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**Research Article**

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# Asymptotic Analysis of a Neo-Hookean Half-Space Deforming under Anti–Plane Shear

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**Abstract** The problem of asymptotic analysis of a Neo-Hookean half space deforming under anti plane shear was investigated. The analysis of the model resulted into a partial differential equation for the determination of stresses and displacement. Analytically obtained closed form solution through asymptomatic analysis in Sobolev space of a Neo-Hookean Half space deforming under anti-plane shear and effort is minimized. Two cases were considered; the first case is a case where flaw is a circular cavity at the centre of the half space at an undeformed configuration. The second case is a case where the void is patched prior to deformation. The patch or filing is done with a material such that a perfect bond is created at the interface. In other words, there is no differential motion at the interface and in either case the analysis leads to an exact closed form solution for stresses and displacement.

**Keywords** Hookean Deformation, Asymptotic Analysis, Anti – Plane Shear

## 1. Introduction

Anti – plane shear deformation problems arise naturally from many real world applications, such as rectilinear steady flow of simple fluids, interface stress effects of nanostructured materials, structures with cracks, layered/composite functioning materials and phase transitions in solids. During the past half century, such problems in finite deformation theory have been subjected to extensively study by both mathematician and engineering scientists. As indicated in the review article by Horgan, [1] anti-plane shear deformations are one of the simplest classes of deformations that solids can undergo. In anti-plane shear of a cylindrical body, the displacement is parallel to the generators of the cylinder and is independent of the axial co-ordinate. In recent years, considerable attention has been paid to the analysis of anti-plane shear deformations within the context of various constitutive theories (linear and non-linear) of solid mechanics. Such studies were largely motivated by the promise of relative analytic simplicity compared with plane problems since the governing equations are a single second order partial differential equation rather than higher-order or coupled systems of partial differential equation. Thus the anti-plane shear problem plays a useful role as a pilot problem, within which various aspects of solutions in solid mechanics may be examined in a particularly simple setting.

The review of Literature shows that various authors have made a number of contributions towards providing a method of analysis that could readily give closed form solutions for the determination of stresses and displacement at various points of various solids deforming under anti-plane shear loading. Atkins, [2] seems to have been the first who studied deformations belonging to this class, for an incompressible isotropic elastic body. He provided an overview in which further analysis in this directions were based. Knowles [3] extended Atkins work by providing the general form that the stored energy should have in order that an isotropic and incompressible material admits states of anti-plane shear. He also used the frame work of anti-plane shear to analyze a body containing a central crack. In the same paper he showed how the three differential equations of equilibrium can be reduced to a single one for the case of generalized Neo-Hookean materials. He also provided a necessary and sufficient conditions on the strain energy density function in order that a compressible isotropic material admits state of anti-plane shear.

Jiang and Knowles [4] determined a special class of a compressible isotropic body that can sustain anti-plane shear. Tsai and Rosakis [5] worked on anisotropic bodies in the dynamical case for a compressible material. They determine the conditions for such materials to admit anti-plane shear deformation. Erumaka, [6] used an asymptotic approach to determine the solution of the boundry value problem which valuate the stresses and displacements in an Ogden solid deforming under anti-plane shear. His solution was considered as a very good approximation to the exact solutions.

Zhou et al, [7] showed that in the vicinity of two collinear cracks perpendicular to the edges of an isotropic strip, the cracks were symmetrical with respect to the centerline of the strip when subjected to anti-plane traction. Li, [8] used the work of Zhou to obtain a closed-form solution for orthotropic strips.

Stress analysis in an isotropic strip weakened by two collinear cracks situated on the centerline under anti-plane shear was carried out by Zhou et al [7]. In the above articles, the application of boundary conditions resulted in a set of integral equations which are solved by the schmidt's method to obtain approximate solutions.

Wu and Dzenis, [9] obtained closed-form solutions which are solved by the factors for an interfacial edge crack between two bonded semi-infinite dissimilar elastic strips. Li, [8] considered an interfacial crack between two bonded dissimilar semi-infinite orthotropic strips where the crack surface was under anti-plane traction. Closed form stress intensity factors were obtained for a strip with either clamped or traction – free boundaries.

Lubarda [10] studied the effect of the couple stresses on the stress concentration around circular inclusions and inhomogeneities subjected to an anti-plane strain field. Couple stress theory has also been used by Shodja and Ghazisaeidi [11] to address the anti-plane problem of circular inhomogeneity in piezoelectric media. Haftbaradaran and Shodja [12] studied elliptic inhomogeneities and inclusions subjected to anti-plane loading within couple stress theory.

## 2. Formulation of the Problem

In this paper, the method proposed by Erumaka, [13] on "finite deformation of a class of Ogden solid under anti-plane shear is employed to provide an exact closed form solution for the boundary value problem arising from the anti-plane shear deformation of a Neo-Hookean half space containing central circular flaws. Two cases are considered. The first is a case in which the flaw is a circular cavity at the centre of the half space at underformed configuration, while the second case is the situation in which the void is patched prior to deformation. The patch or filling is done with a material capable of creating a perfect bond at the interface. The bond is such that there is no differential motion at the interface. In either case the analysis leads to exact closed form solution for the determination of stresses and displacement at every point of the material. In this work, we shall consider an anti-plane shear deformation of a class of an incompressible solid whose strain energy function is given by

$$W = \frac{\mu}{2}(I_1 - 3) \tag{2.1}$$

Such a material is referred to as Neo-Hookean material. An asymptotic method is used to derive an exact closed form solution for the boundary value problem for the anti-plane shear deformation of a Neo-Hookean half space containing central circular flaws. We shall consider two cases. The first is a case in which the flaw is a circular cavity at the centre of the half space at undeformed configuration while the second case is the situation in which the void is patched prior to deformation. The patch or filling is done with a material such that a perfect bond is created at interface. This is to ensure that there is no differential motion at interface. In either case we aim at providing exact closed form solutions for the determination of stresses and displacement. We shall also determine the point of stress concentration indicating the positions of crack initiation.

