Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2020, 7(4):219-230**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

On interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideals in BCH-algebras

Muhammad Zulfiqar[1], Dilawar Nazir[2], Fozia Azhar[3]

Department of Mathematics, GC University Lahore, Pakistan
[1]mzulfiqarshafi@hotmail.com, [2]dilawarnazir@gmail.com, [3]foziafozi11@yahoo.in

Abstract In this paper, the concepts of interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideals in BCH-algebras are introduced and related properties are investigated. Also we define the concept of interval valued anti fuzzy fantastic ideals with thresholds in BCH-algebras and investigate some of its properties.

Keywords BCH-algebras; anti fuzzy fantastic ideal; $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal; interval valued anti fuzzy fantastic ideals with thresholds

1. Introduction

The concept of a BCH-algebra was introduced by Hu and Li [9] and gave examples of proper BCH-algebras [10]. In [22], Saeid et al. initiated notion of fantastic in a BCH-algebra and studied its properties (see [27]). The fundamental concept of fuzzy set given by Zadeh in his pioneering paper [24], of 1965 provides a natural framework for generalizing some of the basic notions of algebra. Extensive applications of fuzzy set theory have been found in various fields, for example, artificial intelligence, computer science, control engineering, expert system, management science, operation research and many others. In [1], Ahmad studied some classification of BCH-algebras.

The notion of anti fuzzy subgroups of groups was introduced by Biswas [6]. In [2], Al-shehri defined the concept of generalized doubt fuzzy ideal in BCI-algebras and studied some of their properties.

Rosenfeld formulated the elements of a theory of fuzzy groups [20], where he introduced the fuzzy subgroup of a group. A new type of fuzzy subgroup, which is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced by Bhakat and Das [5] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [19]. Murali [18] proposed the definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. It was found that the most viable generalization of Rosenfeld's fuzzy subgroup is $(\in, \in \vee q)$-fuzzy subgroup. In [3-4], Bhakat introduced the concept of $(\in \vee q)$-level subsets, $(\in, \in \vee q)$-fuzzy normal, quasi-normal and maximal subgroups. Many researchers utilized these concepts to generalize some concepts of algebra [7-8, 26]. Jun defined the concept of (α, β)-fuzzy subalgebras (ideals) in BCK/BCI-algebras [11-12].

The theory of interval-valued fuzzy sets was proposed forty year ago as a natural extension of fuzzy sets. In [25], Interval valued fuzzy set was introduced by Zadeh, where the values of the membership function is interval of numbers instead of the number. An interval-valued fuzzy set is defined by an interval-valued membership function. In [13], Latha et al. initiated the notion of interval-valued (α, β)-fuzzy subgroups. In [14], Ma et al. introduced the concept of interval-valued $(\in, \in \vee q)$-fuzzy ideals of pseudo MV-algebras. In [15-16], Ma et al. studied $(\in, \in \vee q)$-interval-valued fuzzy ideals in BCI-algebras. Mostafa et al. initiated interval-valued fuzzy KU-ideals in KU-algebras [17]. In [21], Saeid defined the concept of interval-valued fuzzy BG-algebras.

Zhan et al. [26] initiated the notion of interval-valued $(\in, \in \vee q)$-fuzzy filters of pseudo BL-algebras and investigated some of their related properties. Recently Zulfiqar [28], introduced the concept of $(\in, \in \vee q_k)$-fuzzy fantastic ideals in BCI-algebras and investigated its properties.

In this paper, we show that an interval valued fuzzy set $\tilde{\lambda}$ of a BCH-algebra X is an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X if and only if the set $\tilde{\lambda}^{\tilde{t}}$ $(\neq \phi)$ is a fantastic ideal of X, for all $\left[\frac{1-k}{2}, \frac{1-k}{2}\right] < \tilde{t} \leq [1, 1]$. We prove that if $\tilde{\lambda}$ and $\tilde{\eta}$ be an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideals of BCH-algebra X, then ($\tilde{\lambda} \otimes \tilde{\eta}$) is an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X × X. Finally, we prove that if $\zeta : X \to Y$ is an epimomorphism of BCH-algebra and let $\tilde{\lambda}$ and $\tilde{\eta}$ be an interval valued anti fuzzy fantastic ideal of X and Y respectively, then $\zeta(\tilde{\lambda})$ defined by

$$\zeta(\tilde{\lambda})(y) = \inf\{\tilde{\lambda}(x) \mid \zeta(x) = y, \text{ for all } y \in Y\}$$

and $\zeta^{-1}(\tilde{\eta})$ defined by

$$\zeta^{-1}(\tilde{\eta})(x) = \tilde{\eta}(\zeta(x)),$$

for all $x \in X$ are interval valued anti fuzzy fantastic ideals of Y and X, respectively. Furthermore, if $\tilde{\lambda}$ and $\tilde{\eta}$ are interval valued anti fuzzy fantastic ideal with thresholds ($\tilde{\varepsilon}, \tilde{\delta}$), then also $\zeta(\tilde{\lambda})$ and $\zeta^{-1}(\tilde{\eta})$ are interval valued anti fuzzy fantastic ideal with thresholds ($\tilde{\varepsilon}, \tilde{\delta}$).

