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The Sharp Inequality for the Coefficients of Certain Subclass of Analytic Functions Defined by *q*-Derivative

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Abstract The object of the present paper is to give sharp estimates for some initial coefficients of certain subclass of analytic functions defined by q-derivative operator with respect to symmetric points. Here, the Fekete-Szegö problem for this function class is also examined. In addition, we give upper bound estimate for the second Hankel determinant of this class.

Keywords Analytic functions, Coefficient problem, Fekete-Szegö problem, Hankel determinant, q-derivative operator

1. Introduction and Preliminaries

Let A represented the class of analytic functions f on the open unit disk $U = \{z \in, \mathbb{C} : |z| < 1\}$ which given by the following series expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ... = z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathbb{C}. \quad (1)$$

We denote by S the subclass of A consisting of the functions which are also univalent.

Well-known subclasses of S are starlike and convex functions S^* and C , respectively, which defined by

$$S^* = \left\{ f \in S : Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in U \right\}$$

and

$$C = \left\{ f \in S : Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in U \right\}.$$

Some of the important subclasses of S are also $S^*(\alpha)$ and $C(\alpha)$, respectively, starlike and convex functions of order $\alpha \geq 0$. By definition (see for details, [5, 9], also [21])

$$S^*(\alpha) = \left\{ f \in S : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in U \right\},$$

$$C(\alpha) = \left\{ f \in S : Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha, z \in U \right\}.$$

The coefficient problem of the certain subclasses of analytic functions is one of the important problems in theory of the analytic functions. The sharp estimates for coefficients of the functions belonging to certain subclass of analytic functions are still an open problem (see, for example [15, 17]).

It is well-known that, one of the important tools in the theory of analytic functions is the functional $H_2(1)=a_3-a_2^2$, which is known as the Fekete-Szegö functional and one usually considers the further generalized functional $a_3-\mu a_2^2$, where μ is some real or complex number (see [7]). Estimating for the upper bound of $\left|a_3-\mu a_2^2\right|$ is known as the Fekete-Szegö problem. In 1969, Keogh and Merkes [14] solved the Fekete-Szegö problem for the classes starlike and convex functions. Someone can see the Fekete-Szegö problem for the classes of starlike functions and convex functions of order α at special cases in the paper of Orhan et al. [19] On the other hand, recently, Çağlar and Aslan (see [3]) have obtained Fekete-Szegö inequality for a subclass of bi-univalent functions. Also, Zaprawa (see [23, 24]) have studied on Fekete-Szegö problem for some subclasses of bi-univalent functions. In special cases, they solved the Fekete-Szegö problem for the subclasses bi-starlike functions of order α and bi-convex functions of order α.

In 1976, Noonan and Thomas [18] defined the q th Hankel determinant of f for $q\in\mathbb{N}$ by

$$H_q(n)=\begin{vmatrix} a_n & \cdots & a_{n+q-1} \\ . & \cdots & . \\ a_{n+q-1} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Well-known that, the upper bound of the expression $\left|H_2(2)\right|=\left|a_2a_4-a_3^2\right|$ is one important problem in theory of analytic functions. Recently, the upper bounds of $\left|H_2(2)\right|=\left|a_2a_4-a_3^2\right|$ for the bi-starlike and bi-convex functions classes $S_\Sigma^*(\alpha)$ and $C_\Sigma(\alpha)$ were obtained by Deniz et al. [4]. Very soon, Orhan et al. [20] reviewed the study of bounds for the second Hankel determinant for the subclass $M_\Sigma^*(\beta)$ of bi-univalent functions.

Recently, in [22] by Thomas given sharp estimates for some initial coefficients of the so-called gamma starlike functions.

In their studies by many researchers, the $q-$derivative operator used to introduce some subclasses of analytic functions in different ways. For example, in [2] by using the properties of $q-$derivative, shown that $q-$Szasz Mirakyan operators are convex if the involved function is convex, generalizing well known results for $q=1$. Moreover, in [2] shown that $q-$derivatives of these operators converge to derivatives of approximated functions. The effect of the $q-$ derivative operator on the generalized hypergeometric series $r\varphi s(a_1,\ldots a_r;b_1,\ldots b_s;q,z)$ with respect to parameters are discussed in [10].