**3. Asymptotic Analysis of the Problem**

Let the deformation that takes the point $(X_1, X_2, X_3)$ of the undeformed configuration to the point $(x_1, x_2, x_3)$ of the deformed configuration of an isotropic, homogeneous, incompressible, elastic half space deforming under anti-plane shear in Cartesian coordinate be

$$\left.\begin{aligned} x_1 &= X_1 \\ x_2 &= X_2 \\ x_3 &= X_3 + g(X_1, X_2) \end{aligned}\right\} \qquad 3.1$$

where $X_i,\ i = 1,\ 2\ and\ 3$ are the co-ordinates of a point at the undeformed configuration. $x_i,\ i = 1,\ 2\ and\ 3$ are the corresponding co-ordinate at the deformed configurations.g depends continuously on $X_1\ and\ X_2$.

**3.1. Deformation Gradient Tensor $\bar{F}$**

We shall determine the deformation gradient, which describes how material line elements change their length and orientation during deformation. The deformation gradient tensor, $\bar{F}$ is given by

$$\bar{F} = \frac{\partial x_i}{\partial X_R} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \qquad (3.2)$$

$where\ i = 1, 2, 3\ and\ R = 1, 2, 3$

Using (3.1) and (3.2) we obtain the following as components of $\bar{F}$

$$F_{11} = \frac{\partial x_1}{\partial X_1} = 1,\ F_{12} = \frac{\partial x_1}{\partial X_2} = 0,\ F_{13} = \frac{\partial x_1}{\partial X_3} = 0$$

$$F_{21} = \frac{\partial x_2}{\partial X_1} = 0,\ F_{22} = \frac{\partial x_2}{\partial X_2} = 1,\ F_{23} = \frac{\partial x_2}{\partial X_3} = 0$$

$$F_{31} = \frac{\partial x_3}{\partial X_1} = g,_1,\ F_{32} = \frac{\partial x_3}{\partial X_2} = g,_2,\ F_{33} = \frac{\partial x_3}{\partial X_3} = 1$$

Hence (3.2) can be written as the tensor

$$\bar{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g,_1 & g,_2 & 1 \end{pmatrix} \qquad 3.3$$

$$where\ g,_1 = \frac{\partial g}{\partial X_1}\ and\ g,_2 = \frac{\partial g}{\partial X_2}$$

**3.2. Left Cauchy – Green Deformation Tensor $\bar{B}$**

Since the deformation equations are stated on the material or lagrangian mode, we shall obtain the left Cauchy – Green deformation strain tensors $\bar{B}$.

The left Cauchy – Green deformation gradient tensor $\bar{B}$ is a symmetric second order tensor given by (Beatly, 1962).

$$\bar{B} = \bar{F}\,\bar{F}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g,_1 & g,_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & g,_1 \\ 0 & 1 & g,_2 \\ 0 & 0 & 1 \end{pmatrix} \quad 3.3$$

*Therefore*

$$\bar{B} = \begin{pmatrix} 1 & 0 & g,_1 \\ 0 & 1 & g,_2 \\ g,_1 & g,_2 & 1+g_{,1}^2+g_{,2}^2 \end{pmatrix} \quad 3.4$$

$$\bar{B}^2 = \bar{B}.\bar{B} = \begin{pmatrix} 1 & 0 & g,_1 \\ 0 & 1 & g,_2 \\ g,_1 & g,_2 & 1+g,_1^2+g,_2^2 \end{pmatrix}\begin{pmatrix} 1 & 0 & g,_1 \\ 0 & 1 & g,_2 \\ g,_1 & g,_2 & 1+g,_1^2+g,_2^2 \end{pmatrix}$$

$$\bar{B}^2 = \begin{pmatrix} 1+g,_1^2 & g,_1 g,_2 & g,_1\left(2+g,_1^2+g,_2^2\right) \\ g,_1 g,_2 & 1+g,_2 & g,_2\left(2+g,_1^2+g,_2^2\right) \\ g,_1\left(2+g,_1^2+g,_2^2\right) & g,_2\left(2+g,_1^2+g,_2^2\right) & \left(1+3g,_1^2+3g,_2^2+g,_1^4+2g,_1^2\,g,_2^2\right) \end{pmatrix} \quad 3.5$$

### 3.3. The Principal Strain Invariants $I_i,\ i=1,2,3$

The left Cauchy – Green tensor is a second order tensor and it is symmetric, then it has three principal strain invariant denoted by $I_1,\ I_2\ and\ I_3$

It has been established that the strain energy density functions of elastic materials are functions of the corresponding strain invariants. $I_1,\ I_2\ and\ I_3$

The physical interpretation of $I_1,\ I_2\ and\ I_3$ are the hydrostatic stress, shear stress and Von mises stress respectively. It is given in Beatly (1961) as

$$\left.\begin{aligned} I_1 &= tr\bar{B} \\ I_2 &= \frac{1}{2}\left[\left(tr\bar{B}\right)^2 - tr\bar{B}^2\right] \\ I_3 &= \det\bar{B} = J^2 \end{aligned}\right\} \quad 3.6$$

where tr denotes trace, J denotes Jacobian and detB denotes determinant of $\bar{B}$. Using (3.4) and (3.5), we have (3.6) as

$$I_1 = tr\bar{B} = 3+g,_1^2+g,_2^2$$

$$I_2 = \frac{1}{2}\left[\left(tr\bar{B}\right)^2 - tr\bar{B}^2\right]\Big\}$$

$$tr\bar{B}^2 = 3+4g,_1^2+4g,_2^2+g,_1^4+g,_2^4+2g,_1^2\,,g,_2^2$$

$$also\ \left(tr\bar{B}\right)^2 = 9+6g,_1^2+6g,_2^2+g,_1^4+g,_2^4+2g,_1^2\,,g,_2^2$$

