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Research Article

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On the Upper Bound Estimates for the Coefficients of Certain Subclasses of Analytic Functions

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Abstract In this paper, we introduce and investigate subclasses of analytic functions on the open unit disk in complex plane. Here, we give the sufficient, and sufficient and necessary conditions for the functions to belong to these classes. Several interesting geometric properties of the functions belonging to these classes are examined. Some consequences of the results obtained here are also discussed.

Keywords Analytic functions, Starlike function, Convex function, Coefficient bound

1. Introduction and Preliminaries

Let A be the class of functions in the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots = z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathbb{C}. \tag{1.1}$$

which are analytic on the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. It is clear that $f(0) = 0 = f'(0) - 1$ for $f \in A$.

Let T denote the subclass of all functions f in A of the form

$$f(z) = z - a_2 z^2 - a_3 z^3 - \ldots = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{1.2}$$

We denote by S the subclass of A consisting of the functions which are also univalent in U.

Some of the important subclass of S is the class $\Re(\alpha, \beta)$ that is defined as follows [2]

$$\Re(\alpha, \beta) = \{f \in S : Re[f'(z) + \beta z f''(z)] > \alpha, z \in U\}, \alpha \in [0,1), \beta \geq 0.$$

Gao and Zhou [2] investigated the class $\Re(\alpha, \beta)$ and showed some mapping properties of this subclass.

Early, by Alintaş et al. [1] were investigated a subclass $\Re(\alpha, \beta, \gamma)$ of analytic functions consisting of functions f which satisfy the condition

$$f \in T, \left|\frac{1}{\gamma}[f'(z) + \beta z f''(z) - 1]\right| \leq \alpha, z \in U, \beta \in [0,1], \alpha \in [0,1), \gamma \in \mathbb{C}^* = \mathbb{C} - \{0\}.$$

Motivated by the aforementioned works, we define new subclasses of analytic functions as follows.

Definition 1 *A function $f \in A$ given by 1.1 is said to be in the class $\Im(\alpha, \beta, \tau)$ if the following conditions is satisfied*

$$Re\left\{1 + \frac{1}{\tau}[f'(z) + \beta z f''(z) - 1]\right\} > \alpha, \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}, \alpha \in [0,1), \beta \geq 0, z \in U.$$

Definition 2 *A function $f \in A$ given by 1.1 is said to be in the class $\Im(\alpha, \beta) = T\ \Re(\alpha, \beta)$ if the following conditions is satisfied*

$$f \in T\Re(\alpha, \beta, \gamma) \Leftrightarrow Re\{f'(z) + \beta z f''(z)\} > \alpha, \alpha \in [0,1), \beta \geq 0, z \in U.$$

Remark 1 *Choose $\tau = 1$ in Definition 1.1, we have function class $\Im(\alpha, \beta, 1) = \Re(\alpha, \beta), \alpha \in [0,1), \beta \geq 0$; that is,*

$$f \in \Re(\alpha, \beta, \gamma) \Leftrightarrow Re\{f'(z) + \beta z f''(z)\} > \alpha, z \in U.$$

Remark 2 *Choose* $\beta = 0$ *in Definition 1.1, we have function class* $\Im(\alpha, 0, \tau), \alpha \in [0,1), \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$; *that is,*

$$f \in \Im(\alpha, 0, \tau) \Leftrightarrow Re\left\{1 + \frac{1}{\tau}[f'(z) - 1]\right\} > \alpha, z \in U.$$

Remark 3 *Choose* $\beta = 0, \tau = 1$ *in Definition 1.1, we have function class* $\Im(\alpha, 0, \tau) = \Re(\alpha, 0), \alpha \in [0,1)$; *that is,*

$$f \in \Re(\alpha, 0) \Leftrightarrow Re\{f'(z)\} > \alpha, z \in U.$$

Remark 4 *Choose* $\beta = 1$ *in Definition 1.1, we have function class* $\Im(\alpha, 1, \tau), \alpha \in [0,1), \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$; *that is,*

$$f \in \Im_\Sigma(\alpha, 1, \tau) \Leftrightarrow Re\left\{1 + \frac{1}{\tau}[f'(z) + zf''(z) - 1]\right\} > \alpha, z \in U.$$

