Available online www.jsaer.com

Journal of Scientific and Engineering Research, 2020, 7(2):1-14

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Some properties of $(\in, \in \vee q_k)$-fuzzy p-ideals in BF-algebras

Muhammad Zulfiqar[1], Zafar Ullah[2], Saddam Hussain[3]

[1]GC University Lahore, Pakistan
E-mail: mzulfiqarshafi@hotmail.com
[2]Department of Mathematics, University of Education Lahore, Dera Ghazi Khan Campus, Pakistan
E-mail: zafarbhatti@gmail.com
[3]Department of Mathematics, GC University Lahore, Pakistan
E-mail: saddamravian007@gmail.com

Abstract In this paper, we initiate the notion of $(\in, \in \vee q_k)$-fuzzy p-ideal in BF-algebra and investigate some of their associated properties. The idea of implication-based fuzzy p-ideal and implication operators in Lukasiewicz system of continuous-valued logic in BF-algebra is introduced and studied some of their interrelated properties.

Keywords fuzzy p-ideals, BF-algebras

1. Introduction

The theory of BF-algebra was initiated by Walendziak in his pioneering paper [27] and investigated some of their related properties. The theories were enriched by many authors (see [5, 7-9, 17, 21-22, 30]).

The notion of fuzzy set was first introduced by Zadeh in his seminal paper [29], of 1965 provides a natural framework for generalizing some of the fundamental notions of algebra. Extensive applications of fuzzy set theory have been found in different fields, for example, computer science, artificial intelligence, control engineering, management science, operation research, expert system and many others. The notion was applied to the theory of groupoids and groups by Rosenfeld [20], where he introduced the fuzzy subgroup of a group. Since then the literature of various algebraic structures has been fuzzified.

A novel type of fuzzy subgroup, which is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced by Bhakat and Das [3] by using the combined ideas of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which was initiated by Pu and Liu [19]. Murali [18] planned the definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. It was set up that the most viable generalization of Rosenfeld's fuzzy subgroup is $(\in, \in \vee q)$-fuzzy subgroup. Bhakat [1-2] introduced the thought of $(\in \vee q)$-level subsets, $(\in, \in \vee q)$-fuzzy normal, quasi-normal and maximal subgroups. Many researchers utilized these concepts to generalize some concepts of algebra (see [4, 10-14, 28, 32-34]). Davvaz in [6] discussed $(\in, \in \vee q)$-fuzzy subnearrings and ideals. In [10-12], Jun defined the concept of (α, β)-fuzzy subalgebras/ideals in BCK/BCI-algebras, where α, β are any of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$. Zulfiqar introduced the notion of (α, β)-fuzzy positive implicative ideals in BCK-algebras [32]. Generalizing the idea of quasi-coincident of a fuzzy point with a fuzzy set, in [13], Jun defined $(\in, \in \vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras. In [14], Jun et al. discussed $(\in, \in \vee q_k)$-fuzzy ideals in BCK/BCI-algebras. Khan et al. studied order semigroups characterized by $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals [15]. Larimi [16], initiated the notion of $(\in, \in \vee q_k)$-intuitionistic fuzzy ideals of hemirings. Shabir et al. [23], characterized different classes of semigroups by $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals. Shabir and Rehman [24], defined ternary semigroups by $(\in, \in \vee q_k)$-fuzzy ideals. Tang and Xie, studied $(\in, \in \vee q_k)$-fuzzy ideals of ordered semigroups [26]. Shabir and Mahmood [25], defined the concept of semihypergroups characterized by $(\in, \in \vee q_k)$-fuzzy hyperideals. In [31], Zeb et al. studied the characterization of ternary semigroups in terms of $(\in, \in \vee q_k)$-ideals. The notion of $(\in, \in \vee q_k)$-fuzzy fantastic ideals in BCI-algebras was introduced by Zulfiqar in [34] and investigated some of their related properties.

In the current paper, we define the notion of $(\in, \in \vee q_k)$-fuzzy p-ideal in BF-algebra and investigate some of their related properties. The thought of implication-based fuzzy p-ideal and implication operators in Lukasiewicz system of continuous-valued logic in BF-algebra is introduced and discussed some of their related properties.

2. Preliminaries

All over this paper X always represent a BF-algebra without any specification. We also consist of some fundamental aspects that are necessary for this paper.

A BF-algebra X [27] is a general algebra (X, $*$, 0) of type (2, 0) satisfying the following conditions:

(BF-1) $x * x = 0$

(BF-2) $x * 0 = x$

(BF-3) $0 * (x * y) = (y * x)$

for all x, y $\in$ X.

We can define a partial order "$\leq$" on X by $d \leq e$ if and only if $x * y = 0$.

Proposition 2.1. [5, 8, 21, 22] In any BF-algebra X, the following are true:

(i) $0 * (0 * x) = x$

(ii) $0 * x = 0 * y$, then $x = y$

(iii) $x * y = 0$, then $y * x = 0$

(iv) $x * 0 = x$

for all x, y $\in$ X.

Definition 2.2. [5] A non-empty subset I of a BF-algebra X is called an ideal of X if it satisfies the conditions (I1) and (I2), where

(I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$,

for all x, y $\in$ X.