2. Preliminaries

Throughout this paper, X always denotes a BCH-algebra unless otherwise specified. We include some basic definitions and preliminary facts about BCH-algebras which are essential for our results.

Definition 2.1. [9] By a BCH-algebra, we mean an algebra $(X, *, 0)$ of type (2, 0) satisfying the axioms:

(BCH-I) $x * x = 0$

(BCH-II) $x * y = 0$ and $y * x = 0$ imply $x = y$

(BCH-III) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

We can define a partial order "≤" on X by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.2. [27] In any BCH-algebra X, the following are true:

(1) $x * (x * y) \leq y$

(2) $0 * (x * y) = (0 * x) * (0 * y)$

(3) $x * 0 = x$

(4) $x \leq 0$ implies $x = 0$

for all $x, y \in X$.

Definition 2.3. [23] A non-empty subset I of a BCH-algebra X is called an ideal of X if it satisfies (I1) and (I2), where

(I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$,

for all $x, y \in X$.

Definition 2.4. [27] A non-empty subset I of a BCH-algebra X is called a fantastic ideal of X if it satisfies (I1) and (I3), where

(I1) $0 \in I$,

(I3) $(x * y) * z \in I$ and $z \in I$ imply $x * (y * (y * x)) \in I$,
for all $x, y, z \in X$.

A fuzzy set λ of a universe X is a function from X to the unit closed interval [0, 1], that is $\lambda : X \to [0, 1]$. Let λ be a fuzzy set of a BCH-algebra X. For $t \in [0, 1]$, the set

$$\lambda^t = \{x \in X \mid \lambda(x) \leq t\}$$

is called a lower t-level cut of λ.

Definition 2.5. [23] A fuzzy set λ of a BCH-algebra X is called a fuzzy ideal of X if it satisfies (F1) and (F2), where

(F1) $\lambda(0) \geq \lambda(x)$,
(F2) $\lambda(x) \geq \lambda(x * y) \wedge \lambda(y)$,
for all $x, y \in X$.

Definition 2.6. [27] A fuzzy set λ of a BCH-algebra X is called a fuzzy fantastic ideal of X if it satisfies (F1) and (F3), where

(F1) $\lambda(0) \geq \lambda(x)$,
(F3) $\lambda(x * (y * (y * x))) \geq \lambda((x * y) * z) \wedge \lambda(z)$,
for all $x, y, z \in X$.

3. Interval valued anti fuzzy ideals

An interval valued fuzzy set $\tilde{\lambda}$ defined on X is given by [15]

$$\tilde{\lambda} = \{(x, [\lambda^-(x), \lambda^+(x)])\},$$

for all $x \in X$ (briefy, denoted by $\tilde{\lambda} = [\lambda^-, \lambda^+]$) where λ^- and λ^+ are two fuzzy sets in X such that $\lambda^-(x) \leq \lambda^+(x)$, for all $x \in X$.

Let $\tilde{\lambda}(x) = [\lambda^-(x), \lambda^+(x)]$, for all $x \in X$ and let E[0, 1] denotes the family of all closed subintervals of [0, 1]. If $\lambda^-(x) = \lambda^+(x) = c$ (say) where $0 \leq c \leq 1$, then we have $\tilde{\lambda}(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to E[0, 1]. Thus $\tilde{\lambda}(x) \in E[0, 1]$, for all $x \in X$ and therefore the interval valued fuzzy set $\tilde{\lambda}$ is given by

$$\tilde{\lambda} = \{(x, [\lambda^-(x), \lambda^+(x)])\}, \text{ for all } x \in X$$

where $\tilde{\lambda} : X \to E[0, 1]$.

Then, E[0, 1] with $\leq$ is a complete lattice, with $\wedge$ = rmin, $\vee$ = rmax, $\tilde{0} = [0, 0]$ and $\tilde{1} = [1, 1]$ being its least element and the greatest element, respectively.

Let $\tilde{\lambda}$ be an interval valued fuzzy set. Then, for every $[0, 0] < \tilde{t} \leq [1, 1]$, the set

$$\tilde{\lambda}^{\tilde{t}} = \{x \in X \mid \tilde{\lambda}(x) \leq \tilde{t}\}$$

is called the interval valued lower t-level cut of $\tilde{\lambda}$.

Definition 3.1. An interval valued fuzzy set $\tilde{\lambda}$ of a BCH-algebra X is called an interval valued anti fuzzy ideal of X if it satisfies (IVAF1) and (IVAF2), where

(IVAF1) $\tilde{\lambda}(0) \leq \tilde{\lambda}(x)$,
(IVAF2) $\tilde{\lambda}(x) \leq \lambda(x * y) \vee \tilde{\lambda}(y)$,
for all $x, y \in X$.

Definition 3.2. An interval valued fuzzy set $\tilde{\lambda}$ of a BCK-algebra X is called an interval valued anti fuzzy fantastic ideal of X if it satisfies (IVAF1) and (IVAF3), where

(IVAF1) $\tilde{\lambda}(0) \leq \tilde{\lambda}(x)$,

(IVAF3) $\tilde{\lambda}(x * (y * (y * x))) \leq \tilde{\lambda}((x * y) * z) \vee \tilde{\lambda}(z)$,
for all x, y, z ∈ X.

