Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2020, 7(1):185-190**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Some Generating Functions of Modified Bessel Polynomials by Group Theoretic Method

Gamal Ali Qashash

Department of Mathematics, Faculty of Science, Aden University, Yemen
gamalkashash@yahoo.com

Abstract In this paper we obtain generating functions for the modified Bessel polynomials $Y_n^{(\alpha+n)}(x)$ by the application of Weisner's group-theoretic method with the suitable interpretations of n and α of the side polynomials. Whereas, we deployed it to determine the generating relations between the modified Bessel polynomials and with easy way. The ideas in consent with proofs are originated from the book of McBride [7] and we use it to determine some generating relations which involve modified Bessel polynomials. Some particular cases of interest are also discussed.

Keywords Bessel polynomials, generating functions, differential operators, Group theoretic-method

1. Introduction

The unification of generating functions has great importance in connection with ideas and principles of special functions. Group Theoretic Method proposed by Louis Weisner in 1955, and he employed this method to find generating relations for a large class of special functions. Weisner discussed the group-theoretic significance of generating functions for Hypergeometric, Hermite, and Bessel functions in his papers [13, 14] and [15] respectively. McBride [7] deployed Weisner's method to determine the new generating relations that involves Hermite, Bessel, generalized Laguerre, Gegenbauer polynomials. In this directions some important steps has been made by researchers namely Singhal and Srivastava [10], Chaterjea [1, 2] and Chongdar [3].

Now continuing the work in connection with class of generating functions, we extend our ideas to obtain generating relations that involves between the modified Bessel polynomials.

The modified Bessel polynomials is defined by

$$Y_n^{(\alpha+n)}(x) = {}_2F_0(-n, 2n+\alpha-1; -; -\frac{x}{\beta}), \tag{1.1}$$

The Bessel Polynomials $Y_n^{(\alpha)}(x)$ as introduced by [6], (cf.[9] or [11]) is defined by

$$Y_n^{(\alpha)}(x) = {}_2F_0(-n, n+\alpha-1; -; -\frac{x}{\beta}). \tag{1.2}$$

is the solutions of ordinary differential equation:

$$x^2 \frac{d^2y}{dx^2} + ((\alpha+n)x+\beta)\frac{dy}{dx} - n(2n+\alpha-1)y = 0 \tag{1.3}$$

The object of the present paper is to derive some generating functions of modified Bessel polynomials by interpreting n and α simultaneously with the help of Weisner's group theoretic method [7].

2. Group theoretic method

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $(y\frac{\partial}{\partial y})$, α by $(z\frac{\partial}{\partial z})$ and y by $u(x,y,z)$ in (1.3), we get the following partial differential equation

$$x^2\frac{\partial^2 u}{\partial x^2}+xz\frac{\partial^2 u}{\partial z\partial x}+xy\frac{\partial^2 u}{\partial y\partial x}+\beta\frac{\partial u}{\partial x}-2y^2\frac{\partial^2 u}{\partial y^2}-y\frac{\partial u}{\partial y}-yz\frac{\partial^2 u}{\partial y\partial z}=0 \tag{2.1}$$

Thus $u_1(x,y,z)=Y_n^{(\alpha+n)}(x)y^n z^\alpha$ is a solution of the differential equation (2.1). We now seek linearly independent differential operators A, B and C such that:

$$A\left[Y_n^{(\alpha+n)}(x)y^n z^\alpha\right]=a_n Y_n^{(\alpha+n)}(x)y^n z^\alpha \tag{2.2}$$

$$B\left[Y_n^{(\alpha+n)}(x)y^n z^\alpha\right]=b_n Y_{n-1}^{(\alpha+2)+(n-1)}(x)y^{n-1}z^{\alpha+2} \tag{2.3}$$

$$C\left[Y_n^{(\alpha+n)}(x)y^n z^\alpha\right]=c_n Y_{n+1}^{(\alpha-2)+(n+1)}(x)y^{n+1}z^{\alpha-2} \tag{2.4}$$

where a_n, b_n *and* c_n are coefficients involving α, β *and* n. This necessitates the bringing into use of the recurrence relations.

$$DY_n^{(\alpha+n)}(x)=\frac{n}{x}\left[Y_n^{(\alpha+n)}(x)-Y_{n-1}^{(\alpha+n+1)}(x)\right] \tag{2.5}$$

and

$$DY_n^{(\alpha+n)}(x)=\frac{1}{x^2}\left[\beta Y_{n+1}^{(\alpha+n-1)}(x)-(\beta+2nx+\alpha x-x)Y_n^{(\alpha+n)}(x)\right] \tag{2.6}$$

where D is the differential operator $D=\frac{d}{dx}$.