Therefore,

$$I_2 = \frac{1}{2}\left\{9+6g,_1^2+6g,_2^2+g,_1^4+g,_2^4+2g,_1^2\, g,_2^2-\left(3+4g,_1^2+4g,_2^2+g,_1^4+g,_2^4+2g,_1^2\,,g,_2^2\right)\right\}$$

$$I_2 = 3+g,_1^2+g,_2^2 = \mathrm{I}_1$$

$$I_3 = \det B = 1.\begin{vmatrix} 1 & g,_2 \\ g,_2 & 1+g,_1^2+g,_2^2 \end{vmatrix} - 0.\begin{vmatrix} 0 & g,_2 \\ g,_1 & 1+g,_1^2+g,_2^2 \end{vmatrix} + g,_1\begin{vmatrix} 0 & 1 \\ g,_1 & g,_2 \end{vmatrix}$$

$$= 1.\left(1+g,_1^2+g,_2^2-g,_2^2\right)+g,_1\left(-g,_1\right)$$

$$= 1+g,_1^2-g,_1^2 = 1$$

$I_3 = 1$, shows that the material is undiverging and incompressible deformation. We have

$$\left.\begin{aligned} &\mathrm{I}_1 = 3+g,_1^2+g,_2^2 \\ &\mathrm{I}_2 = 3+g,_1^2+g,_2^2 \left(NOTE\ \mathrm{I}_1 = \mathrm{I}_2\right) \\ &\mathrm{I}_3 = 1 \end{aligned}\right\} \qquad (3.7)$$

**3.4. Stress Tensor $\bar{\tau}$**

The Cauchy stress tensor $\bar{\tau}$for incompressible material is given by (Holzapfel, [14]) as

$$\bar{\tau} = -\rho I + 2W_1\overline{B} - 2W_2\overline{B}^{-1} \qquad 3.8$$

where I is the unit tensor, ρ is the hydrostatic pressure which insures incompressibility,

$$W_1 = \frac{\partial W}{\partial I_1}$$

$$W_2 = \frac{\partial W}{\partial I_2}$$

and W is the strain energy density function of the material considered. In this paper, we will consider the Neo-Hookean material whose strain energy density function is given by:

$$W = \frac{\mu}{2}\left(I_1 - 3\right) \qquad (3.9)$$

where μ is the shear modulus

$$W_i = \frac{\partial W}{\partial I_i},\ i = 1,\ 2$$

$$\left.\begin{aligned} &For\ i=1,\ W_1 = \frac{\partial W}{\partial I_1} = \frac{\mu}{2} \\ &For\ i=2,\ W_2 = \frac{\partial W}{\partial I_2} = 0 \end{aligned}\right\} \qquad (3.10)$$

$I_1 = I_2$ we can use either $I_1$ or $I_2$ but not both. If $I_1$ is used then $I_2 = 0$

Putting (3.10) in (3.8), the stress tensor reduces to

$$\bar{\tau} = -\rho I + 2W_1\overline{\mathrm{B}}$$

$$\bar{\tau} = -\rho I + 2\frac{\mu}{2}\overline{\mathrm{B}} \qquad \left[\text{since } W_1 = \mu/2\right]$$

$$\bar{\tau} = \mu\overline{\mathrm{B}} - \rho I \qquad (3.11)$$

where,

$$\mathrm{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\ unit\ tensor \tag{3.12}$$

$$\overline{\mathrm{B}}=\begin{pmatrix}1&0&g,_1\\0&1&g,_2\\g,_1&g,_2&1+g,_1^2+g,_2^2\end{pmatrix} \tag{3.13}$$

Putting (3.12) and (3.13) in (3.11).

$$\bar{\tau}=\begin{pmatrix}\mu&0&\mu g,_1\\0&\mu&\mu g,_2\\\mu g,_1&\mu g,_2&\mu+\mu g,_1^2+\mu g,_2^2\end{pmatrix}-\begin{pmatrix}\rho&0&0\\0&\rho&0\\0&0&\rho\end{pmatrix}$$

$$\bar{\tau}=\begin{pmatrix}\mu-\rho&0&\mu g,_1\\0&\mu-\rho&\mu g,_2\\\mu g,_1&\mu g,_2&\mu-\rho+\mu g,_1^2+\mu g,_2^2\end{pmatrix} \tag{3.14}$$

In Cartesian co-ordinate, $\bar{\tau}$ is given as

$$\bar{\tau}=\begin{pmatrix}\tau_{xx}&\tau_{xy}&\tau_{xz}\\\tau_{yx}&\tau_{yy}&\tau_{yz}\\\tau_{zx}&\tau_{zy}&\tau_{zz}\end{pmatrix} \tag{3.15}$$

Comparing (3.15) and (3.14) we have the component form of the stresses as

$$\left.\begin{aligned}&\tau_{xx}=\tau_{yy}=\mu-\rho\\&\tau_{xy}=\tau_{yx}=0\\&\tau_{xz}=\tau_{zx}=\mu g,_1\\&\tau_{yz}=\tau_{zy}=\mu g,_2\\&\tau_{zz}=\mu\left(1+g,_1^2+g,_2^2\right)-\rho\end{aligned}\right\} \tag{3.16}$$

The works in literature shows that the analysis of problems of anti-plane shear deformations are better handled in cylindrical polar coordinates. Hence, we need to transform the field equation to the cylindrical polar coordinates.