For $q\in(0,1)$, in the fundamental paper [12] by Jackson introduced $q-$derivative operator of a function f as follows

$$D_q f(z)=\begin{cases} \dfrac{f(z)-f(qz)}{(1-q)z}, & if \quad z\neq 0, \\ f'(0), & if \quad z=0. \end{cases} \tag{2}$$

It follows from (2) that

$$D_q z^n=[n]_q z^{n-1}, n\in\mathbb{N}, \tag{3}$$

where

$$[n]_q=\sum_{k=1}^{n} q^{k-1}$$

is the $q-$analogue of the natural numbers (which is called the basic number n). Also, it can be easily shown that $[n]_q=\dfrac{1-q^n}{1-q}$, $[0]_q=0,\ [1]_q=1$, $\lim\limits_{q\to1^-}[n]_q=n$, $D_q\left(zD_qf(z)\right)=D_qf(z)+zD_q^2f(z)$. For more properties of the operator D_q see [6, 11, 13] .

For $f\in A$, we can easily see that

$$D_qf(z)=1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}. \tag{4}$$

It follows from (2) that $\lim\limits_{q\to1^-}D_qf(z)=f'(z)$.

Let $S_s^*(\alpha),\alpha\in[0,1)$ be the subclass of S consisting of the functions given by (1) satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z)-f(-z)}\right)>\alpha, z\in U.$$

In [8] by Goel and Mehrok introduced a subclass of $S_s^*(A,B)$ as follows

$$S_s^*(A,B)=\left\{f\in S:\frac{2zf'(z)}{f(z)-f(-z)}\prec\frac{1+Az}{1+Bz}, z\in U\right\}, -1\le B<A\le1.$$

Now, we introduce a subclass $S_{q,s}^*(\alpha)$ of the analytic functions defined by $q-$derivative operator with respect to symmetric points.

Definition 1 *A function $f\in S$ given by (1) is said to be in the class $S_{q,s}^*(\alpha), q\in(0,1), \alpha\in[0,1)$ if the following condition is satisfied*

$$Re\left(\frac{2zD_qf(z)}{f(z)-f(-z)}\right)>\alpha, z\in U.$$

From this definition, we can easily write $\lim\limits_{q\to1^-}S_{q,s}^*(\alpha)=S_s^*(\alpha)$.

Remark 1 *Choose $\alpha=0$ in the Definition 1, we have function class $S_{q,s}^*, q\in(0,1)$.*

Remark 2 *Choose $q\to1^-$ in the Definition 1, we have function class $S_s^*(\alpha), \alpha\in[0,1)$.*

Remark 3 *Choose $q\to1^-$ and $\alpha=0$ in the Definition 1, we have function class S_s^*.*

In this paper, we give sharp estimates for some initial coefficients of the subclass $S_{q,s}^*(\alpha)$ of analytic functions. Here, the Fekete-Szegö problem for this function class is also examined. In addition, we give upper bound estimate for the second Hankel determinant of this class.
To prove our main results, we shall need the following lemmas concerning functions with real part (see e. g. [1, 16]).

Denote by P the set of functions p analytic in U with power series $p(z)=1+\sum_{n=1}^{\infty}p_nz^n$ and satisfying $Rep(z)>0$ for $z\in U$.

Lemma 1 *Let* $p \in P$, *then* $|p_n| \le 2$ *is sharp for each* $n = 1,2,3,\ldots$ *and*

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$
$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)w$$

for some complex valued x *and w with* $|x| \le 1$ *and* $|w| \le 1$.

Lemma 2 *Let* $p \in P$, *then* $|p_n| \le 2$ *is sharp for each* $n = 1,2,3,\ldots$ *and*

$$\left| p_2 - \frac{v}{2} p_1^2 \right| \le \max\{2, 2|v-1|\} = \begin{cases} 2, & \text{if } 0 \le v \le 2, \\ 2|v-1|, & \text{elsewhere.} \end{cases}$$

2. The Coefficient Inequalities for the Class $S_{q,s}^*(\alpha)$

In this section, we give the following theorem on the sharp estimates for the some initial coefficients of the function class $S_{q,s}^*(\alpha)$.