Definition 2.3. A non-empty subset I of a BF-algebra X is called a p-ideal of X if it satisfies the conditions (I1) and (I3), where

(I1) $0 \in I$,

(I3) $(x * z) * (y * z) \in I, y \in I \Rightarrow x \in I$,

for all x, y, z $\in$ X.

We now review some fuzzy logic concepts. Recall that the real unit interval [0, 1] with the totally ordered relation "$\leq$" is a complete lattice, with $\wedge$ = min and $\vee$ = max, 0 and 1 being the least element and the greatest element, respectively.

A fuzzy set λ of a universe X is a function from X into the unit closed interval [0, 1], that is $\lambda : X \to [0, 1]$. For a fuzzy set λ of a BF-algebra X and $t \in (0, 1]$, the crisp set

$$\lambda_t = \{x \in X \mid \lambda(x) \geq t\}$$

is called the level subset of λ [5].

Definition 2.4. [5] A fuzzy set λ of a BF-algebra X is called a fuzzy ideal of X if it satisfies the conditions (F1) and (F2), where

(F1) $\lambda(0) \geq \lambda(x)$,

(F2) $\lambda(x) \geq \lambda(x * y) \wedge \lambda(y)$,

for all $x, y \in X$.

Definition 2.5. A fuzzy set λ of a BF-algebra X is called a fuzzy p-ideal of X if it satisfies the conditions (F1) and (F3), where

(F1) $\lambda(0) \geq \lambda(x)$,

(F3) $\lambda(x) \geq \lambda((x * z) * (y * z)) \wedge \lambda(y)$,

for all $x, y, z \in X$.

Example 2.6. Assume X = {0, *j*, *k*, *l*} be a BF-algebra with Cayley table as follows [5]:

$*$	0	j	k	l
0	0	j	k	l
j	j	0	l	k
k	k	l	0	j
l	l	k	j	0

We define a fuzzy subset λ in X by $\lambda(0) = 0.78$, $\lambda(j) = 0.65$, $\lambda(k) = 0.65$ and $\lambda(l) = 0.78$ By calculations gives that λ is a fuzzy p-ideal of X.

A fuzzy set λ of a BF-algebra X having the form

$$\lambda(y) = \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set λ in a set X, Pu and Liu [19] gave meaning to the symbol $x_t \alpha \lambda$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp., quasi-coincident with) a fuzzy set λ, written as $x_t \in \lambda$ (resp. $x_t q\lambda$) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$). By $x_t \in \vee q \lambda$ ($x_t \in \wedge q \lambda$) we mean that $x_t \in \lambda$ or $x_t q\lambda$ ($x_t \in \lambda$ and $x_t q\lambda$). For all $t_1, t_2 \in [0, 1]$, $\min\{t_1, t_2\}$ and $\max\{t_1, t_2\}$ will be denoted by $t_1 \wedge t_2$ and $t_1 \vee t_2$, respectively.

In what follows let α and β denote any one of $\in$, q, $\in \vee q$, $\in \wedge q$ and $\alpha \neq \in \wedge q$ unless otherwise specified. To say that $x_t \overline{\alpha} \lambda$ means that $x_t \alpha \lambda$ does not hold.

Let k denote an arbitrary element of [0, 1) unless otherwise specified. Now we define

(i) $x_t q_k \lambda$ if $\lambda(x) + t + k > 1$.

(ii) $x_t \in \vee q_k \lambda$ if $x_t \in \lambda$ or $x_t q_k \lambda$.

(iii) $x_t \overline{\alpha} \lambda$ if $x_t \alpha \lambda$ does not exist for $\alpha \in \{q_k, \in \vee q_k\}$.

3. (α, β)-fuzzy p-ideals

In this section, we initiate the idea of (α, β)-fuzzy p-ideals in a BF-algebra and investigate some of their properties, where α, β are any one of $\in$, q_k, $\in \vee q_k$, $\in \wedge q_k$ unless otherwise specified.

Definition 3.1. A fuzzy set λ of a BF-algebra X is called an (α, β)-fuzzy subalgebra of X, where $\alpha \neq \in \wedge q_k$, if it satisfies the condition

$$x_{t_1}\ \alpha\, \lambda,\ y_{t_2}\ \alpha\, \lambda \ \Rightarrow\ (x * y)_{t_1 \wedge t_2}\ \beta\, \lambda$$

for all $t_1, t_2 \in (0, 1]$ and $x, y \in X$.

Let λ be a fuzzy set of a BF-algebra X such that $\lambda(x) \leq \frac{1-k}{2}$ for all $x \in X$. Let $x \in X$ and $t \in (0, 1]$ be such that

$$x_t \in \wedge q_k \lambda.$$

Then

$$\lambda(x) \geq t \text{ and } \lambda(x) + t + k > 1.$$

It follows that

$$2\lambda(x) + k = \lambda(x) + \lambda(x) + k \geq \lambda(x) + t + k > 1.$$

This implies that $\lambda(x) > \frac{1-k}{2}$. This means that

$$\{x_t \mid x_t \in \wedge q_k \lambda\} = \phi.$$

Therefore, the case $\alpha = \in \wedge q_k$ in the above definition is omitted.