The relationship between the Cartesian and the cylindrical polar is given by

$$\left.\begin{aligned}&X_1=r\cos\theta\\&X_2=r\sin\theta\\&X_3=Z\\&r=\left(X_1^2+X_2^2\right)^{1/2}\\&\Theta=\tan^{-1}\left(\frac{X_2}{X_1}\right)\end{aligned}\right\} \tag{3.17}$$

Let g be a function of r and $\theta$ that is

$$g = g(r,\theta) \tag{3.18}$$

$$now\ g,_1 = \frac{\partial g(X_1, X_2)}{\partial X_1} = \frac{\partial g(r,\theta)}{\partial r}.\frac{\partial r}{\partial X_1} + \frac{\partial g(r,\theta)}{\partial \theta}.\frac{\partial \theta}{\partial X_1} \tag{3.19}$$

From (3.17) we have

$$\frac{\partial r}{\partial X_1} = \cos\theta \ and \ \frac{\partial \theta}{\partial X_1} = -\frac{\sin\theta}{r} \tag{3.20}$$

Putting (3.20) in (3.19)

$$g,_1 = \cos\theta g_r - \frac{\sin\theta}{r} g_\theta \tag{3.21}$$

Squaring (3.21) we have

$$g,_1^2 = \cos^2\theta g_r^2 - \frac{2}{r}\cos\theta\sin\theta g_r g_\theta + \frac{1}{r^2}\sin^2\theta g_\theta^2 \tag{3.22}$$

Also,

$$g,_2 = \frac{\partial g(X_1, X_2)}{\partial X_2} = \frac{\partial g(r,\theta)}{\partial r}.\frac{\partial r}{\partial X_2} + \frac{\partial g(r,\theta)}{\partial \theta}.\frac{\partial \theta}{\partial X_2} \tag{3.23}$$

From (3.17) we have

$$\frac{\partial r}{\partial X_2} = \sin\theta \ and \ \frac{\partial \theta}{\partial X_2} = \frac{\cos\theta}{r} \tag{3.24}$$

Putting (3.24) in (3.23)

$$g,_2 = \sin\theta g_r + \frac{\cos\theta}{r} g_\theta \tag{3.25}$$

Squaring (3.25) we have

$$g,_2^2 = \sin^2\theta g_r^2 + \frac{2}{r}\cos\theta\sin\theta g_r g_\theta + \frac{1}{r^2}\cos^2\theta g_\theta^2 \tag{3.26}$$

Substituting (3.21), (3.22), (3.25) and (3.26) into (3.16)

$$\left.\begin{array}{l} \tau_{xx} = \tau_{yy} = \mu - \rho \\ \tau_{xy} = \tau_{yx} = 0 \\ \tau_{xz} = \tau_{zx} = \mu\left(\cos\theta g_r - \frac{1}{r}\sin\theta g_\theta\right) \\ \tau_{yz} = \tau_{zy} = \mu\left(\sin\theta g_r + \frac{1}{r}\cos\theta g_\theta\right) \\ \tau_{zz} = \mu\left(1 + g_r^2 + \frac{1}{r^2} g_\theta^2\right) - \rho \end{array}\right\} \tag{3.27}$$

The coordinate transformation between Cartesian and cylindrical polar coordinate (3.28) is given by the tensor

$$\bar{\tau}^* = \bar{R}\bar{\tau}\bar{R}^T \tag{3.28}$$

*where*

$$\overline{R} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.29)$$

$$\overline{R}^T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.30)$$

$$\bar{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \quad (3.31)$$

Putting (3.31), (3.30) and (3.29) into (3.28)

$$\bar{\tau}^* = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.32)$$

$$\bar{\tau}^* = \begin{pmatrix} \cos\theta\tau_{xx} + \sin\theta\tau_{yx} & \cos\theta\tau_{xy} + \sin\theta\tau_{yy} & \cos\theta\tau_{xz} + \sin\theta\tau_{yz} \\ -\sin\theta\tau_{xx} + \cos\theta\tau_{yx} & -\sin\theta\tau_{xy} + \cos\theta\tau_{yy} & -\sin\theta\tau_{xz} + \cos\theta\tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.33)$$

From (3.27)

$$\tau_{xy} = \tau_{yx} = 0$$

Therefore, (3.33) reduces to

$$\bar{\tau}^* = \begin{pmatrix} \cos\theta\tau_{xx} & \sin\theta\tau_{yy} & \sin\theta\tau_{xz} + \sin\theta\tau_{yz} \\ -\sin\theta\tau_{xx} & \cos\theta\tau_{yy} & -\sin\theta\tau_{xz} + \cos\theta\tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bar{\tau}^* = \begin{pmatrix} \cos^2\theta\tau_{xx} + \sin^2\theta\tau_{yy} & -\cos\theta\sin\theta\tau_{xx} + \cos\theta\sin\theta\tau_{yy} & \cos\theta\tau_{xz} + \sin\theta\tau_{yz} \\ -\cos\theta\sin\theta\tau_{xx} + \cos\theta\sin\theta\tau_{yy} & \sin^2\theta\tau_{xx} + \cos^2\theta\tau_{yy} & -\sin\theta\tau_{xz} + \cos\theta\tau_{yz} \\ \cos\theta\tau_{zx} + \sin\theta\tau_{zy} & -\sin\theta\tau_{zx} + \cos\theta\tau_{zy} & \tau_{zz} \end{pmatrix} \quad (3.34)$$

In components

$$\bar{\tau}^* = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix} \quad (3.35)$$