With the help of (2.5) and (2.6), it follows form (2.2), (2.3) and (2.4) that

$$R_1=y\frac{\partial}{\partial y} \tag{2.7}$$

$$R_2=z\frac{\partial}{\partial z} \tag{2.8}$$

$$R_3=xy^{-1}z^2\frac{\partial}{\partial x}-z^2\frac{\partial}{\partial y} \tag{2.9}$$

$$R_4=x^2y\,z^{-2}\frac{\partial}{\partial x}+2xy^2z^{-2}\frac{\partial}{\partial y}+xyz^{-1}\frac{\partial}{\partial z}+(\beta-x)yz^{-2} \tag{2.10}$$

Which satisfy the following rules:

$$R_1\left[Y_n^{(\alpha+n)}(x)y^n z^\alpha\right]=nY_n^{(\alpha+n)}(x)y^n z^\alpha \tag{2.11}$$

$$R_2\left[Y_n^{(\alpha+n)}(x)y^n z^\alpha\right]=\alpha Y_n^{(\alpha+n)}(x)y^n z^\alpha \tag{2.12}$$

$$R_3\left[Y_n^{(\alpha+n)}(x)y^n z^\alpha\right]=-nY_{n-1}^{(\alpha+n+1)}(x)y^{n-1}z^{\alpha+2} \tag{2.13}$$

$$R_4\left[Y_n^{(\alpha+n)}(x)y^n z^\alpha\right]=\beta Y_{n+1}^{(\alpha+n-1)}(x)y^{n+1}z^{\alpha-2} \tag{2.14}$$

Now, we use the commutator notation [A,B] = (AB - BA). Henceforth, we consider a function $u=u(x,y,z)$, is a l^3 function in three independent variables x, y and z. Therefore, an action of a commutator operator on u is that

[A,B]u = (AB - BA)u = A(Bu) - B(Au).

Then we have the operators R_1, R_2, R_3 *and* R_4, which satisfy the following commutator relations:

$[R_1, R_2] = 0, \quad [R_1, R_3] = -R_3, \quad [R_1, R_4] = R_4,$
$[R_2, R_3] = 2R_3, \quad [R_2, R_4] = -2R_4, \quad [R_3, R_4] = -\beta.$

Hence, we define a linear partial differential operator L as

$$L = x^2 \frac{\partial^2}{\partial x^2} + xz \frac{\partial^2}{\partial z \partial x} + xy \frac{\partial^2}{\partial y \partial x} + \beta \frac{\partial}{\partial x} - 2y^2 \frac{\partial^2}{\partial y^2} - y \frac{\partial}{\partial y} - yz \frac{\partial^2}{\partial y \partial z}$$

Which can be express as:

$$xL = R_3 R_4 + \beta R_1 + \beta,$$

It can be easy verified that the operator $R_i \; (i = 1, 2, 3, 4)$ commute with xL,
i.e., $[xL, R_i] = 0$.

3. Extended form of the group of Operators

In this section, we extend the operators $R_1, R_2, R_3 \; and \; R_4$, which we defined in the previous section to the exponential form. Consider an arbitrary function $f = f(x, y, z)$ in three independent variables. Also, we consider the arbitrary constants $a, b, c \; and \; d$. $\exp(aR_1), \exp(bR_2), \exp(cR_3) \; and \exp(dR_4)$ are called the extended form of the transformation groups generated by $R_1, R_2, R_3 \; and \; R_4$. For doing this business, we will follow the method suggested by Weisner [13]. One may refers McBride [7] for similar kind of analysis.