Definition 3.2. A fuzzy set λ of a BF-algebra X is called an (α, β)-fuzzy ideal of X, where $\alpha \neq \in \wedge q_k$, if it satisfies the conditions (A) and (B), where

(A) $x_t\, \alpha\, \lambda \ \Rightarrow\ 0_t\, \beta\, \lambda$,

(B) $(x * y)_{t_1}\ \alpha\, \lambda,\ y_{t_2}\ \alpha\, \lambda \ \Rightarrow\ x_{t_1 \wedge t_2}\ \beta\, \lambda$,

for all $t, t_1, t_2 \in (0, 1]$ and $x, y \in X$.

Definition 3.3. A fuzzy set λ of a BF-algebra X is called an (α, β)-fuzzy p-ideal of X, where $\alpha \neq \in \wedge q_k$, if it satisfies the conditions (A) and (C), where

(A) $x_t\, \alpha\, \lambda \ \Rightarrow\ 0_t\, \beta\, \lambda$,

(C) $((x * z) * (y * z))_{t_1}\ \alpha\, \lambda,\ y_{t_2}\ \alpha\, \lambda \ \Rightarrow\ x_{t_1 \wedge t_2}\ \beta\, \lambda$,

for all $t, t_1, t_2 \in (0, 1]$ and $x, y, z \in X$.

Theorem 3.4. Every (α, β)-fuzzy p-ideal of a BF-algebra X is an (α, β)-fuzzy ideal of X.

Proof. Let λ be an (α, β)-fuzzy p-ideal of X. Then for all $t_1, t_2 \in (0, 1]$ and $x, y, z \in X$, we have

$$((x * z) * (y * z))_{t_1}\ \alpha\ \lambda,\ y_{t_2}\ \alpha\ \lambda \ \Rightarrow\ x_{t_1 \wedge t_2}\ \beta\ \lambda.$$

Putting z = 0 in above, we get

$$((x * 0) * (y * 0))_{t_1}\ \alpha\ \lambda,\ y_{t_2}\ \alpha\ \lambda \ \Rightarrow\ x_{t_1 \wedge t_2}\ \beta\ \lambda.$$

$$(x * y)_{t_1}\ \alpha\ \lambda,\ y_{t_2}\ \alpha\ \lambda \ \Rightarrow\ x_{t_1 \wedge t_2}\ \beta\ \lambda \quad \text{(By BF-2)}$$

This means that λ satisfies the condition (B). Combining with (A) implies that λ is an (α, β)-fuzzy ideal of X.

Theorem 3.5. A fuzzy set λ of a BF-algebra X is a fuzzy p-ideal of X if and only if λ is an $(\in, \in)$-fuzzy p-ideal of X.

Proof. Suppose λ is a fuzzy p-ideal of X and $x_t \in \lambda$ for $x \in X$ and $t \in (0, 1]$. Then $\lambda(x) \geq t$. By Definition 2.5, $\lambda(0) \geq \lambda(x)$, we have $\lambda(0) \geq t$, that is $0_t \in \lambda$. Let x, y, z $\in$ X and t, r $\in$ (0, 1] be such that

$$((x * z) * (y * z))_t \in \lambda \text{ and } y_t \in \lambda.$$

Then

$$\lambda((x * z) * (y * z)) \geq t \text{ and } \lambda(y) \geq r.$$

By Definition 2.5, we have

$$\lambda(x) \geq \lambda((x * z) * (y * z)) \wedge \lambda(y) \geq t \wedge r.$$

This implies that $x_{t \wedge r} \in \lambda$. This shows that λ is an $(\in, \in)$-fuzzy p-ideal of X.

Conversely, assume that λ is an $(\in, \in)$-fuzzy p-ideal of X. Suppose there exists $x \in X$ such that $\lambda(0) < \lambda(x)$. Select $t \in (0, 1]$ such that $\lambda(0) < t \leq \lambda(x)$. Then $x_t \in \lambda$ but $0_t \,\overline{\in}\, \lambda$, which is a contradiction. Hence $\lambda(0) \geq \lambda(x)$, for all $x \in X$. Now suppose there exist x, y, z $\in$ X such that

$$\lambda(x) < \lambda((x * z) * (y * z)) \wedge \lambda(y).$$

Select $t \in (0, 1]$ such that

$$\lambda(x) < t \leq \lambda((x * z) * (y * z)) \wedge \lambda(y).$$

Then $((x * z) * (y * z))_t \in \lambda$ and $y_t \in \lambda$ but $x_t \,\overline{\in}\, \lambda$, which is a contradiction. Hence

$$\lambda(x) \geq \lambda((x * z) * (y * z)) \wedge \lambda(y).$$

This shows that λ is a fuzzy p-ideal of X.

4. $(\in, \in \vee q_k)$-fuzzy ideals

In this section, we discuss $(\in, \in \vee q_k)$-fuzzy ideal in a BF-algebra and study their related properties. Let k denote an arbitrary element of [0, 1) unless otherwise specified. We defined $x_t q_k \lambda$ if $\lambda(x) + t + k > t$, $x_t \in \vee q_k \lambda$ if $x_t \in \lambda$ or $x_t q_k \lambda$.