Comparing (3.35) and (3.34) we have the following

$$\left.\begin{aligned} &\tau_{rr} = \cos^2\theta\tau_{xx} + \sin^2\theta\tau_{yy} \\ &\tau_{r\theta} = \tau_{\theta r} = \left(\tau_{yy} - \tau_{xx}\right)\cos\theta\sin\theta \\ &\tau_{rz} = \tau_{zr} = \cos\theta\tau_{xz} + \sin\theta\tau_{yz} \\ &\tau_{\theta z} = \tau_{z\theta} = \cos\theta\tau_{yz} - \sin\theta\tau_{xz} \\ &\tau_{\theta\theta} = \sin^2\theta\tau_{xx} + \cos^2\theta\tau_{yy} \\ &\tau_{zz} = \tau_{zz} \end{aligned}\right\} \quad (3.36)$$

Substituting (3.27) into (3.36) we have

$$\left.\begin{aligned} &\tau_{rr} = \tau_{\theta\theta} = \mu - \rho \\ &\tau_{r\theta} = \tau_{\theta r} = 0 \\ &\tau_{rz} = \tau_{zr} = \mu g_r \\ &\tau_{\theta z} = \tau_{z\theta} = \frac{\mu}{r} g_\theta \\ &\tau_{zz} = \mu\left(1 + g_r^2 + \frac{1}{r^2} g_\theta^2\right) - \rho \end{aligned}\right\} \quad (3.37)$$

**3.5. Equation of Equilibrium**

The equilibrium equation for the isotropic homogenous deformation is given by

$.div\ \bar{\tau}^* = \nabla.\bar{\tau}^* = 0$ (3.38)

Which in component form is the same as

$$\left.\begin{aligned} &\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + B_r = 0 \\ &\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + B_\theta = 0 \\ &\frac{\partial\tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + B_z = 0 \end{aligned}\right\} \quad (3.39)$$

Where $B_r,\ B_\theta\ and\ B_z$ are the components of the body forces in $r,\ \theta\ and\ z$ directions respectively. From (3.37) and (3.39) the only non-zero component of equilibrium equation is the axial component given by

$$\frac{\partial\tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z} + \frac{\partial\tau_{rz}}{r} + B_z = 0 \quad (3.40)$$

In the absences of body force, (3.40) reduces to

$$\partial\frac{\tau_{rz}}{\partial r} + \frac{1}{r}\partial\frac{\tau_{\theta z}}{\partial\theta} + \partial\frac{\tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r} = 0 \quad (3.41)$$

From (3.37)

$$\partial\frac{\tau_{rz}}{\partial r} = \mu g_{rr},\ \partial\frac{\tau_{\theta z}}{\partial\theta} = \frac{\mu g_{\theta\theta}}{r},\ \partial\frac{\tau_{zz}}{\partial z} = 0$$

Then, equation of equilibrium becomes

$$\mu g_{rr} + \frac{\mu}{r} g_r + \frac{\mu g_{\theta\theta}}{r^2} = 0$$

$$\mu\left( g_{rr} + \frac{1}{r} g_r + \frac{1}{r^2} g_{\theta\theta} \right) = 0 \tag{3.42}$$

Since $\mu \neq 0$ we have that (3.42) now becomes

$$g_{rr} + \frac{1}{r} g_r + \frac{1}{r^2} g_{\theta\theta} = 0 \tag{3.43}$$

**4. Application of the Problem - Boundry Value Problem I**

Consider a solid in a state of strain characterized by the strain energy density function (3.9), containing a central circular cavity of radius '$a$' and under an anti-plane shear loading.

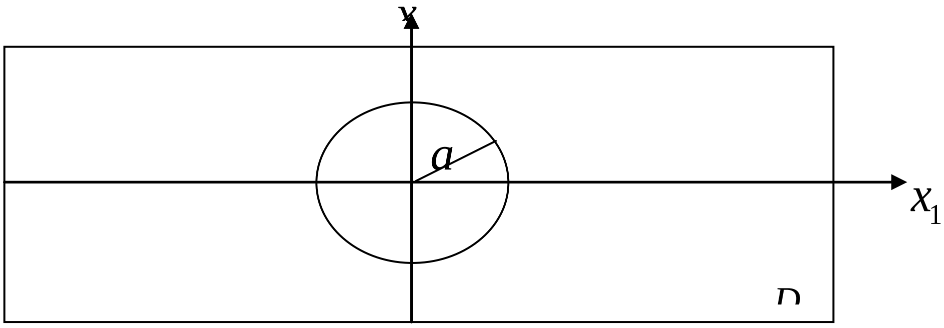


*Figure 1: A plane cross section of the solid*

The surface D is traction free. The boundary conditions are that at infinity the stress shall approximate that of simple shear and that the boundary of the hole $r = a$ shall be traction free. The corresponding boundary conditions here are given as:

$$\left. \begin{aligned} g &= kr\sin\theta, & r &\to \infty \\ g_r &= 0, & r &= a \end{aligned} \right\} \tag{4.1.1}$$

Therefore, we need to solve equation (3.43) subject to the boundry condition (4.1). we seek for an asymptotic solution g in $W^{1,2}$ (Sobolev space of order 2) which approximates the solution of the boundary value problem.
Now let

$$g = k\left( r + \frac{b}{r} + \frac{c}{r^2} \right)\sin\theta \Big\} \tag{4.1.2}$$

g(r,Θ) $\leq r < \infty$, $0 \leq \theta \leq 2\pi$, b and c are to be determined and k is the magnitude of shear as $r \to \infty$. It is easy to see that the assumed form of g automatically satisfies the boundary condition (4.1.1a). We now consider boundary condition (4.1.1b). Differentiating (4.1.2) with respect to r we have

$$g_r = k\left( 1 - \frac{b}{r^2} - \frac{2c}{r^3} \right)\sin\theta \tag{4.1.3}$$