Theorem 1 *Let the function* f *given by (1) belong to the class* $S_{q,s}^*(\alpha)$. *Then, for three initial coefficients of the function* f *, we have*

$$|a_2| \le \frac{2(1-\alpha)}{[2]_q}, |a_3| \le \frac{2(1-\alpha)}{[3]_q - 1}, |a_4| \le \frac{2(1-\alpha)}{[4]_q}\left(1 + \frac{2(1-\alpha)}{[3]_q - 1}\right). \tag{5}$$

All the inequalities obtained here are sharp.

Proof. Assume that $f \in S_{q,s}^*(\alpha), \alpha \in [0,1), q \in (0,1)$. Then,

$$Re\left(\frac{2zD_q f(z)}{f(z) - f(-z)}\right) > \alpha, z \in U. \tag{6}$$

It follows from that

$$\frac{2zD_q f(z)}{f(z) - f(-z)} = \alpha + (1-\alpha)p(z), z \in U, \tag{7}$$

where $p \in P$.

From (7) and (3), we obtain

$$Z + \sum_{n=2}^{\infty} [n]_q a_n z^n = \left(z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1}\right)\left(\alpha + (1-\alpha)\left(1 + \sum_{n=1}^{\infty} p_n z^n\right)\right). \tag{8}$$

As a result of simple simplification from (8) comparing the coefficients of the like power of z in the both sides, we get

$$a_2 = \frac{1-\alpha}{[2]_q} p_1, \quad a_3 = \frac{1-\alpha}{[3]_q - 1} p_2, \quad a_4 = \frac{1-\alpha}{[4]_q}\left(p_3 + \frac{1-\alpha}{[3]_q - 1} p_1 p_2\right). \tag{9}$$

Using inequalities $|p_n| \le 2$, from Lemma 2, for each $n = 1,2,\ldots$ from the above equalities gives

$$|a_2| \le \frac{2(1-\alpha)}{[2]_q}, |a_3| \le \frac{2(1-\alpha)}{[3]_q - 1} \text{ and } |a_4| \le \frac{2(1-\alpha)}{[4]_q}\left(1 + \frac{2(1-\alpha)}{[3]_q - 1}\right). \tag{10}$$

Thus, the inequalities of the theorem are provided.

To see that inequalities obtained in the theorem are sharp, we note that equalities are attained in the inequalities for $|a_n|, n=2,3,4$ when $p_1=p_2=p_3=2$. For example, for the function f chosen from the condition

$$\frac{2zD_q f(z)}{f(z)-f(-z)}=\frac{1+(1-2\alpha)z}{1-z}, z\in U,$$

the equalities are attained in the inequalities for $|a_n|, n=2,3,4$.

Thus, the proof of Theorem 1 is completed.

From the Theorem 1, we arrive at the following results.

Corollary 1 *Let the function* f *given by (1) belong to the class* $S^*_{q,s}$ *. Then, for the coefficients of the function* f *, we have*

$$|a_2|\le\frac{2}{[2]_q}, |a_3|\le\frac{2}{[3]_q-1} \text{ and } |a_4|\le\frac{2([3]_q+1)}{([3]_q-1)[4]_q}$$

All the inequalities obtained here are sharp.

Corollary 2 *Let the function* f *given by (1) belong to the class* $S^*_s(\alpha)$*. Then, for the coefficients of the function* f *, we have*

$$|a_n|\le 1-\alpha, n=2,3 \text{ and } |a_4|\le\frac{(1-\alpha)(2-\alpha)}{2}.$$

All the inequalities obtained here are sharp.

Corollary 3 *Let the function* f *given by (1) belong to the class* S^*_s *. Then, for the coefficients of the function* f *, we have*

$$|a_n|\le 1, n=2,3,4 \ .$$

All the inequalities obtained here are sharp.

3. The Fekete-Szegö Problem for the function class $S^*_{q,s}(\alpha)$

In this section, we will prove the following theorem on the Fekete-Szegö problem of the function class $S^*_{q,s}(\alpha)$.