Remark 5 *Choose* $\beta = 1, \tau = 1$ *in Definition 1.1, we have function class* $\Im(\alpha, 1, \tau), \alpha \in [0,1), \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$; *that is,*

$$f \in \Re(\alpha, 1) \Leftrightarrow Re\left(f'(z) + zf''(z)\right) > \alpha, z \in U.$$

Remark 6 *Choose* $\beta = 0$ *in Definition 1.2, we have function class* $\Im(\alpha, 0) = T\Re(\alpha, 0), \alpha \in [0,1)$; *that is,*

$$f \in \Re(\alpha, 0) \Leftrightarrow Re\{f'(z)\} > \alpha, z \in U.$$

Remark 7 *Choose* $\beta = 1$ *in Definition 1.2, we have function class* $\Im(\alpha, 1) = T\Re(\alpha, 1), \alpha \in [0,1)$; *that is,*

$$f \in \Re(\alpha, 1) \Leftrightarrow Re\{f'(z) + zf''(z)\} > \alpha, \alpha \in [0,1), z \in U.$$

In this paper, two new subclasses $\Im(\alpha, \beta, \tau)$ and $T\,\Re(\alpha, \beta, \gamma)$ of the analytic functions in the open unit disk are introduced. The sufficient, and sufficient and necessary conditions for the functions to belong to these classes were given. The various geometric properties of the functions belonging to these classes are also examined. Some consequences of the results obtained here are also discussed.

2. Coefficient bound estimates

In this section, we will examine some inclusion results of the subclasses $\Im(\alpha, \beta, \tau)$ and $T\,\Re(\alpha, \beta, \gamma)$ of the analytic functions in the open unit disk. Furthermore, we give coefficient bound estimates for the functions belonging to these subclasses.

A sufficient condition for the functions in the class $\Im(\alpha, \beta, \tau)$ is given by the following theorem.

Theorem 1 *Let* A. *Then, the function* f *belongs to the class* $\Im(\alpha, \beta, \tau)$ *if the following condition is satisfied*

$$\sum_{n=2}^{\infty} n[1 + (n-1)\beta]a_n \leq (1 - \alpha)|\tau|. \tag{2.1}$$

The result obtained here is sharp.

Proof. Assume that $f \in \Im(\alpha, \beta, \tau), \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}, \alpha \in [0,1), \beta \geq 0$. From the Definition 1, we have

$$Re\left\{1 + \frac{1}{\tau}[f'(z) + \beta zf''(z) - 1]\right\} > \alpha \tag{2.2}$$

It is clear that condition (2.2) is satisfied if provided following condition

$$\left|\frac{1}{\tau}[f'(z) + \beta zf''(z) - 1]\right| \leq 1 - \alpha. \tag{2.3}$$

In that case, it suffices to show that satisfied condition (2.3).

From (1.1), by simple computation, we write

$$\left|\frac{1}{\tau}[f'(z) + \beta zf''(z) - 1]\right| = \left|\frac{1}{\tau}\sum_{n=2}^{\infty} n[1 + (n-1)\beta]a_n z^{n-1}\right|$$
$$\leq \frac{1}{|\tau|}\sum_{n=2}^{\infty} n[1 + (n-1)\beta]|a_n|.$$

The last expression in the last inequality is bounded above by $1 - \alpha$ if and only if

$$\sum_{n=2}^{\infty} n[1 + (n-1)\beta]|a_n| \leq (1 - \alpha)|\tau|. \tag{2.4}$$

According to this, the inequality (2.3) is true if condition (2.4) is satisfied. So, the inequality (2.1) is provided.