In order to find the extended form of the group generated by operators $R_1, R_2, R_3 \; and \; R_4$, we will change the form of differential operator $R = R_1 \frac{\partial}{\partial x} + R_2 \frac{\partial}{\partial y} + R_3 \frac{\partial}{\partial z} + R_0$ to $E = R_1 \frac{\partial}{\partial x} + R_2 \frac{\partial}{\partial y} + R_3 \frac{\partial}{\partial z}$, and finally by change of variable to $D = \frac{\partial}{\partial X}$. If $\phi(x, y, z)$ is any solution of $C[\phi(x, y, z)] = 0$, then

$$\phi^{-1} R \phi = R_1 \frac{\partial}{\partial x} + R_2 \frac{\partial}{\partial y} + R_3 \frac{\partial}{\partial z} = E.$$

The extended form of the group generated by $R_i \; (i = 1, 2, 3, 4)$ are given by

$$e^{aR_1} f(x, y, z) = f(x, e^a y, z) \tag{3.1}$$

$$e^{bR_2} f(x, y, z) = f(x, y, e^b z) \tag{3.2}$$

$$e^{cR_3} f(x, y, z) = f\left(\frac{xy}{y - cz^2}, y - cz^2, z \right) \tag{3.3}$$

$$e^{dR_4} f(x, y, z) = \left(1 - d\frac{xy}{z^2}\right) \exp\left(d \frac{\beta y}{z^2}\right) f\left(\frac{x}{1 - d(xy/z^2)}, \frac{y}{(1 - d\, xy/z^2)^2}, \frac{z}{1 - d(xy/z^2)} \right) \tag{3.4}$$

where $a, b, c \; and \; d$ are arbitrary constants.

We proceed to determine $e^{cR_3} e^{bR_4}$. Hence, we consider its action on an arbitrary function $f(x, y, z)$.

$$e^{cR_3} e^{dR_4} f(x, y, z) = \left(1 - d\frac{xy}{z^2}\right) \exp\left(d \frac{\beta y}{z^2}\right) f(\eta, \mu, \xi) \tag{3.5}$$

where

$$\eta = \frac{xy}{(1 - d(xy/z^2))(y - cz^2)}, \quad \mu = \frac{(y - cz^2)}{(1 - d\, xy/z^2)^2} \text{ and } \xi = \frac{z}{1 - d(xy/z^2)}.$$

4. Generating functions

In this section we have determined the new generating relations between modified Bessel polynomials. To obtain the generating function for { $Y_n^{(\alpha+n)}(x)$ }, we now transform $f(x,y,z)$ by means of the operator $e^{cR_3}e^{bR_4}$. We shall consider the following cases:

Case 1:

If we substitute c = 0 and d = 1 in (3.5), then it will gives us

$$e^{R_4} f(x,y,z) = \left(1-\frac{xy}{z^2}\right)\exp\left(\frac{\beta y}{z^2}\right) f\left(\frac{x}{1-(xy/z^2)}, \frac{y}{(1-xy/z^2)^2}, \frac{z}{1-(xy/z^2)}\right). \tag{4.1}$$

Hence, if we take $f(x,y,z) = Y_n^{(\alpha+n)}(x) y^n z^\alpha$ and putting $\frac{y}{z^2} = w$ then it will be result into following relation

$$e^{R_4} f(x,y,z) = (1-wx)^{(1-2n-\alpha)} \exp(w\beta)\, Y_n^{(\alpha+n)}\left(\frac{x}{1-wx}\right) y^n z^\alpha . \tag{4.2}$$

Also, we notice from (2.14) that

$$R_4\left[Y_n^{(\alpha+n)}(x) y^n z^\alpha\right] = \beta Y_{n+1}^{(\alpha+n-1)}(x) y^{n+1} z^{\alpha-2}$$

On other hand, we can expand left hand side of (4.2) in series form and then repeated application of (2.14) on the same side of (4.2), we get

$$\sum_{p=0}^{\infty} \frac{(\beta)^p}{p!} Y_{n+p}^{(\alpha+n-p)}(x)\, y^{n+p} z^{\alpha-2p} \tag{4.3}$$

Let us put $\frac{y}{z^2} = w$ in above equation, we get

$$\sum_{p=0}^{\infty} \frac{(w\beta)^p}{p!} Y_{n+p}^{(\alpha+n-p)}(x)\, y^n z^\alpha \tag{4.4}$$

Equating the result (4.2) and (4.4) we get

$$(1-wx)^{(1-2n-\alpha)} \exp(w\beta)\, Y_n^{(\alpha+n)}\left(\frac{x}{1-wx}\right) = \sum_{p=0}^{\infty} \frac{(w\beta)^p}{p!} Y_{n+p}^{(\alpha+n-p)}(x) . \tag{4.5}$$

If we put $\alpha = \alpha - 2n$ in (4.5), we get

$$(1-wx)^{(1-\alpha)} \exp(w\beta)\, Y_n^{(\alpha-n)}\left(\frac{x}{1-wx}\right) = \sum_{p=0}^{\infty} \frac{(w\beta)^p}{p!} Y_{n+p}^{(\alpha-n-p)}(x) \tag{4.6}$$

this was derived by [4, 5].