Definition 3.3. An interval valued fuzzy set $\tilde{\lambda}$ of a BCH-algebra X of the form

$$\tilde{\lambda}(y) = \begin{cases} \tilde{t}\ (\neq [0, 0]) & \text{if } y = x, \\ [0, 0] & \text{if } y \neq x, \end{cases}$$

is said to be the interval valued fuzzy point with support x and value $\tilde{t}$ and is denoted by $x_{\tilde{t}}$. An interval value fuzzy point $x_{\tilde{t}}$ is said to belong to (resp. be quasicoincident with) an interval valued fuzzy set $\tilde{\lambda}$, written as $x_{\tilde{t}} \in \tilde{\lambda}$ (resp. $x_{\tilde{t}}\, q\, \tilde{\lambda}$) if $\tilde{\lambda}(x) \geq \tilde{t}$ (resp. $\tilde{\lambda}(x) + \tilde{t} > [1, 1]$).

If $x_{\tilde{t}} \in \tilde{\lambda}$ or (resp. and) $x_{\tilde{t}}\, q\, \tilde{\lambda}$, then we write $x_{\tilde{t}} \in \vee q\, \tilde{\lambda}$ (resp. $x_{\tilde{t}} \in \wedge\, q\, \tilde{\lambda}$).

4. Interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideals

In this section, we introduce the concept of interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideals in BCH-algebras and investigate some of their related properties.

Let k denote an arbitrary element of $0 \leq k < 1$ unless otherwise specified. By $x_{\tilde{t}}\, q_k\, \tilde{\lambda}$, we mean $\tilde{\lambda}(x) + \tilde{t} + k > 1$, $[0, 0] < \tilde{t} \leq \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$. The notation $x_{\tilde{t}} \in \vee\, q_k\, \tilde{\lambda}$ means that $x_{\tilde{t}} \in \tilde{\lambda}$ or $x_{\tilde{t}}\, q_k\, \tilde{\lambda}$.

Definition 4.1. An interval valued fuzzy set $\tilde{\lambda}$ of a BCH-algebra X is called an interval valued $(\in, \in \vee q_k)$-anti fuzzy ideal of X if it satisfies (A) and (B), where

(A) $\tilde{\lambda}(0) \leq \tilde{\lambda}(x) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$,

(B) $\tilde{\lambda}(x) \leq \tilde{\lambda}(x * y) \vee \tilde{\lambda}(y) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$,

for all x, y ∈ X.

Definition 4.2. An interval valued fuzzy set $\tilde{\lambda}$ of a BCH-algebra X is called an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X if it satisfies (A) and (C), where

(A) $\tilde{\lambda}(0) \leq \tilde{\lambda}(x) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$,

(C) $\tilde{\lambda}(x * (y * (y * x))) \leq \tilde{\lambda}((x * y) * z) \vee \tilde{\lambda}(z) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$,

for all x, y, z ∈ X.

Example 4.3. Let X = {0, *a*, b, c} in which ∗ is defined as follows:

∗	0	*a*	b	c
0	0	0	0	0

a	*a*	0	*a*	0
b	b	b	0	0
c	c	b	*a*	0

Then (X, ∗, 0) is a BCH-algebra. We define an interval valued fuzzy set $\tilde{\lambda}$ in X by $\tilde{\lambda}$ (0) = [0.44, 0.45], $\tilde{\lambda}$ (*a*) = $\tilde{\lambda}$ (b) = [0.62, 0.67], $\tilde{\lambda}$ (c) = [0.32, 0.37]. By simple calculations show that $\tilde{\lambda}$ is an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X for k = 0.

Theorem 4.4. An interval valued fuzzy set $\tilde{\lambda}$ of a BCH-algebra X is an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X if and only if the set $\tilde{\lambda}^{\tilde{t}}$ ($\neq \phi$) is a fantastic ideal of X, for all $\left[\frac{1-k}{2}, \frac{1-k}{2}\right] < \tilde{t} \leq [1, 1]$.

Proof. Let $\tilde{\lambda}$ be an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X. Then

$$\tilde{\lambda}(0) \leq \tilde{\lambda}(x) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right].$$

For all x ∈ $\tilde{\lambda}^{\tilde{t}}$, we have $\tilde{\lambda}$ (x) ≤ $\tilde{t}$. Thus

$$\tilde{\lambda}(0) \leq \tilde{t} \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right] = \tilde{t}$$

So that 0 ∈ $\tilde{\lambda}^{\tilde{t}}$. Let x, y, z ∈ X be such that

$$(x * y) * z \in \tilde{\lambda}^{\tilde{t}} \text{ and } z \in \tilde{\lambda}^{\tilde{t}}.$$

Then

$$\tilde{\lambda}((x * y) * z) \leq \tilde{t} \text{ and } \tilde{\lambda}(z) \leq \tilde{t}.$$

Then by Definition 5.1(C), we have

$$\tilde{\lambda}(x * (y * (y * x))) \leq \tilde{\lambda}((x * y) * z) \vee \tilde{\lambda}(z) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$
$$\leq \tilde{t} \vee \tilde{t} \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$
$$\leq \tilde{t} \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$
$$= \tilde{t}$$

This implies that

$$x * (y * (y * x)) \in \tilde{\lambda}^{\tilde{t}}$$

and so $\tilde{\lambda}^{\tilde{t}}$ is a fantastic ideal of X.