Substituting (4.1.3 ) into (4.1.1b) we obtain

$$k\left( 1 - \frac{b}{a^2} - \frac{2c}{a^3} \right)\sin\theta = 0 \tag{4.1.4}$$

Since $k \neq 0$ *and* $\sin\theta$ is not identically zero we have

$$1-\frac{b}{a^2}-\frac{2c}{a^3}=0 \qquad (4.1.5)$$

This implies that

$$a^3=ab+2c \qquad (4.1.6)$$

$$or \qquad b=\frac{a^3-2c}{a} \qquad (4.1.7)$$

Substituting (4.1.7) into (4.1.2) we have

$$g=k\left[r+\frac{a^2}{r}+c\left(\frac{1}{r^2}-\frac{2}{ar}\right)\right]\sin\theta \qquad (4.1.8)$$

where '$a$' is the radius of the central cavity and c is the parameter to be determined. From (4.1.8), we obtain $g_r, g_{rr}$ *and* $g_{\theta\theta}$ as

$$g_r=k\left[1-\frac{a^2}{r^2}+c\left(\frac{-2}{r^3}+\frac{2}{ar^2}\right)\right]\theta\sin\theta \qquad (4.1.9)$$

$$g_{rr}=k\left[1-\frac{2a^2}{r^3}+c\left(\frac{6}{r^4}-\frac{4}{ar^3}\right)\right]\sin\theta \qquad (4.1.10)$$

$$g_\theta=k\left[r+\frac{a^2}{r}+c\left(\frac{1}{r^2}-\frac{2}{ar}\right)\right]\cos\theta \qquad (4.1.11)$$

$$g_{\theta\theta}=-k\left[r+\frac{a^2}{r}+c\left(\frac{1}{r^2}-\frac{2}{ar}\right)\right]\sin\theta \qquad (4.1.12)$$

Substituting (4.1.19), (4.1.10) and (4.1.12) into left hand side of (3.43)

$$k\left[\frac{2a^2}{r^3}+c\left(\frac{6}{r^4}-\frac{4}{ar^3}\right)\right]\sin\theta+\frac{k}{r}\left[1-\frac{a^2}{r^2}+c\left(\frac{-2}{ar^2}-\frac{2}{r^3}\right)\right]\sin\theta$$

$$-\frac{k}{r^2}\left[r+\frac{a^2}{r}+c\left(\frac{1}{r^2}-\frac{2}{ar}\right)\right]\sin\theta=0 \qquad (4.1.13)$$

$$k\left[\frac{2a^2}{r^3}+\frac{6c}{r^4}-\frac{`4c}{ar^3}+\frac{1}{r}-\frac{a^2}{r^3}+\frac{2c}{ar^3}-\frac{`2c}{r^4}-\frac{1}{r}-\frac{a^2}{r^3}-\frac{c}{r^4}+\frac{2c}{ar^3}\right]\sin\theta=0$$

Which simplifies to

$$\varepsilon(r,\theta,c)=\frac{3kc}{r^4}\sin\theta \qquad (4.1.14)$$

Equation (4.1.14) is the error term

**4.1. Minimization of the Error**

If the assumed g is the exact solution of the boundry value problem, the substitution of (4.9), (4.10) and (4.12) into (3.43) would have satisfied the equilibrium equation, rather, it gave us a deviation which is an error.

We seek to minimize this error in the Sobolev space (space of infinitely differentiable functions). The advantage of minimizing in the sobolev space is that both the function and its rate of change are controlled.

The norm on Sobolev space of order two in polar coordinate is given as

$$\|\varepsilon(r,\theta,c)\|=\sqrt{\int_0^{2\pi}\int_a^{\infty}\left(\varepsilon^2+\varepsilon_r^2+\frac{1}{r^2}\varepsilon_\theta^2\right)rdr\ d\theta} \qquad (4.1.15)$$

where $\varepsilon_r$ is the partial derivative of the $\varepsilon$ with respect to $r$

$\varepsilon_\theta$ is the partial derivative of $\varepsilon$ with respect to $\theta$

Squaring (4.1.14) we have

$$\varepsilon^2 = k^2 \sin^2\theta\left(\frac{9c^2}{r^8}\right) \quad (4.1.16)$$

Differentiating (4.1.14) with respect to r

$$\varepsilon_r = -k\left(\frac{12c}{r^5}\right)\ \sin\theta \quad (4.1.17)$$

Squaring (1.1.17) we have

$$\varepsilon_r^2 = k^2 \sin^2\theta\left(\frac{144c^2}{r^{10}}\right) \quad (4.1.18)$$

Differentiating (4.1.14) with respect to $\theta$

$$\varepsilon_\theta = k\left(\frac{3c}{r^4}\right)\ \cos\theta \quad (4.1.19)$$

Squaring (4.1.19) we have

$$\varepsilon_\theta^2 = k^2 \cos^2\theta\left(\frac{9c^2}{r^8}\right) \quad (4.1.20)$$