To see that inequality obtained in the theorem is sharp, it is sufficient to see that inequality is provided as equality for the function given below

$$f_n(z) = z + \frac{(1-\alpha)|\tau|}{n[1+(n-1)\beta]}z^n, z \in U$$

for every $n = 2,3, \ldots$.

Thus the proof of Theorem 1 is completed.

From the Theorem 1, we obtain the following results.

Corollary 1 *The function f definition by (1.1) belongs to the class $\Re(\alpha,\beta)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty} n[1+(n-1)\beta]|a_n| \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z) = z + \frac{1-\alpha}{n[1+(n-1)\beta]} z^n, z \in U$$

for every $n = 2,3,\ldots$.

Corollary 2 *The function f definition by (1.1) belongs to the class $\Im(\alpha,0,\tau)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty} n|a_n| \leq (1-\alpha)|\tau|.$$

The result is sharp for the function

$$f_n(z) = z + \frac{(1-\alpha)|\tau|}{n} z^n, z \in U$$

for every $n = 2,3,\ldots$.

Corollary 3 *The function f definition by (1.1) belongs to the class $\Re(\alpha,0)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty} n|a_n| \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z) = z + \frac{1-\alpha}{n} z^n, z \in U$$

for every $n = 2,3,\ldots$.

Corollary 4 *The function f definition by (1.1) belongs to the class $\Im(\alpha,1,\tau)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty} n^2|a_n| \leq (1-\alpha)|\tau|.$$

The result is sharp for the function

$$f_n(z) = z + \frac{(1-\alpha)|\tau|}{n^2} z^n, z \in U$$

for every $n = 2,3,\ldots$.

Corollary 5 *The function f definition by (1.1) belongs to the class $\Re(\alpha,1)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty} n^2|a_n| \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z) = z + \frac{(1-\alpha)}{n^2} z^n, z \in U$$

for every $n = 2,3,\ldots$.

For the function in the class $T\,\Re(\alpha,\beta)$, the converse of Theorem 1 is also true. So, the following theorem gives the sufficient and necessary condition for the functions belonging to the class $T\,\Re(\alpha,\beta)$.

Theorem 2 *Let $f \in T$. Then, the function f belongs to the class $T\,\Re(\alpha,\beta,\gamma)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty} n[1+(n-1)\beta]a_n \leq 1-\alpha.$$

The result obtained here is sharp.

Proof. In view of Theorem 1, we need only to prove the necessity of the Theorem 2 Assume that $f \in T\Re(\alpha,\beta), \alpha \in [0,1), \beta \geq 0$, which is equivalent to $f \in T$ and

$$Re\left(f'(z) + \beta z f''(z)\right) > \alpha, z \in U, \quad (2.5)$$

From (1.2) and (2.5) by simple computation, we get

$$Re\{z - \sum_{n=2}^{\infty} n[1+(n-1)\beta]a_n z^{n-1}\} > \alpha, z \in U.$$

The expression

$$z - \sum_{n=2}^{\infty} n[1+(n-1)\beta]a_n z^{n-1}$$

is real if choose z real. Thus, from the previous inequality letting $z \to 1^-$ through real values, we obtain

$$1-\sum_{n=2}^{\infty} n[1+(n-1)\beta]a_n \geq \alpha;$$

so that,

$$\sum_{n=2}^{\infty} n[1+(n-1)\beta]a_n \leq 1-\alpha$$

This completed the proof of the necessity of theorem.

To see that inequality obtained in the theorem is sharp, it is sufficient to see that inequality is provided as equality for the function given below

$$f_n(z)=z-\frac{(1-\alpha)}{n[1+(n-1)\beta]}z^n, z\in U$$

for every $n=2,3,\ldots$.

Thus, the proof of Theorem 2 is completed.

From the Theorem 2, we can readily deduce the following results.

Corollary 6 *The function f definition by (1.2) belongs to the class $T\Re(\alpha,0)$ if and only if*

$$\sum_{n=2}^{\infty} n|a_n| \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z)=z-\frac{(1-\alpha)}{n}z^n, z\in U$$

for every $n=2,3,\ldots$.