Conversely, assume that $\tilde{\lambda}^{\tilde{t}}$ ($\neq \phi$) is a fantastic ideal of X for all $\left[\frac{1-k}{2}, \frac{1-k}{2}\right] < \tilde{t} \leq [1, 1]$ and there exists x_1 ∈ X such that

$$\tilde{\lambda}(0) > \tilde{\lambda}(x_1) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right].$$

Hence $\tilde{\lambda}(0) > \tilde{\lambda}(x_1)$. Setting

$$\tilde{s}_0 = \frac{1}{2}\{\tilde{\lambda}(0) + \tilde{\lambda}(x_1)\}.$$

Then

$$[0, 0] \le \tilde{\lambda}(x_1) < \tilde{s}_0 < \tilde{\lambda}(0) \le [1, 1].$$

It follows that

$$x_1 \in \tilde{\lambda}^{\tilde{s}_0} \text{ and } \tilde{\lambda}^{\tilde{s}_0} \ne \phi.$$

Since $\tilde{\lambda}^{\tilde{s}_0}$ is a fantastic ideal of X, so $0 \in \tilde{\lambda}^{\tilde{s}_0}$. Thus

$$\tilde{\lambda}(0) \le \tilde{s}_0,$$

a contradiction. Hence

$$\tilde{\lambda}(0) \le \tilde{\lambda}(x) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right],$$

for all $x \in X$. Suppose there exists $x_1, y_1, z_1 \in X$ be such that

$$\tilde{\lambda}(x_1 * (y_1 * (y_1 * x_1))) > \tilde{\lambda}((x_1 * y_1) * z_1) \vee \tilde{\lambda}(z_1) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right].$$

We have

$$\tilde{\lambda}(x_1 * (y_1 * (y_1 * x_1))) > \tilde{\lambda}((x_1 * y_1) * z_1) \vee \tilde{\lambda}(z_1).$$

Let

$$\tilde{s}_0 = \frac{1}{2}\{\tilde{\lambda}(x_1 * (y_1 * (y_1 * x_1))) + (\tilde{\lambda}((x_1 * y_1) * z_1) \vee \tilde{\lambda}(z_1))\}.$$

Then

$$\tilde{s}_0 < \tilde{\lambda}(x_1 * (y_1 * (y_1 * x_1))) \text{ and } [0, 0] \le \tilde{\lambda}((x_1 * y_1) * z_1) \vee \tilde{\lambda}(z_1) < \tilde{s}_0 \le [1, 1].$$

Thus

$$\tilde{s}_0 > \tilde{\lambda}((x_1 * y_1) * z_1) \text{ and } \tilde{s}_0 > \tilde{\lambda}(z_1).$$

This implies that

$$(x_1 * y_1) * z_1 \in \tilde{\lambda}^{\tilde{s}_0} \text{ and } z_1 \in \tilde{\lambda}^{\tilde{s}_0}.$$

As $\tilde{\lambda}^{\tilde{s}_0}$ is a fantastic ideal of X, it follows that

$$x_1 * (y_1 * (y_1 * x_1)) \in \tilde{\lambda}^{\tilde{s}_0}.$$

Thus

$$\tilde{\lambda}(x_1 * (y_1 * (y_1 * x_1))) \le \tilde{s}_0,$$

a contradiction. Hence

$$\tilde{\lambda}(x * (y * (y * x))) \le \tilde{\lambda}((x * y) * z) \vee \tilde{\lambda}(z) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right].$$

Therefore, $\tilde{\lambda}$ is an interval valued $(\in, \in \vee\, q_k)$-anti fuzzy fantastic ideal of X.

Next, we characterize of an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal in BCH-algebras.

Theorem 4.5. Let $\tilde{\lambda}$ be an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of a BCH-algebra X. Then the following inequality holds:

(D) $\tilde{\lambda}(x * (y * (y * x))) \leq \tilde{\lambda}(x * y) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$.

Proof. Let $\tilde{\lambda}$ be an $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X. Then by taking z = 0 in (C), we have

$$\tilde{\lambda}(x * (y * (y * x))) \leq \tilde{\lambda}((x * y) * 0) \vee \tilde{\lambda}(0) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$

$$\leq \tilde{\lambda}(x * y) \vee \tilde{\lambda}(0) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right] \quad \text{(by Proposition 2.2(3))}$$

By using condition (A) of Definition 5.1, we get

$$\tilde{\lambda}(x * (y * (y * x))) \leq \tilde{\lambda}(x * y) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right].$$

Theorem 4.6. Every $(\in, \in \vee q_k)$-anti fuzzy ideal $\tilde{\lambda}$ of a BCH-algebra X satisfying the condition (D) of the Theorem 4.5 is an $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X.