Theorem 2 *Let the function given* f *by (1) be in the class* $S^*_{q,s}(\alpha)$ *and* $\mu\in\mathbb{C}$*. Then,*

$$|a_3-\mu a_2^2|\le\frac{2(1-\alpha)}{[3]_q-1}\begin{cases} 1 & \text{if } \left|\frac{[2]_q^2}{2(1-\alpha)([3]_q-1)}-\mu\right|\le\frac{[2]_q^2}{2(1-\alpha)([3]_q-1)}, \\ \left|1-\frac{2(1-\alpha)([3]_q-1)}{[2]_q^2}\mu\right| & \text{if } \left|\frac{[2]_q^2}{2(1-\alpha)([3]_q-1)}-\mu\right|\ge\frac{[2]_q^2}{2(1-\alpha)([3]_q-1)}. \end{cases}$$

The inequalities obtained here are sharp.

Proof. Let $f \in S_{q,s}^{*}(\alpha), \alpha \in [0,1), q \in (0,1)$ and $\mu \in \mathbb{C}$.

From (9), we find that

$$a_3 - \mu a_2^2 = \frac{(1-\alpha)}{[3]_q - 1}\left[p_2 - \frac{(1-\alpha)([3]_q - 1)}{[2]_q^2}\mu p_1^2 \right]. \tag{11}$$

Substituting the expression $p_2 = \frac{1}{2}\left[p_1^2 + (4 - p_1^2)x\right]$, from Lemma 1, in (11) and using triangle inequality, putting $t = |p_1|$ and $|x| = \eta$, we can easily obtain that

$$\left|a_3 - \mu a_2^2\right| \le \frac{1-\alpha}{2([3]_q - 1)}\left[d_1(t) + d_2(t)\eta\right] = \Psi(\eta), \tag{12}$$

where

$$d_1(t) = \left|1 - \frac{2(1-\alpha)([3]_q - 1)}{[2]_q^2}\mu\right| t^2 \ge 0 \text{ and} d_2(t) = 4 - t^2 \ge 0.$$

Since $\Psi'(\eta) \ge 0$, the function $\Psi(\eta)$ is increasing function in $[0,1]$. Therefore, the maximum of the function $\Psi(\eta)$ occurs at $\eta = 1$; that is,

$$\Psi(\eta) \le \max\{\Psi(\eta) : \eta \in [0,1]\} = \Psi(1) = \frac{1-\alpha}{2([3]_q - 1)}\left[d_1(t) + d_2(t)\right]. \tag{13}$$

Let us define the function $H : [0,2] \to \mathbb{R}$ as follows

$$H(t) = d_1(t) + d_2(t). \tag{14}$$

By substitution of the expression $d_1(t)$ and $d_2(t)$ in (14), the function H, writable as

$$H(t) = C(\alpha, q, \mu)t^2 + 4, \tag{15}$$

where

$$C(\alpha, q, \mu) = \left|1 - \frac{2(1-\alpha)([3]_q - 1)}{[2]_q^2}\mu\right| - 1$$

It is clear that the function H is an increasing function if $C(\alpha, q, \mu) \ge 0$ and decreasing function if $C(\alpha, q, \mu) \le 0$.

Therefore,

$$H(t) \le \max\{H(t) : t \in (0,2)\} = \begin{cases} 4C(\alpha, q, \mu) + 4 & if\ C(\alpha, q, \mu) \ge 0, \\ 4 & if\ C(\alpha, q, \mu) \ge 0. \end{cases} \tag{16}$$

Thus, from (12)-(16) we obtain the following inequality for $\left|a_3 - \mu a_2^2\right|$

$$\left|a_3 - \mu a_2^2\right| \le \frac{2(1-\alpha)}{[3]_q - 1}\begin{cases} C(\alpha, q, \mu) + 1 & if\ C(\alpha, q, \mu) \ge 0, \\ 1 & if\ C(\alpha, q, \mu) \le 0. \end{cases}$$

This completes the proof inequaliies of theorem.

To see that inequalities obtained in theorem are sharp, we note that the equality is attained in the first inequality, when $p_2 = 2$ and $p_1 = 0$. Also, the equality is attained in the second inequality, when $p_2 = p_1 = 2$.