Definition 4.1. A fuzzy set λ of a BF-algebra X is called an $(\in, \in \vee q_k)$-fuzzy ideal of X if it satisfies the conditions (D) and (E), where

(D) $x_t \in \lambda \;\Rightarrow\; 0_t \in \vee q_k \lambda$,

(E) $(x * y)_t \in \lambda,\ y_r \in \lambda \;\Rightarrow\; x_{t \wedge r} \in \vee q_k \lambda$,

for all t, r $\in$ (0, 1] and x, y $\in$ X.

Theorem 4.2. The conditions (D) and (E) in Definition 4.1, are equivalent to the following conditions, respectively:

(F) $\lambda(0) \geq \lambda(x) \wedge \dfrac{1-k}{2}$,

(G) $\lambda(x) \geq \lambda(x * y) \wedge \lambda(y) \wedge \dfrac{1-k}{2}$,

for all x, y $\in$ X.

Proof. Straightforward.

Lemma 4.3. For any $(\in, \in \vee q_k)$-fuzzy ideal λ of a BF-algebra X, if $x \leq y$ then

$$\lambda(x) \geq \lambda(y) \wedge \frac{1-k}{2}.$$

Proof. Straightforward.

Lemma 4.4. Let λ be an $(\in, \in \vee q_k)$-fuzzy ideal of a BF-algebra X. Then

$$x * y \le z \text{ implies } \lambda(x) \ge \lambda(y) \wedge \lambda(z) \wedge \frac{1-k}{2}.$$

Proof. Straightforward.

Proposition 4.5. A fuzzy set λ of a BF-algebra X is an $(\in, \in \vee q_k)$-fuzzy ideal of X if and only if the set λ_t is an ideal of X for all $0 < t \le \frac{1-k}{2}$.

Proof. Straightforward.

Theorem 4.6. Every fuzzy ideal of a BF-algebra X is an $(\in, \in \vee q_k)$-fuzzy ideal of X.

Proof. Straightforward.

5. $(\in, \in \vee q_k)$-fuzzy p-ideals

In this section, we define the notion of $(\in, \in \vee q_k)$-fuzzy p-ideal in a BF-algebra and investigate some of their related properties.

Definition 5.1. A fuzzy set λ of a BF-algebra X is called an $(\in, \in \vee q_k)$-fuzzy p-ideal of X if it satisfies the conditions (D) and (H), where

(D) $x_t \in \lambda \Rightarrow 0_t \in \vee q_k\lambda$,

(H) $((x * z) * (y * z))_t \in \lambda, y_r \in \lambda \Rightarrow x_{t \wedge r} \in \vee q_k\lambda$,

for all t, r $\in$ (0, 1] and x, y, z $\in$ X.

Theorem 5.2. The condition (H) is equivalent to the condition (I), where

(I) $\lambda(x) \ge \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2}$

for all x, y, z $\in$ X.

Proof. (H) $\Rightarrow$ (I)

Suppose λ satisfies condition (H). On the contrary, assume that there exist x, y, z $\in$ X such that

$$\lambda(x) < \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2}.$$

Choose t $\in$ (0, 1] such that

$$\lambda(x) < t \le \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2}.$$

Then

$$((x * z) * (y * z))_t \in \lambda, y_t \in \lambda \text{ but } x_t \overline{\in} \lambda$$

and

$$\lambda(x) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1.$$

So $x_t \overline{\in \vee q_k} \lambda$. This is a contradiction. Hence

$$\lambda(x) \ge \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2}.$$

(I) $\Rightarrow$ (H)

Assume that

$$\lambda(x) \geq \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2}.$$

Let

$$((x * z) * (y * z))_t \in \lambda,\ y_r \in \lambda \text{ for all } t, r \in (0, 1].$$

Then

$$\lambda((x * z) * (y * z)) \geq t \text{ and } \lambda(y) \geq r.$$

Now

$$\lambda(x) \geq \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2}$$

$$\geq t \wedge r \wedge \frac{1-k}{2}.$$

If $t \wedge r \leq \frac{1-k}{2}$, then $\lambda(x) \geq t \wedge r$. So $x_{t \wedge r} \in \lambda$.

If $t \wedge r > \frac{1-k}{2}$, then

$$\lambda(x) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1.$$

This implies that $x_{t \wedge r}\, q_k \lambda$. Hence

$$x_{t \wedge r} \in \vee\, q_k \lambda.$$

Corollary 5.3. A fuzzy set λ of a BF-algebra X is an $(\in, \in \vee q_k)$-fuzzy p-ideal of X if it satisfies the conditions (F) and (I).

Example 5.4. Let X = {0, 1, 2, 3} be a BF-algebra with Caylay table as follows [5]:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Let λ be a fuzzy subset in X defined by $\lambda(0) = 0.99$, $\lambda(1) = \lambda(2) = \lambda(3) = 0.88$. By route calculations show that λ is an $(\in, \in \vee q_k)$-fuzzy p-ideal of X for k = 0.1.

