0.5]) of X.

(P) Let $I = I_g$. Then $\tilde{\lambda}$ is a [0.5, 0.5]-implication-based interval valued fuzzy quasi-associative ideal of a BRK-algebra X if and only if $\tilde{\lambda}$ is an interval valued fuzzy quasi-associative ideal with thresholds ($\tilde{\varepsilon}$ = [0, 0], $\tilde{\delta}$ = [0.5, 0.5]) of X.

(Q) Let $I = I_{cg}$. Then $\tilde{\lambda}$ is a [0.5, 0.5]-implication-based interval valued fuzzy quasi-associative ideal of a BRK-algebra X if and only if $\tilde{\lambda}$ is an interval valued fuzzy quasi-associative ideal with thresholds ($\tilde{\varepsilon}$ = [0, 0], $\tilde{\delta}$ = [0.5, 0.5]) of X.

Proof. We only prove (P) and the proofs of (O) and (Q) are similar.

Let $\tilde{\lambda}$ be a [0.5, 0.5]-implication-based interval valued fuzzy quasi-associative ideal of X. Then by Corollary 5.3, we have

(*a*) $I_g(\tilde{\lambda}(x), \tilde{\lambda}(0)) \geq [0.5, 0.5]$ and

(b) $I_g(\tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y), \tilde{\lambda}(x * z)) \geq [0.5, 0.5]$.

From (*a*), we have

$$\tilde{\lambda}(0) \geq \tilde{\lambda}(x) \text{ or } \tilde{\lambda}(x) > \tilde{\lambda}(0) \geq [0.5, 0.5].$$

Thus

$$\tilde{\lambda}(0) \geq \tilde{\lambda}(x) \wedge [0.5, 0.5].$$

It follows that

$$\tilde{\lambda}(0) \vee [0, 0] = \tilde{\lambda}(0) \geq \tilde{\lambda}(x) \wedge [0.5, 0.5].$$

From (b), we have

$$\tilde{\lambda}(x * z) \geq \tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y),$$

or

$$\tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y) > \tilde{\lambda}(x * z) \geq [0.5, 0.5].$$

It follows that

$$\tilde{\lambda}(x * z) \vee [0, 0] = \tilde{\lambda}(x * z) \geq \tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5].$$

This shows that $\tilde{\lambda}$ is an interval valued fuzzy quasi-associative ideal with thresholds ($\tilde{\varepsilon} = [0, 0]$, $\tilde{\delta} = [0.5, 0.5]$) of X.

Conversely, assume that $\tilde{\lambda}$ is an interval valued fuzzy quasi-associative ideal with thresholds ($\tilde{\varepsilon} = [0, 0]$, $\tilde{\delta} = [0.5, 0.5]$) of X, then we have (*a*)

$$\tilde{\lambda}(0) = \tilde{\lambda}(0) \vee [0, 0] \geq \tilde{\lambda}(x) \wedge [0.5, 0.5]$$

and (b)

$$\tilde{\lambda}(x * z) = \tilde{\lambda}(x) \vee [0, 0] \geq \tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5].$$

From (*a*), if $\tilde{\lambda}(x) \wedge [0.5, 0.5] = \tilde{\lambda}(x)$, then

$$I_g(\tilde{\lambda}(x), \tilde{\lambda}(0)) = [1, 1] \geq [0.5, 0.5].$$

Otherwise

$$I_g(\tilde{\lambda}(x), \tilde{\lambda}(0)) \geq [0.5, 0.5].$$

From (b), if

$$\tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5] = \tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y),$$

then

$$I_g(\tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y), \tilde{\lambda}(x * z)) = [1, 1] \geq [0.5, 0.5].$$

Otherwise

$$I_g(\tilde{\lambda}(x * (y * z)) \wedge \tilde{\lambda}(y), \tilde{\lambda}(x * z)) \geq [0.5, 0.5].$$

Therefore, $\tilde{\lambda}$ is a [0.5, 0.5]-implication-based interval valued fuzzy quasi-associative ideal of X.

6. Conclusion

To investigate the structure of an algebraic system, we see that interval valued fuzzy quasi-associative ideal with special properties always play a fundamental role.

In this paper, we define the concept of $(\in, \in \vee q)$-interval valued fuzzy quasi-associative ideal in BRK-algebra and investigate some related properties. The concepts of implication-based interval valued fuzzy ideal, implication-based interval valued fuzzy quasi-associative ideal and implication operators in Lukasiewicz system of continuous-valued logic in BRK-algebra are introduced.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of interval valued fuzzy BRK-algebras and their applications in other branches of algebra. In future study of interval valued fuzzy BRK-algebras, perhaps the following topics are worth to be considering:

(1) To characterize other classes of BRK-algebras by using this notion;
(2) To apply this notion to some other algebraic structures;
(3) To consider these results to some possible applications in computer sciences and information systems in the future.

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