Substituting (4.1.16), (4.1.18) and (4.1.20) into (4.1.15)

$$\left\|\varepsilon(r,\theta,c)\right\| = \left\{k^2c^2\int_0^{2\pi}\int_a^\infty\left[\left(\frac{9}{r^8}+\frac{144}{r^{10}}\right)\sin^2\theta + \frac{1}{r^2}\left(\frac{9}{r^8}\right)\cos^2\theta\right]r dr d\theta\right\}^{\frac{1}{2}}$$

$$\left\|\varepsilon(r,\theta,c)\right\| = \left\{k^2c^2\int_0^{2\pi}\int_a^\infty\left[\left(\frac{9}{r^7}+\frac{144}{r^9}\right)\sin^2\theta + \left(\frac{9}{r^9}\right)\cos^2\theta\right]dr d\theta\right\}^{\frac{1}{2}}$$

$$\left\|\varepsilon(r,\theta,c)\right\| = \left\{k^2c^2\left[\int_a^\infty\left(\frac{9}{r^7}+\frac{144}{r^9}\right)dr.\int_0^{2\pi}\sin^2\theta d\theta + \int_a^\infty\left(\frac{9}{r^9}\right)dr.\int_0^{2\pi}\cos^2\theta d\theta\right\}^{\frac{1}{2}}$$

$$\left\|\varepsilon(r,\theta,c)\right\| = \left\{k^2c^2\left[\lim_{r\to\infty}\left(\frac{-3}{2r^6}-\frac{18}{r^8}\right)\Big|_a^r.\frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta)\,d\theta + \lim_{r\to\infty}\left(\frac{9}{8r^8}\right)\Big|_a^r.\frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)\right\}^{\frac{1}{2}}$$

$$\left\|\varepsilon(r,\theta,c)\right\| = \left\{k^2c^2\left[\left(\frac{3}{2a^6}+\frac{18}{a^8}\right)\pi - \left(\frac{9}{8a^8}\right)\pi\right]\right\}^{\frac{1}{2}}$$

$$\left\|\varepsilon(r,\theta,c)\right\| = \left\{\pi k^2c^2\left[\frac{3}{2a^6}+\frac{18}{a^8}-\frac{9}{8a^8}\right]\right\}^{\frac{1}{2}} \quad (4.1.21)$$

This error is minimum when $\frac{\partial}{\partial c}\left\|\varepsilon(r,\theta,c)\right\| = 0$

Therefore (4.21) becomes

$$\frac{\partial}{\partial c}\left\|\varepsilon(r,\theta,c)\right\| = \frac{1}{2}\frac{\left[2\pi k^2 c\left(\frac{3}{2a^6}+\frac{18}{a^8}-\frac{9}{8a^8}\right)\right]}{\left[\pi k^2 c^2\left(\frac{3}{2a^6}+\frac{18}{a^8}-\frac{9}{8a^8}\right)\right]^{\frac{1}{2}}} = 0$$

$$\pi k^2\left[c\left(\frac{3}{2a^6}+\frac{18}{a^8}-\frac{9}{8a^8}\right)\right] = 0$$

$$\text{since } \pi k^2 \neq 0$$

Then

$$c\left(\frac{3}{2a^6}+\frac{18}{a^8}-\frac{9}{8a^8}\right) = 0$$

Therefore

$$c = 0 \tag{4.1.22}$$

Substituting (4.1.22) into (4.1.7) we have

$$b = a^2 \tag{4.1.23}$$

Substituting (4.1.22) and (4.1.23) into (4.1.2) we have

$$g = k\left(r+\frac{a^2}{r}\right)\sin\theta \tag{4.1.24}$$

This is exact.

Using (4.1.24) we obtain $g_r$ *and* $g_\theta$ as

$$g_r = k\left(1-\frac{a^2}{r^2}\right)\sin\theta$$

$$g_\theta = k\left(r-\frac{a^2}{r}\right)\cos\theta$$

Now substituting (4.1.26) and (4.1.25) into (3.38) we have the stress components as

$$\tau_{rr} = \tau_{\theta\theta} = \mu - \rho\sin\theta$$

$$\tau_{rz} = \tau_{\theta r} = 0$$

$$\tau_{rz} = \tau_{zr} = \mu k\left(1-\frac{a^2}{r^2}\right)\sin\theta$$

$$\tau_{\theta z} = \tau_{z\theta} = \frac{\mu k}{r}\left(r+\frac{a^2}{r}\right)\cos\theta$$

$$\tau_{zz} = \mu\left[1+k^2+\frac{k^2a^4}{r^4}-\frac{2k^2a^2}{r^2}\cos 2\theta\right]$$

**4.2. Boundary Value Problem II**

We now consider the case in which the void is patched with a bonding material. The patch is such that there is no differential movement between the bonding fibre and the material. Again the system is subjected to an anti-plane loading.

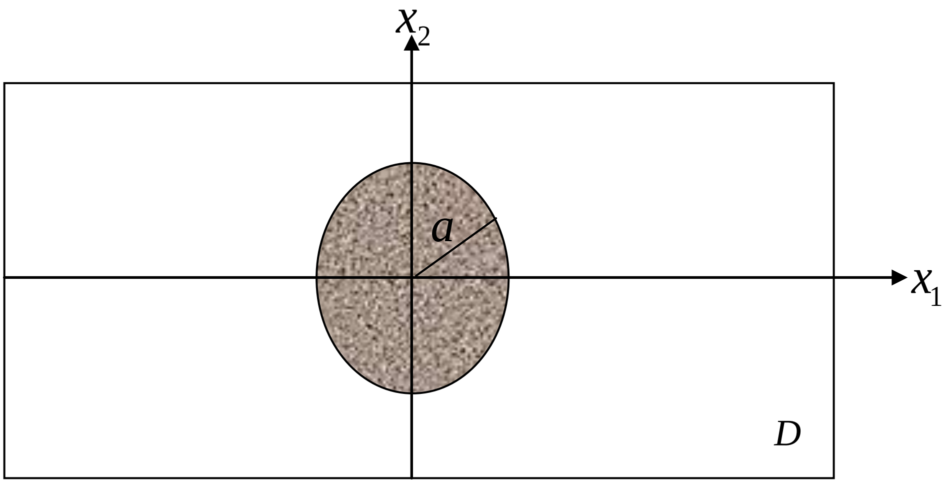


*Figure 2: A plane cross section of the solid*

In this case the boundary conditions are

$$\left.\begin{aligned} g &= kr\sin\theta, \quad & r &\to \infty \\ g &= 0, & r &= a \end{aligned}\right\} \qquad (4.2.1)$$