Theorem 5.5. Every $(\in, \in)$-fuzzy p-ideal of a BF-algebra X is an $(\in, \in \vee q_k)$-fuzzy p-ideal of X.

Proof. Straightforward.

Corollary 5.6. Every fuzzy p-ideal of a BF-algebra X is an $(\in, \in \vee q_k)$-fuzzy p-ideal of X.

Proof. By Theorem 3.5, every fuzzy p-ideal of a BF-algebra X is an $(\in, \in)$-fuzzy p-ideal of X. Hence by above Theorem 5.5, every fuzzy p-ideal of a BF-algebra X is an $(\in, \in \vee q_k)$-fuzzy p-ideal of X.

Now, we characterize $(\in, \in \vee q_k)$-fuzzy p-ideals by their level sets.

Theorem 5.7. A fuzzy set λ of a BF-algebra X is an $(\in, \in \vee q_k)$-fuzzy p-ideal of X if and only if the set λ_t ($\neq \phi$) is an p-ideal of X for all $0 < t \leq \frac{1-k}{2}$.

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy p-ideal of X and $0 < t \leq \frac{1-k}{2}$. If $\lambda_t \neq \phi$, then $x \in \lambda_t$. This implies that $\lambda(x) \geq t$. By condition (F), we have

$$\lambda(0) \geq \lambda(x) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2} = t.$$

Thus $\lambda(0) \geq t$. Hence $0 \in \lambda_t$. Let $(x * z) * (y * z) \in \lambda_t$ and $y \in \lambda_t$. Then

$$\lambda((x * z) * (y * z)) \geq t \quad \text{and} \quad \lambda(y) \geq t.$$

By condition (I), we have

$$\begin{aligned} \lambda(x) &\geq \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2} \\ &\geq t \wedge t \wedge \frac{1-k}{2} \\ &= t \wedge \frac{1-k}{2} \\ &= t. \end{aligned}$$

Thus $\lambda(x) \geq t$, that is, $x \in \lambda_t$. Therefore λ_t is a p-ideal of X.

Conversely, assume that λ is a fuzzy set of X such that $\lambda_t (\neq \phi)$ is an p-ideal of X for all $0 < t \leq \frac{1-k}{2}$. Let $x \in X$ be such that

$$\lambda(0) < \lambda(x) \wedge \frac{1-k}{2}.$$

Select $0 < t \leq \frac{1-k}{2}$ such that

$$\lambda(0) < t \leq \lambda(x) \wedge \frac{1-k}{2}.$$

Then $x \in \lambda_t$ but $0 \notin \lambda_t$, a contradiction. Hence

$$\lambda(0) \geq \lambda(x) \wedge \frac{1-k}{2}.$$

Now assume that $x, y, z \in X$ such that

$$\lambda(x) < \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2}.$$

Select $0 < t \leq \frac{1-k}{2}$ such that

$$\lambda(x) < t \leq \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2}.$$

Then $((x * z) * (y * z))$ and y are in λ_t but $x \notin \lambda_t$, a contradiction. Hence

$$\lambda(x) \geq \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2}.$$

This shows that λ is an $(\in, \in \vee q_k)$-fuzzy p-ideal of X.

Theorem 5.8. Every $(\in, \in \vee q_k)$-fuzzy p-ideal of a BF-algebra X is an $(\in, \in \vee q_k)$-fuzzy ideal of X.

Proof. The proof follows from Theorem 3.4.

Theorem 5.9. The intersection of any family of $(\in, \in \vee q_k)$-fuzzy p-ideals of a BF-algebra X is an $(\in, \in \vee q_k)$-fuzzy p-ideal of X.

Proof. Let $\{\lambda_i\}_{i \in I}$ be a family of $(\in, \in \vee q_k)$-fuzzy p-ideals of a BF-algebra X and x $\in$ X. So

$$\lambda_i(0) \geq \lambda_i(x) \wedge \frac{1-k}{2}$$

for all i $\in$ I. Thus

$$\begin{aligned}(\wedge_{i \in I} \lambda_i)(0) &= \wedge_{i \in I} (\lambda_i(0)) \\ &\geq \wedge_{i \in I} (\lambda_i(x) \wedge \frac{1-k}{2}) \\ &= (\wedge_{i \in I} \lambda_i)(x) \wedge \frac{1-k}{2}.\end{aligned}$$

Thus

$$(\wedge_{i \in I} \lambda_i)(0) \geq (\wedge_{i \in I} \lambda_i)(x) \wedge \frac{1-k}{2}.$$

Let x, y, z $\in$ X. Since each λ_i is an $(\in, \in \vee q_k)$-fuzzy p-ideal of X. So

$$\lambda_i(x) \geq \lambda_i((x * z) * (y * z)) \wedge \lambda_i(y) \wedge \frac{1-k}{2}$$

for all i $\in$ I. Thus

$$\begin{aligned}(\wedge_{i \in I} \lambda_i)(x) &= \wedge_{i \in I} (\lambda_i(x)) \\ &\geq \wedge_{i \in I} (\lambda_i((x * z) * (y * z)) \wedge \lambda_i(y) \wedge \frac{1-k}{2}) \\ &= (\wedge_{i \in I} \lambda_i)((x * z) * (y * z)) \wedge (\wedge_{i \in I} \lambda_i)(y) \wedge \frac{1-k}{2}\end{aligned}$$

Thus

$$(\wedge_{i \in I} \lambda_i)(x) \geq (\wedge_{i \in I} \lambda_i)((x * z) * (y * z)) \wedge (\wedge_{i \in I} \lambda_i)(y) \wedge \frac{1-k}{2}.$$

Hence, $\wedge_{i \in I} \lambda_i$ is an $(\in, \in \vee q_k)$-fuzzy p-ideal of X.