Proof. Suppose $\tilde{\lambda}$ is an $(\in, \in \vee q_k)$-anti fuzzy ideal of X. For any x, y in X, by conditions (D) of Theorem 4.5, we have

$$\tilde{\lambda}(x * (y * (y * x))) \leq \tilde{\lambda}(x * y) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right].$$

Since $\tilde{\lambda}$ is an $(\in, \in \vee q_k)$-anti fuzzy ideal of X, we have

$$\tilde{\lambda}(x * (y * (y * x))) \leq \tilde{\lambda}((x * y) * z) \vee \tilde{\lambda}(z) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right] \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$

$$\leq \tilde{\lambda}((x * y) * z) \vee \tilde{\lambda}(z) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$

Therefore $\tilde{\lambda}$ is an $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X.

Theorem 4.7. Let $\tilde{\lambda}$ be an $(\in, \in \vee q_k)$-anti fuzzy ideal of a BCH-algebra X. Then $\tilde{\lambda}$ is an $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X if and only if $\tilde{\lambda}$ satisfies the condition

$$\tilde{\lambda}(x * (y * (y * x))) \leq \tilde{\lambda}(x * y) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right] \quad (1)$$

Proof. Suppose $\tilde{\lambda}$ is an $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X. It follows from Theorem 4.5.

Conversely, assume that $\tilde{\lambda}$ satisfies (1). As $\tilde{\lambda}$ is an $(\in, \in \vee q_k)$-anti fuzzy ideal of X, hence

$$\tilde{\lambda}(x * y) \leq \tilde{\lambda}((x * y) * z) \vee \tilde{\lambda}(z) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right] \quad (2)$$

for all x, y, z $\in$ X. Combining (1) and (2), we get

$$\tilde{\lambda}(x * (y * (y * x))) \leq \tilde{\lambda}((x * y) * z) \vee \tilde{\lambda}(z) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right] \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$

$$\leq \tilde{\lambda}((x * y) * z) \vee \tilde{\lambda}(z) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$

Hence $\tilde{\lambda}$ is an $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X.

Definition 4.8. Let $\tilde{\lambda}$ and $\tilde{\eta}$ be interval valued fuzzy subsets of a set S. Then interval valued anti Cartesian product of $\tilde{\lambda}$ and $\tilde{\eta}$ is defined by

$$(\tilde{\lambda} \otimes \tilde{\eta})(x, y) = \tilde{\lambda}(x) \vee \tilde{\eta}(y) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right],$$

for all x, y $\in$ S.

Theorem 4.9. Let $\tilde{\lambda}$ and $\tilde{\eta}$ be an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideals of BCH-algebra X. Then $(\tilde{\lambda} \otimes \tilde{\eta})$ is an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X $\times$ X.

Proof. Let (x, y) $\in$ X $\times$ X. Then

$$(\tilde{\lambda} \otimes \tilde{\eta})(0, 0) = \tilde{\lambda}(0) \vee \tilde{\eta}(0) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$

$$\leq (\tilde{\lambda}(x) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]) \vee (\tilde{\eta}(y) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$

$$= \tilde{\lambda}(x) \vee \tilde{\eta}(y) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$

$$= (\tilde{\lambda} \otimes \tilde{\eta})(x, y)$$

Suppose $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$. Then

$$\{(\tilde{\lambda} \otimes \tilde{\eta})(((x_1, x_2) * (y_1, y_2)) * (z_1, z_2)) \vee (\tilde{\lambda} \otimes \tilde{\eta})(z_1, z_2) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]\}$$

$$= \{(\tilde{\lambda} \otimes \tilde{\eta})((x_1 * y_1, x_2 * y_2) * (z_1, z_2)) \vee (\tilde{\lambda} \otimes \tilde{\eta})(z_1, z_2) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]\}$$

$$= \{(\tilde{\lambda} \otimes \tilde{\eta})((x_1 * y_1) * z_1), (x_2 * y_2) * z_2)) \vee (\tilde{\lambda} \otimes \tilde{\eta})(z_1, z_2) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]\}$$

$$= \{\tilde{\lambda}((x_1 * y_1) * z_1) \vee \tilde{\eta}((x_2 * y_2) * z_2) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]\} \vee \{\tilde{\lambda}(z_1) \vee \tilde{\eta}(z_2) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]\} \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$

$$= \{\tilde{\lambda}((x_1 * y_1) * z_1) \vee \tilde{\lambda}(z_1) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]\} \vee \{\tilde{\eta}((x_2 * y_2) * z_2) \vee \tilde{\eta}(z_2) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]\} \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$

$$\leq \tilde{\lambda}(x_1 * (y_1 * (y_1 * x_1))) \vee \tilde{\eta}(x_2 * (y_2 * (y_2 * x_2))) \vee \left[\frac{1-k}{2}, \frac{1-k}{2}\right]$$

(by Definition 4.2 (C))

$= (\tilde{\lambda} \otimes \tilde{\eta})(x_1 * (y_1 * (y_1 * x_1)), x_2 * (y_2 * (y_2 * x_2)))$ (by Definition 4.8)

Therefore, $(\tilde{\lambda} \otimes \tilde{\eta})$ is an interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideal of X × X.