Corollary 7 *The function f definition by (1.2) belongs to the class $T\Re(\alpha,1)$ if and only if*

$$\sum_{n=2}^{\infty} n^2|a_n| \leq 1-\alpha.$$

The result is sharp for the function

$$f_n(z)=z-\frac{(1-\alpha)}{n^2}z^n, z\in U$$

for every $n=2,3,\ldots$.

On the coefficient bounds of the functions belonging to the class $T\Re(\alpha,\beta)$, we give the following result.

Lemma 1 *Let the function f definition by (1.2) belongs to the class $T\Re(\alpha,\beta)$. Then,*

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\alpha}{2(1+\beta)} \text{ and } \sum_{n=2}^{\infty} na_n \leq \frac{1-\alpha}{1+\beta}.$$

Proof. Assume that $f\in T\Re(\alpha,\beta), \alpha\in[0,1), \beta\geq 0$.

In this case, according to Theorem 2, we get

$$2(1+\beta)\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n[1+(n-1)\beta]a_n \leq 1-\alpha,$$

which equivalent to the first inequality of the lemma.

Similarly, we write

$$(1+\beta)\sum_{n=2}^{\infty} na_n \leq \sum_{n=2}^{\infty} n[1+(n-1)\beta]a_n \leq 1-\alpha;$$

that is,

$$(1+\beta)\sum_{n=2}^{\infty} na_n \leq 1-\alpha,$$

which immediately yields the second assertion of the lemma.

Thus, the proof of Lemma 1 is completed.

From the Lemma 1, we can easily obtain the following results.

Corollary 8 *Let the function f definition by (1.2) belongs to the class $T\Re(\alpha,0)$. Then,*

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\alpha}{2} \text{ and } \sum_{n=2}^{\infty} na_n \leq 1-\alpha$$

Corollary 9 *Let the function f definition by (1.2) belongs to the class $T\Re(\alpha,1)$. Then,*

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\alpha}{4} \text{ and } \sum_{n=2}^{\infty} na_n \leq \frac{1-\alpha}{2}$$

The following corollary is direct result of Theorem2.

Corollary 10 *If $f\in T\Re(\alpha,\beta)$, then*

$$a_n \leq \frac{1-\alpha}{n[1+(n-1)\beta]}, n=2,3,\ldots.$$

Remark 8 *Numerous consequences of the Corollary 10 can indeed be deduced by specializing the various parameters involved.*

Now, we give following properties of the class $T\Re(\alpha,\beta)$.

Lemma 2 *The subclass* $T\Re(\alpha,\beta)$ *of the analytic functions is convex set.*

Proof. Assume that each of the functions $f, g \in T\Re(\alpha,\beta), \alpha \in [0,1), \beta \geq 0,$ with $g(z) = z - \sum_{n=2}^{\infty} b_n$. Then, for $\lambda \in [0,1]$, we write

$$\varphi(z) = \lambda f(z) + (1-\lambda)g(z) = z - \sum_{n=2}^{\infty} c_n,$$

where $c_n = \lambda a_n + (1-\lambda)b_n, n = 2,3,\ldots$.

Using necessary part of the Theorem 2.2, we write

$$\sum_{n=2}^{\infty} n[1+(n-1)\beta]c_n = \lambda \sum_{n=2}^{\infty} n[1+(n-1)\beta]a_n + (1-\lambda)\sum_{n=2}^{\infty} n[1+(n-1)\beta]b_n$$
$$\leq \lambda(1-\alpha) + (1-\lambda)(1-\alpha) = 1-\alpha;$$

that is,

$$\sum_{n=2}^{\infty} n[1+(n-1)\beta]c_n \leq 1-\alpha.$$

Next, using the sufficiently part of the Theorem 2, we have $\varphi \in T(\alpha,\beta;\gamma)$.

Thus, the proof of Lemma 2 is completed.

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