Theorem 5.10. The union of any family of $(\in, \in \vee q_k)$-fuzzy p-ideals of a BF-algebra X is an $(\in, \in \vee q_k)$-fuzzy p-ideal of X.

Proof. Straightforward.

6. Implication-based fuzzy p-ideals

In this section, we define the concept of implication-based fuzzy p-ideal in a BF-algebra and explore some of their properties.

Fuzzy propositional calculus is an extension of the Aristotelean propositional calculus. In fuzzy propositional calculus the truth set is taken [0, 1] instead of {0, 1}, which is the truth set in Aristotelean propositional calculus. In fuzzy logic some of the operators, like $\wedge, \vee, \neg, \rightarrow$ can be defined by using truth tables. One can also use the extension principle to obtain the definitions of these operators.

In fuzzy logic the truth value of a fuzzy proposition P is denoted by [P]: In the following we give fuzzy logic and its corresponding set theoretical notations, which we will use in the Thesis hereafter.

$[x \in \lambda] = \lambda(x)$
$[x \notin \lambda] = 1 - \lambda(x)$
$[P \wedge Q] = [P] \wedge [Q]$
$[P \vee Q] = [P] \vee [Q]$
$[P \rightarrow Q] = 1 \wedge (1 - [P] + [Q])$
$[\forall x P(x)] = \wedge [P(x)]$
$\models P$ if and only if $[P] = 1$ for all valuations.

Of course, various implication operators can be similarly defined. We consider in the following some important implication operators:

Name	Definition of Implication Operators
Early Zadeh	$I_m(x, y) = \max\{1- x, \min\{x, y\}\}$
Lukasiewicz	$I_a(x, y) = \min\{1, 1- x + y\}$
Standard Star (Godel)	$I_g(x, y) = \begin{cases} 1 & \text{if } x \le y, \\ y & \text{if } x > y, \end{cases}$
Contraposition of Godel	$I_{cg}(x, y) = \begin{cases} 1 & \text{if } x \le y, \\ 1-x & \text{if } x > y, \end{cases}$
Gaines-Rescher	$I_{gr}(x, y) = \begin{cases} 1 & \text{if } x \le y, \\ 0 & \text{if } x > y, \end{cases}$
Kleene-Dienes	$I_b(x, y) = \max\{1- x, y\}$

where x is the degree of truth (or degree of membership) of the premise and y is the respective value for the consequence and I is the resulting degree of truth for the implication. The "quality" of these implication operators could be evaluated either by empirically or by axiomatically methods.

In the following definition, we consider the implication operators in the Lukasiewicz system of continuous-valued logic.

Definition 6.1. A fuzzy set λ of a BF-algebra X is called a fuzzifying p-ideal of X if it satisfies the conditions (J) and (K), where

(J) $\models [x \in \lambda] \rightarrow [0 \in \lambda]$,

(K) $\models [((x * z) * (y * z) \in \lambda] \wedge [y \in \lambda] \rightarrow [x \in \lambda]$,

for all $x, y, z \in X$.

Clearly, Definition 6.1 is equivalent to Definition 2.5. Therefore a fuzzifying p-ideal is an ordinary fuzzy p-ideal.

In [28] the concept of t-tautology is given, $\models_t P$ if and only if $[P] \ge t$, for all valuations. Now we define implication-based fuzzy p-ideal of a BF-algebra.

Definition 6.2. Let λ be a fuzzy set of a BF-algebra X and $t \in (0, 1]$ be a fixed number. Then λ is called a t-implication-based fuzzy p-ideal of X if it satisfies the conditions (J) and (L), where

(J) $\models_t [x \in \lambda] \rightarrow [0 \in \lambda]$,

(L) $\models_t [((x * z) * (y * z) \in \lambda] \wedge [y \in \lambda] \rightarrow [x \in \lambda]$,

for all $x, y, z \in X$.

Corollary 6.3. Let I be an implication operator, $t \in (0, 1]$ be a fixed number and λ be a fuzzy set of X. Then λ is a t-implication-based fuzzy p-ideal of a BF-algebra X if and only if the following conditions hold:

(M) $I(\lambda(x), \lambda(0)) \ge t$,

(N) $I(\lambda((x * z) * (y * z)) \wedge \lambda(y), \lambda(x)) \ge t$,

for all $x, y, z \in X$.

Let λ be a fuzzy set of a BF-algebra X. Then we have the following results:

Theorem 6.4.