Definition 4.10. Let $[0, 0] \leq \tilde{\varepsilon} < \tilde{\delta} \leq [1, 1]$ and $\tilde{\varepsilon} < \tilde{\delta}$. Then an interval valued fuzzy set $\tilde{\lambda}$ of a BCH-algebra X is called an interval valued anti fuzzy fantastic ideal with thresholds $(\tilde{\varepsilon}, \tilde{\delta})$ of X if it satisfies (E) and (F), where

(E) $\tilde{\lambda}(0) \wedge \tilde{\delta} \leq \tilde{\lambda}(x) \vee \tilde{\varepsilon}$,

(F) $\tilde{\lambda}(x * (y * (y * x))) \wedge \tilde{\delta} \leq \tilde{\lambda}((x * y) * z) \vee \tilde{\lambda}(z) \vee \tilde{\varepsilon}$,

for all x, y, z ∈ X.

Example 4.11. Let X = {0, 1, 2, 3, 4} be a BCH-algebra with the following Cayley table:

∗	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let $\tilde{\lambda}$ be an interval valued fuzzy set in X defined by $\tilde{\lambda}(0) = [0.51, 0.57]$, $\tilde{\lambda}(1) = [0, 0]$, $\tilde{\lambda}(2) = [1, 1]$ and $\tilde{\lambda}(3) = \tilde{\lambda}(4) = [0.82, 0.89]$. By simple calculations show that $\tilde{\lambda}$ is an interval valued anti fuzzy fantastic ideal with thresholds $\tilde{\varepsilon} = [0.51, 0.57]$ and $\tilde{\delta} = [0.82, 0.89]$ of X.

Theorem 4.12. An interval valued fuzzy set $\tilde{\lambda}$ of a BCH-algebra X is an interval valued anti fuzzy fantastic ideal with thresholds $(\tilde{\varepsilon}, \tilde{\delta})$ of X if and only if $\tilde{\lambda}^{\tilde{t}}$ $(\neq \phi)$ is a fantastic ideal of X for all $\tilde{\varepsilon} \leq \tilde{t} < \tilde{\delta}$.

Proof. Let $\tilde{\lambda}$ be an interval valued anti fuzzy fantastic ideal with thresholds $(\tilde{\varepsilon}, \tilde{\delta})$ of X and $\tilde{\varepsilon} \leq \tilde{t} < \tilde{\delta}$. Let $x \in \tilde{\lambda}^{\tilde{t}}$. Then

$$\tilde{\lambda}(x) \leq \tilde{t}.$$

So

$$\begin{aligned} \tilde{\lambda}(0) \wedge \delta &\leq \tilde{\lambda}(x) \vee \tilde{\varepsilon} \\ &\leq \tilde{t} \vee \tilde{\varepsilon} \\ &\leq \tilde{t} \\ &< \tilde{\varepsilon}. \end{aligned}$$

Thus

$$\tilde{\lambda}(0) \leq \tilde{t}.$$

We get $0 \in \tilde{\lambda}^{\tilde{t}}$. Let $(x * y) * z \in \tilde{\lambda}^{\tilde{t}}$ and $z \in \tilde{\lambda}^{\tilde{t}}$. Then

$$\tilde{\lambda}((x * y) * z) \leq \tilde{t} \text{ and } \tilde{\lambda}(z) \leq \tilde{t}.$$

Now we have

$$\begin{aligned} \tilde{\lambda}(x * (y * (y * x))) \wedge \tilde{\delta} &\leq \tilde{\lambda}((x * y) * z) \vee \tilde{\lambda}(z) \vee \tilde{\varepsilon} \\ &\leq \tilde{t} \vee \tilde{t} \vee \tilde{\varepsilon} \\ &\leq \tilde{t} \vee \tilde{\varepsilon} \\ &\leq \tilde{t} \end{aligned}$$

$$< \tilde{\varepsilon}$$

This implies that

$$\tilde{\lambda}\,(x * (y * (y * x))) < \tilde{t}\ ,$$

and so

$$x * (y * (y * x)) \in \tilde{\lambda}^{\tilde{t}}\ .$$

Hence $\tilde{\lambda}^{\tilde{t}}$ is a fantastic ideal of X.

Conversely, let $\tilde{\lambda}$ be an interval valued fuzzy set of X such that $\tilde{\lambda}^{\tilde{t}}$ $(\neq \phi)$ is a fantastic ideal of X for all $\tilde{\varepsilon} \leq \tilde{t} < \tilde{\delta}$. It is easy to verify that

$$\tilde{\lambda}\,(0) \wedge \tilde{\delta} \leq \tilde{\lambda}\,(x) \vee \tilde{\varepsilon}$$

for all x ∈ X. If there exist x, y, z ∈ X such that

$$\tilde{\lambda}\,(x * (y * (y * x))) \wedge \tilde{\delta} > \tilde{\lambda}\,((x * y) * z) \vee \tilde{\lambda}\,(z) \vee \tilde{\varepsilon} = \tilde{t}\ .$$

Then

$$\tilde{\varepsilon} \leq \tilde{t} < \tilde{\delta},\ \tilde{\lambda}\,(x * (y * (y * x))) > \tilde{t}\ ,\ (x * y) * z \in \tilde{\lambda}^{\tilde{t}} \text{ and } z \in \tilde{\lambda}^{\tilde{t}}\ .$$