Thus, the proof of Theorem 2 is completed.

From the Theorem 2, we obtain the following results.

Corollary 4 *Let the function given* f *by (1) be in the class* $S_{q,s}^{*}$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \frac{2}{[3]_q-1}\begin{cases} 1 & \text{if } \left|\frac{[2]_q^2}{2([3]_q-1)}-\mu\right| \le \frac{[2]_q^2}{2([3]_q-1)}, \\ 1-\frac{2([3]_q-1)}{[2]_q^2}\mu & \text{if } \left|\frac{[2]_q^2}{2([3]_q-1)}-\mu\right| \ge \frac{[2]_q^2}{2([3]_q-1)}. \end{cases}$$

Corollary 5 *Let the function given f by (1) be in the class $S_s^*(\alpha)$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3-\mu a_2^2\right| \le (1-\alpha)\begin{cases} 1 & \text{if } |1-(1-\alpha)\mu| \le 1, \\ 1-(1-\alpha)\mu & \text{if } |1-(1-\alpha)\mu| \ge 1. \end{cases}$$

Corollary 6 *Let the function given f by (1) be in the class S_s^* and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} 1 & \text{if } |1-\mu| \le 1, \\ |1-\mu| & \text{if } |1-\mu| \ge 1. \end{cases}$$

In the case $\mu \in \mathbb{R}$, Theorem 2 can be given as follows.

Theorem 3 *Let the function given f by (1) be in the class $S_{q,s}^*(\alpha)$ and $\mu \in \mathbb{R}$. Then,*

$$\left|a_3-\mu a_2^2\right| \le \frac{2(1-\alpha)}{[3]_q-1}\begin{cases} 1-\frac{2(1-\alpha)([3]_q-1)}{[2]_q^2}\mu & \text{if } \mu \le 0, \\ 1 & \text{if } 0 \le \mu \le \frac{[2]_q^2}{(1-\alpha)([3]_q-1)}, \\ \frac{2(1-\alpha)([3]_q-1)}{[2]_q^2}\mu-1 & \text{if } \frac{[2]_q^2}{(1-\alpha)([3]_q-1)} \le \mu. \end{cases}$$

The inequalities obtained here are sharp.

Proof. Assume that $f \in S_{q,s}^*(\alpha), \alpha \in [0,1), q \in (0,1)$ and $\mu \in \mathbb{R}$.

From (9), we find that

$$a_3-\mu a_2^2 = \frac{1-\alpha}{[3]_q-1}\left[p_2-\frac{(1-\alpha)([3]_q-1)}{[2]_q^2}\mu p_1^2\right].$$

The expression for $a_3-\mu a_2^2$, we write as follows:

$$a_3-\mu a_2^2 = \frac{1-\alpha}{[3]_q-1}\left(p_2-\frac{\nu}{2}p_1^2\right), \tag{17}$$

with $\nu = 2(1-\alpha)([3]_q-1)\mu/[2]_q^2$.

Using Lemma 2 to equality (17), we obtain the following inequality for $\left|a_3-\mu a_2^2\right|$

$$\left|a_3-\mu a_2^2\right| \le \frac{2(1-\alpha)}{[3]_q-1}\begin{cases} 1 & \text{if } 0 \le \frac{(1-\alpha)([3]_q-1)}{[2]_q^2}\mu \le 1, \\ \left|\frac{2(1-\alpha)([3]_q-1)}{[2]_q^2}\mu-1\right| & \text{if } \frac{(1-\alpha)([3]_q-1)}{[2]_q^2}\mu \notin [0,1]. \end{cases}$$

This completes the proof inequaliies of theorem.

To see that inequalities obtained in theorem are sharp, we note that the equality is attained in the first inequality, when $p_1 = p_2 = 2$. The equality is attained in the second inequality, when $p_1 = 0$ and $p_2 = 2$. Also, the equality is attained in the third inequality, when $p_1 = p_2 = 2$.

Thus, the proof of Theorem 3 is copleted.

Notation 1 *It should be noted that Theorem 3 could also be given as a direct result of Theorem 2 But, here we have given shorter proof of the Theorem 3 for the real value of* $\mu \in \mathbb{R}$.