(O) Let $I = I_{gr}$. Then λ is a 0.5-implication-based fuzzy p-ideal of a BF-algebra X if and only if λ is a fuzzy p-ideal with thresholds (ε = 0, δ = 1) of X.

(P) Let $I = I_g$. Then λ is a $\frac{1-k}{2}$-implication-based fuzzy p-ideal of a BF-algebra X if and only if λ is an $(\in, \in \vee q_k)$-fuzzy p-ideal with thresholds (ε = 0, δ = $\frac{1-k}{2}$) of X.

(Q) Let $I = I_{cg}$. Then λ is a $\frac{1-k}{2}$-implication-based fuzzy p-ideal of a BF-algebra X if and only if λ is an $(\in, \in \vee q_k)$-fuzzy p-ideal with thresholds (ε = $\frac{1-k}{2}$, δ = 1) of X.

Proof. We only prove (P) and the proofs of (O) and (Q) are similar.

Let λ be a $\frac{1-k}{2}$-implication-based fuzzy p-ideal of X. Then by Corollary 6.3, we have

(*a*) $I_g(\lambda(x), \lambda(0)) \geq \frac{1-k}{2}$ and

(b) $I_g(\lambda((x * z) * (y * z)) \wedge \lambda(y), \lambda(x)) \geq \frac{1-k}{2}$.

From (*a*), we have

$$\lambda(0) \geq \lambda(x) \text{ or } \lambda(x) > \lambda(0) \geq \frac{1-k}{2}.$$

Thus

$$\lambda(0) \geq \lambda(x) \wedge \frac{1-k}{2}.$$

It follows that

$$\lambda(0) \vee 0 = \lambda(0) \geq \lambda(x) \wedge \frac{1-k}{2}.$$

From (b), we have

$$\lambda(x) \geq \lambda((x * z) * (y * z)) \wedge \lambda(y),$$

or

$$\lambda((x * z) * (y * z)) \wedge \lambda(y) > \lambda(x)) \geq \frac{1-k}{2}.$$

It follows that

$$\lambda(x) \vee 0 = \lambda(x) \geq \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2}.$$

This shows that λ is an $(\in, \in \vee q_k)$-fuzzy p-ideal with thresholds (ε = 0, δ = $\frac{1-k}{2}$) of X.

Conversely, assume that λ is an $(\in, \in \vee q_k)$-fuzzy p-ideal with thresholds (ε = 0, δ = $\frac{1-k}{2}$) of X, then we have (*a*)

$$\lambda(0) = \lambda(0) \vee 0 \geq \lambda(x) \wedge \frac{1-k}{2},$$

and (b)

$$\lambda(x) = \lambda(x) \vee 0 \geq \lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2}.$$

From (*a*), if $\lambda(x) \wedge \frac{1-k}{2} = \lambda(x)$, then

$$I_g(\lambda(x), \lambda(0)) = 1 \geq \frac{1-k}{2}.$$

Otherwise

$$I_g(\lambda(x), \lambda(0)) \geq \frac{1-k}{2}.$$

From (b), if

$$\lambda((x * z) * (y * z)) \wedge \lambda(y) \wedge \frac{1-k}{2} = \lambda((x * z) * (y * z)) \wedge \lambda(y),$$

then

$$I_g(\lambda((x * z) * (y * z)) \wedge \lambda(y), \lambda(x)) = 1 \geq \frac{1-k}{2}.$$

Otherwise

$$I_g(\lambda((x * z) * (y * z)) \wedge \lambda(y), \lambda(x)) \geq \frac{1-k}{2}.$$

Therefore, λ is a $\frac{1-k}{2}$-implication-based fuzzy p-ideal of X.

7. Conclusion

In learn of fuzzy algebraic system, we observe that the fuzzy p-ideals with extraordinary properties for all time play an essential role.

The purpose of this paper is to define the concept of $(\in, \in \vee q_k)$-fuzzy p-ideal in BF-algebra and some related properties are investigated. The notion of implication-based fuzzy p-ideal and implication operators in Lukasiewicz system of continuous-valued logic in BF-algebra is introduced and discussed some of their connected properties.

We think that the research along this direction can be continued, and in fact, a few results in this paper have already constituted a foundation for extra investigation relating to the more progress of fuzzy BF-algebras and their applications in other branches of algebra. In the future study of fuzzy BF-algebras, perhaps the following topics are worth to be considered:

(1) To describe other classes of BF-algebras by using this concept;

(2) To apply this idea to some further algebraic structures;

(3) To consider these results to some possible applications in computer sciences and information systems in the future.

References

[1]. S. K. Bhakat, $(\in \vee q)$-1eve1 subsets, Fuzzy Sets and Systems, 103 (1999), 529-533.

[2]. S. K. Bhakat, $(\in, \in \vee q)$-fuzzy normal, quasinormal and maximal subgroups, Fuzzy Sets and Systems, 112 (2000), 299-312.

[3]. S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy subgroups, Fuzzy Sets and Systems, 80 (1996), 359-368.