So

$$x * (y * (y * x)) \in \tilde{\lambda}^{\tilde{t}}\ .$$

This implies that

$$\tilde{\lambda}_F\,(x * (y * (y * x))) \leq \tilde{t}\ .$$

This contradicts with

$$\tilde{\lambda}\,(x * (y * (y * x))) > \tilde{t}\ .$$

Thus

$$\tilde{\lambda}\,(x * (y * (y * x))) \wedge \tilde{\delta} \leq \tilde{\lambda}\,((x * y) * z) \vee \tilde{\lambda}\,(z) \vee \tilde{\varepsilon}\ ,$$

for all x, y, z ∈ X. Therefore $\tilde{\lambda}$ is an interval valued anti fuzzy fantastic ideal with thresholds $(\tilde{\varepsilon}, \tilde{\delta})$ of X.

Theorem 4.13. Let $\zeta : X \rightarrow Y$ be an epimomorphism of BCH-algebra and let $\tilde{\lambda}$ and $\tilde{\eta}$ be an interval valued anti fuzzy fantastic ideal of X and Y respectively. Then $\zeta(\tilde{\lambda})$ defined by

$$\zeta(\tilde{\lambda})(y) = \inf\{\tilde{\lambda}\,(x) \mid \zeta(x) = y, \text{ for all } y \in Y\}$$

and $\zeta^{-1}(\tilde{\eta})$ defined by

$$\zeta^{-1}(\tilde{\eta})\,(x) = \tilde{\eta}\,(\zeta(x)),$$

for all x ∈ X are interval valued anti fuzzy fantastic ideals of Y and X, respectively. Furthermore, if $\tilde{\lambda}$ and $\tilde{\eta}$ are interval valued anti fuzzy fantastic ideal with thresholds $(\tilde{\varepsilon}, \tilde{\delta})$, then also $\zeta(\tilde{\lambda})$ and $\zeta^{-1}(\tilde{\eta})$ are interval valued anti fuzzy fantastic ideal with thresholds $(\tilde{\varepsilon}, \tilde{\delta})$.

Proof. Suppose $\tilde{\lambda}$ is an interval valued anti fuzzy fantastic ideal with thresholds $(\tilde{\varepsilon}, \tilde{\delta})$ and let $x_1, y_1, z_1 \in Y$. Since ζ is an epimomorphism, we have

$$x_1 = \zeta(x),\ y_1 = \zeta(y) \text{ and } z_1 = \zeta(z),$$

for some x, y, z ∈ X. Then

$\min\{\zeta(\tilde{\lambda})(x_1 * (y_1 * (y_1 * x_1))),\ \tilde{\delta}\ \}$

$= \min\{\inf\{\tilde{\lambda}\,(x * (y * (y * x)))\} \mid \zeta(x * (y * (y * x))) = x_1 * (y_1 * (y_1 * x_1))),\ \tilde{\delta}\ \}$

$= \inf\{\min\{\tilde{\lambda}(x * (y * (y * x))), \tilde{\delta}\} \mid \zeta(x * (y * (y * x))) = x_1 * (y_1 * (y_1 * x_1))\}$

$\leq \inf\{\max\{\tilde{\lambda}((x * y) * z), \tilde{\lambda}(z), \tilde{\varepsilon}\} \mid \zeta((x * y) * z) = (x_1 * y_1) * z_1, \zeta(z) = z_1\}$

$= \max\{\inf\{\tilde{\lambda}((x * y) * z) \mid \zeta((x * y) * z) = (x_1 * y_1) * z_1\}, \{\inf\{\tilde{\lambda}(z) \mid \zeta(z) = z_1\}, \tilde{\varepsilon}\}$

$= \max\{\zeta(\tilde{\lambda})((x_1 * y_1) * z_1), \zeta(\tilde{\lambda})(z_1), \tilde{\varepsilon}\}$

Hence $\zeta(\tilde{\lambda})$ is an interval valued anti fuzzy fantastic ideal with thresholds $(\tilde{\varepsilon}, \tilde{\delta})$. Similarly, if $\tilde{\eta}$ is an interval valued anti fuzzy fantastic ideal with thresholds $(\tilde{\varepsilon}, \tilde{\delta})$, then for any x, y, z $\in$ X.

$$\zeta^{-1}(\tilde{\eta})(x_1 * (y_1 * (y_1 * x_1))) \wedge \tilde{\delta} = \tilde{\eta}(\zeta(x_1 * (y_1 * (y_1 * x_1)))) \wedge \tilde{\delta}$$
$$\leq \tilde{\eta}(\zeta((x_1 * y_1) * z_1)) \vee \tilde{\eta}(\zeta(z_1)) \vee \tilde{\varepsilon}$$
$$= \zeta^{-1}(\tilde{\eta})((x * y) * z) \vee \zeta^{-1}(\tilde{\eta})(z) \vee \tilde{\varepsilon}.$$

6. Conclusion

In the study of fuzzy algebraic system, we see that the interval valued anti fuzzy fantastic ideals with special properties always play a central role.