Choose $\mu = 0$ in Theorem 3, we obtain the following inequality for $|a_3|$, which confirm the inequality obtained in Theorem 1 and Corollaries 1,2 and 3, respectively.

Corollary 7 *The following inequalities are provided:*

If $f \in S^*_{q,s}(\alpha)$ then,

$$|a_3| \le \frac{2(1-\alpha)}{[3]_q - 1}.$$

If $f \in S^*_{q,s}$ then,

$$|a_3| \le \frac{2}{[3]_q - 1}.$$

If $f \in S^*_s(\alpha)$ then,

$$|a_3| \le (1-\alpha).$$

If $f \in S^*_s$ then,

$$|a_3| \le 1.$$

4. The Second Hankel Determinant of the function class $S^*_{q,s}(\alpha)$

In this section, we prove the following theorem on upper bound of the second Hankel determinant of the function class $S^*_{q,s}(\alpha)$.

Theorem 4 *Let the function given* f *by (1) be in the class* $S^*_{q,s}(\alpha)$ *. Then,*

$$|a_2 a_4 - a_3^2| \le \frac{4(1-\alpha)^2}{[2]_q [4]_q}\left[1 + \frac{[2]_q [4]_q}{([3]_q - 1)^2}\right].$$

Proof. Let $f \in S^*_{q,s}(\alpha), \alpha \in [0,1), q \in (0,1)$ .Then, from (9) we have

$$a_2 a_4 - a_3^2 = \frac{(1-\alpha)^2}{[2]_q [4]_q}\left\{p_1 p_3 + \frac{1-\alpha}{[3]_q - 1} p_1^2 p_2 - \frac{[2]_q [4]_q}{([3]_q - 1)^2} p_2^2\right\}$$

$$= \frac{(1-\alpha)^2}{[2]_q [4]_q}\left\{p_1 p_3 + \frac{[2]_q [4]_q}{([3]_q - 1)^2}\left[\frac{(1-\alpha)([3]_q - 1)}{[2]_q [4]_q} p_1^2 - p_2\right] p_2\right\}$$

that is

$$a_2a_4-a_3^2=\frac{(1-\alpha)^2}{[2]_q[4]_q}\left\{p_1p_3+\frac{[2]_q[4]_q}{([3]_q-1)^2}\left[\frac{\upsilon}{2}p_1^2-p_2\right]p_2\right\},$$

with $\upsilon=\frac{2(1-\alpha)([3]_q-1)}{[2]_q[4]_q}$.

Apply triangle inequality, we obtain

$$|a_2a_4-a_3^2|\le\frac{(1-\alpha)^2}{[2]_q[4]_q}\left\{|p_1p_3|+\frac{[2]_q[4]_q}{([3]_q-1)^2}\left|\frac{\upsilon}{2}p_1^2-p_2\right||p_2|\right\}$$

Since $0\le\upsilon=\frac{2(1-\alpha)([3]_q-1)}{[2]_q[4]_q}\le 2$ for all $q\in(0,1)$ and $\alpha\in[0,1)$, using Lemma 2 to the last equality, we get

$$|a_2a_4-a_3^2|\le\frac{4(1-\alpha)^2}{[2]_q[4]_q}\left[1+\frac{[2]_q[4]_q}{([3]_q-1)^2}\right]$$

Thus, the proof of Theorem 4 is completed.
From the Theorem 4, we obtain the following results.

Corollary 8 *Let the function given* f *by (1) be in the class* $S_{q,s}^*$ *. Then,*

$$|a_2a_4-a_3^2|\le\frac{4}{[2]_q[4]_q}\left[1+\frac{[2]_q[4]_q}{([3]_q-1)^2}\right].$$

Corollary 9 *Let the function given* f *by (1) be in the class* $S_s^*(\alpha)$ *. Then,*

$$|a_2a_4-a_3^2|\le\frac{3(1-\alpha)^2}{2}.$$

Corollary 10 *Let the function given* f *by (1) be in the class* S_s^* *. Then,*

$$|a_2a_4-a_3^2|\le\frac{3}{2}.$$

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