[4]. P. S. Das, Fuzzy groups and 1eve1 subgroups, J. Math. Anal. Appl., 84 (1981), 264-269.
[5]. A. K. Dutta and D. Hazarika, $(\in, \in \vee q)$-fuzzy ideals of BF-algebra, International Mathematical Forum, 8 (2013), 1253-1261.
[6]. B. Davvaz, $(\in, \in \vee q)$-fuzzy subnearrings and ideals, Soft Comput., 10 (2006), 206-211.
[7]. A. R. Hadipour and A. B. Saeid, Fuzzy soft BF-algebras, Indian Journal of Science and Technology, 6(3) (2013), 4200-4204.
[8]. A. R. Hadipour, On generalized fuzzy BF-algebras, FUZZ-IEEE, (2009), 1672-1676.
[9]. A. R. Hadipour, Double-framed Soft BF-algebras, Indian Journal of Science and Technology, 7(4)(2014), 491-496.
[10]. Y. B. Jun, On (α, β)-fuzzy ideals of BCK/BCI-algebras, Sci. Math. Japon., 60 (2004), 613-617.
[11]. Y. B. Jun, On (α, β)-fuzzy subalgebras of BCK/BCI-algebras, Bull. Korean Math. Soc., 42 (2005), 703-711.
[12]. Y. B. Jun, Fuzzy subalgebras of type (α, β) in BCK/BCI-algebras, Kyungpook Math. J., 47 (2007), 403-410.
[13]. Y. B. Jun, Genera1izations of $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras, Comput. Math. Appl., 58(2009), 1383-1390.
[14]. Y. B. Jun, K. J. Lee and C. H. Park, New types of fuzzy ideals in BCK/BCI-algebras, Comput. Math. Appl., 60 (2010), 771-785.
[15]. A. Khan, Y. B. Jun, N. H. Sarmin and F. M. Khan, Ordered semigroups characterized by $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals, Neural Comput. & Applic., 21 (2012), S121-S132.
[16]. M. A. Larimi, On $(\in, \in \vee q_k)$-intuitionistic fuzzy ideals of hemirings, World App1ied Sciences Journal, 21 (2013), 54-67.
[17]. P. Muralikrishna and M. Chandramouleeswaran, Study on N-ideal of a BF-algebras, International Journal of Pure and App1ied Mathematics, 83(4) (2013), 607-612.
[18]. V. Murali, Fuzzy points of equivalent FSs, Inform. Sci., 158 (2004), 277-288.
[19]. P. M. Pu and Y. M. Liu, Fuzzy topology I: neighourhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl., 76 (1980), 571-599.
[20]. A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., 35 (1971), 512-517.
[21]. M . Renugha, M . Sivasakthi and M . Chellam,, Cubic BF-A1gebra, International Journal of Innovative Research in Advanced Engineering, 1(7)(2014), 48-52.
[22]. A. B. Saeid and M. A. Rezvani, On fuzzy BF-algebras, International Mathematical Forum, 4(1) (2009), 13-25.
[23]. M. Shabir, Y. B. Jun and Y. Nawaz, Semigroups characterized by $(\in, \in \vee q_k)$-fuzzy ideals, Comput. Math. Appl., 60 (2010), 1473-1493.
[24]. M. Shabir and N. Rehman, Characterizations of ternary semigroups by $(\in, \in \vee q_k)$-fuzzy ideals, Iranian Journal of Science & Technology, A3 (2012), 395-410.
[25]. M. Shabir and T. Mahmood, Semihypergroups characterized by $(\in, \in \vee q_k)$- fuzzy hyperideals, Inf. Sci. Lett., 2(2) (2013), 101-121.
[26]. J. Tang and X. Y. Xie, On $(\in, \in \vee q_k)$-fuzzy ideals of ordered semigroups, Fuzzy Inf. Eng. (2013), 1 (2013), 57-67.
[27]. A. Walendziak, On BF-algebras, Mathematica Slovaca, 57 (2007), 119-128.
[28]. M. S. Ying, On standard models of fuzzy modal logics, Fuzzy Sets and Systems, 26 (1988), 357-363.
[29]. L. A. Zadeh, Fuzzy sets, Inform. and Control, 8 (1965), 338-353.
[30]. S. A. N. Zadeh, A. Radfar and A. B. Saied, On BF-algebras and QS-algebras, The Journal of Mathematics and Computer Science, 5(1) (2012), 7-21.

[31]. A. Zeb, G. Zaman, I. A. Shah and A. Khan, Characterization of ternary semigroups in terms of $(\in, \in \vee q_k)$ ideals, Mathematica Aeterna, 2(3) (2012), 227-246.

[32]. M. Zulfiqar, Some properties of (α, β)-fuzzy positive implicative ideals in BCK-algebras, Acta Scientiarum. Technology, 35(2) (2013), 371-377.

[33]. M. Zulfiqar, Commutative fuzzy ideals of BCH-algebras and other superior 1eve1s of fuzzyfication, Mathematical Reports, 3(2014), 331-372.

[34]. M. Zulfiqar, Characterizations of $(\in, \in \vee q_k)$-fuzzy fantastic ideals in BCI-algebras, Kuwait Journal of Science, 41(1) (2014), 35-64.