In this paper, the concepts of interval valued $(\in, \in \vee q_k)$-anti fuzzy fantastic ideals in BCH-algebras are introduced and related properties are investigated. Also we define the concept of interval valued anti fuzzy fantastic ideals with thresholds in BCH-algebras and investigate some of its properties.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of interval valued fuzzy BCH-algebras and their applications in other branches of algebra. In the future study of interval valued fuzzy BCH-algebra perhaps the following topics are worth to be considered:

(1) To characterize other classes of BCH-algebras by using this notion;
(2) To apply this notion to some other algebraic structures;
(3) To consider these results to some possible applications in computer sciences and information systems in the future.

References

[1]. B. Ahmad, On classification of BCH-algebras, Marh. Apon., 35(1990), 801-805.
[2]. N. O. Al-Shehrie, Generalized doubt fuzzy ideals of BCI-algebras, World Applied Sciences Journal, 10 (2010), 1114-1118.
[3]. S. K. Bhakat, $(\in \vee q)$-level subsets, Fuzzy Sets and Systems, 103 (1999), 529 -533.
[4]. S. K. Bhakat, $(\in, \in \vee q)$-fuzzy normal, quasinormal and maximal subgroups, Fuzzy Sets and System, 112 (2000), 299-312.
[5]. S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy subgroups, Fuzzy Sets and System, 80 (1996), 359-368.
[6]. R. Biswas, Fuzzy subgroups and anti fuzzy subgroups, Fuzzy Sets and Systems, 35(1990), 121-124.
[7]. R. Biswas, Rosenfeld's fuzzy subgroups with interval valued membership functions, Fuzzy Sets and Systems, 63 (1994), 87-90.
[8]. H. Hedayati, Connections between generalized fuzzy ideals and sub-implicative ideals of BCI-algebras, IAENG Int. J. Appl. Math., 41(1) (2011), 17-22.
[9]. Q. P. Hu and X. Li, On BCH-algebras, Math. Sem. Notes, 11(1983), 313-320.
[10]. Q. P. Hu and X. Li, On proper BCH-algebras, Math. Japon., 30(1985), 659-661.
[11]. Y. B. Jun, On (α, β)-fuzzy ideals of BCK/BCI-algebras, Sci. Math. Japon., 60 (2004), 613-617.
[12]. Y. B. Jun, On (α, β)-fuzzy subalgebras of BCK/BCI-algebras, Bull. Korean Math. Soc., 42 (2005), 703-711.

[13]. K. B. Latha, D. R. P. Williams and E. Chandrasekar, Interval-valued (α, β)-fuzzy subgroups I, Mathematica, Tome 52 (75) (2010), 177-184.

[14]. X. Ma, J. Zhan and Y. B. Jun, Interval-valued $(\in, \in \vee q)$-fuzzy ideals of pseudo MV-algebras, Int. J. Fuzzy Syst., 10 (2008), 84-91.

[15]. X. Ma, J. Zhan, B. Davvaz and Y. B. Jun, Some kinds of $(\in, \in \vee q)$-interval valued fuzzy ideals of BCI-algebras, Inform. Sci., 178 (2008), 3738-3754.

[16]. X. Ma, J. Zhan and Y. B. Jun, Some types of $(\in, \in \vee q)$-interval valued fuzzy ideals of BCI-algebras, Iranian J. of fuzzy Systems, 6 (2009), 53-63.

[17]. S. M. Mostafa, M. A. Abd-Elnaby and O. R. Elgendy, Interval valued fuzzy KU-ideals in KU-algebras, Int. Math. Forum, 6 (64) (2011), 3151-3159.

[18]. V. Murali, Fuzzy points of equivalent fuzzy subsets, Inform. Sci., 158 (2004), 277-288.

[19]. P. M. Pu and Y. M. Liu, Fuzzy topology I: neighourhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl., 76 (1980), 571-599.

[20]. A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., 35 (1971), 512-517.

[21]. A. B. Saeid, Some results on interval-valued fuzzy BG-algebras, Proceedings of World Academy of Science, Engineering and Technology, 5 (2005), 183-186.

[22]. A. B. Saeid, A. Namdar, and R.A. Borzooei, Ideal theory of BCH-algebras, World Appl. Sci. J., 7 (11) (2009), 1446-1455.

[23]. A. B. Saeid, A. Namdar and M. K. Rafsanjani, Fuzzy n-fold ideals in BCH-algebras, Neural Comput & Applic, 19 (2010), 775-783.

[24]. L. A. Zadeh, Fuzzy sets, Inform. and Control, 8 (1965), 338-353.

[25]. L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inform. Sci., 8 (1975), 199-249.

[26]. J. Zhan, W. A. Dudek and and Y. B. Jun, Interval-valued $(\in, \in \vee q)$-fuzzy filters of pseudo BL-algebras, Soft Computing - A Fusion of Foundations, Methodologies and Applications, 13 (2008), 13-21.

[27]. M. Zulfiqar, Some characterizations of ($\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta}$)-fuzzy fantastic ideals in BCH-algebras, Acta Sci. Tech., 35 (2013), 123-129.

[28]. M. Zulfiqar, Characterizations of ($\in, \in \vee q_k$)-fuzzy fantastic ideals in BCI-algebras, accepted.