Once again we let the solution be

$$g = k\left(r + \frac{b}{r} + \frac{c}{r^2}\right)\sin\theta, \quad r > 0 \qquad (4.2.2)\}$$

At the interface, $r = a,\ g = 0$ gives

$$k\left(a + \frac{b}{a} + \frac{c}{a^2}\right)\sin\theta = 0$$

Hence we have

$$a^3 + ab + c = 0$$

Which gives

$$b = \frac{-\left(a^3 + c\right)}{a} \qquad (4.2.3)$$

Therefore

$$g = k\left(r - \frac{a^3 + c}{ar} + \frac{c}{r^2}\right)\sin\theta$$

$$= k\left[r - \frac{a^2}{r} + c\left(\frac{1}{r^2} - \frac{1}{ar}\right)\sin\theta\right] \qquad (4.2.4)$$

From (4.2.4) we obtain $g_r,\ g_{rr},\ and\ g_{\theta\theta}$ as

$$g_r = k\left[1 + \frac{a^2}{r^2} + c\left(\frac{-2}{r^3} + \frac{1}{ar^2}\right)\right]\sin\theta \qquad (4.2.5)$$

$$g_{rr} = k\left[\frac{2a^2}{r^3} + c\left(\frac{6}{r^4} - \frac{2}{ar^3}\right)\right]\sin\theta \qquad (4.2.6)$$

$$g_\theta = k\left[r - \frac{a^2}{r} + c\left(\frac{1}{r^2} - \frac{1}{ar}\right)\right]\cos\theta$$

$$g_{\theta\theta} = -k\left[r + \frac{a^2}{r} + c\left(\frac{1}{r^2} - \frac{1}{ar}\right)\right]\sin\theta \qquad (4.2.7)$$

Substituting (4.25-4.27) into (3.43) we obtain

$$k\left[\frac{-2a^2}{r^3}+c\left(\frac{6}{r^4}-\frac{2}{ar^3}\right)\right]\sin\theta+\frac{k}{r}\left[1+\frac{a}{r^2}+c\left(\frac{-2}{r^3}+\frac{1}{ar^2}\right)\right]\sin\theta$$

$$-\frac{k}{r^2}\left[r-\frac{a^2}{r}+c\left(\frac{1}{r^2}-\frac{1}{ar}\right)\right]\sin\theta\ =0$$

$$=\frac{k}{r}-\frac{k}{r}-\frac{2ka^2}{r^3}+\frac{ka^2}{r^3}+\frac{ka^2}{r^3}+\frac{6kc}{r^4}-\frac{2kc}{r^4}-\frac{kc}{r^4}-\frac{2kc}{ar^3}+\frac{kc}{ar^3}+\frac{kc}{ar^3}$$

$$=\frac{3kc}{r^4} \qquad (4.2.8)$$

(4.2.8) gives the error. Minimizing again in the sobolev space we see from previous analysis that $c=0$ for minimum error. Consequently we have

$$b=-a^2 \qquad (4.2.9)$$

So that

$$g=k\left(r-\frac{a^2}{r}\right)\sin\theta \qquad (4.2.10)$$

as the displacement and

$$\tau_{rz}=\mu k\left(1+\frac{a^2}{r^2}\right)\sin\theta \qquad (4.2.11)$$

$$\tau_{\theta z}=\frac{\mu k}{r}\left(r-\frac{a^2}{r}\right)\cos\theta \qquad (4.2.12)$$

$$\tau_{zz}=\mu\left(1+k^2+\frac{k^2a^4}{r^4}-\frac{2k^2a^2}{r^2}\cos 2\theta\right) \qquad (4.2.13)$$

are the stresses.

## 5. Conclusion

In paper, we have used the asymptotic method to simplify the analysis of a Neo-Hookean half-space deforming under anti-plane share loading. Two major problems were solved. The first is a half plane containing a central circular cavity and the second is the same half space with the circular hole filled with some inclusion in such a manner that there is no differential movement between the solid inclusion and the original material. In either case the exact deformation functions and stress functions were got. For the first case these are given in equations (4.1.24) through (4.1.27) and for the second case they are given in equations (4.2.10) through (4.2.13). It is easy to see the superiority of this method over the method of separation of variables and other methods.

Here, one can easily determine the exact displacement and stresses at every plane section of the material by mere substitutions of the coordinates of points desired. The method, however, has assumed that the dimensions of the material are very large in comparison with the dimension of the hole. Consequently, the end effect including shape are neglected. The process also permits the use of any fibre to fill the central cavity since the only requirement is that the interface should be strongly bonded. We must note here that the homogeneity property of the material permit the analysis to be done at every cross section of the solid. Consequently, this finds its application in the construction industries where metal sheets are needed. The commonest are failure resulting from fixing of bolt connecting metal sheets to columns beams or other frames. The minimization of the error in the sobolev norm is advantageous since it controls both the functions and its rate of change.

From the solutions, we see that the maximum stress for the solid with a central cavity occurs when $\theta = 0$ while it occurs when $\theta = \frac{\pi}{2}$, when the cavity is filled. These therefore, suggest points of possible development of cracks. The observations above show that the result can be applied to the analysis of problem of crack and crack extensions.

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