Case 2:

If we choose c = 1 and d= 0 in (3.5) we get

$$e^{R_3} f(x,y,z) = f\left(\frac{xy}{y-z^2}, y-z^2, z\right). \tag{4.7}$$

Since, we have

$$R_3\left[Y_n^{(\alpha+n)}(x) y^n z^\alpha\right] = -nY_{n-1}^{(\alpha+n+1)}(x) y^{n-1} z^{\alpha+2}$$

For the purpose to use it repeatedly, we expand the left hand side of (4.7) in series form and then simplifying, we obtain the following relation

$$\sum_{k=0}^{\infty} \frac{(-n)_k}{k!} Y_{n-k}^{(\alpha+n+k)}(x)\, y^{n-k} z^{\alpha+2k}$$

Let us replace $\frac{z^2}{y} = w$ in above simplified form so that we able to determine one more new generating function for the modified Bessel polynomials, which is as given below.

$$(1-w)^n Y_n^{(\alpha+n)}\left(\frac{x}{1-w}\right) = \sum_{k=0}^{\infty} \frac{w^k}{k!}(-n)_k Y_{n-k}^{(\alpha+n+k)}(x). \quad (4.8)$$

If we put $\alpha = \alpha-(m+n)$ in (4.8), we get

$$(1-w)^n Y_n^{(\alpha-m)}\left(\frac{x}{1-w}\right) = \sum_{k=0}^{\infty} \frac{w^k}{k!}(-n)_k Y_{n-k}^{(\alpha-m+k)}(x) \quad (4.9)$$

Which obtained by [8]

Case 3: putting c=1, d=w in (3.5), we get

$$e^{R_3}e^{R_4} f(x,y,z) = \left(1-\frac{wxy}{z^2}\right)\exp\left(\frac{\beta wy}{z^2}\right) f\left(\frac{xy}{(1-wxy/z^2)(y-z^2)}, \frac{y-z^2}{(1-xy/z^2)^2}, \frac{z}{1-(wxy/z^2)}\right)$$

$$= \left(1-\frac{wxy}{z^2}\right)\exp\left(\frac{\beta wy}{z^2}\right)\left(\frac{y-z^2}{(1-wxy/z^2)^2}\right)^n \left(\frac{z}{1-(wxy/z^2)}\right)^{\alpha} Y_n^{(\alpha+n)}\left(\frac{xy}{(1-wxy/z^2)(y-z^2)}\right)$$

(4.10)

and we write exponential operators in series form so that we have the following relation.

$$e^{R_3}e^{R_4}[Y_n^{(\alpha+n)}(x)y^n z^{\alpha}] = \sum_{p,k=0}^{\infty} \frac{(\beta)^p(-n)_k}{p!k!} Y_{n+p-k}^{(\alpha+n+k-p)}(x)\, y^{n+p-k} z^{\alpha+2k-2p} \quad (4.11)$$

Equating (4.10) and (4.11) and then putting $z^2 = t^{-1}$ and $\alpha = \alpha-(n+m)$, we get

$$(1-wxyt)^{=(1+\alpha+n-m)} \exp(\beta wyt)\left(1-\frac{1}{yt}\right)^n Y_n^{(\alpha-m)}\left(\frac{xyt}{(1-wxyt)(yt-1)}\right)$$

$$= \sum_{p,k=0}^{\infty} \frac{(\beta)^p(-n)_k}{p!k!} Y_{n+p-k}^{(\alpha-m+k-p)}(x)(yt)^{p-k} \quad (4.12)$$

Which obtained by [8]

Case 4:

Now we assign the value to arbitrary constants c and d. Let us take c = 1 and d = 1, so that (3.5) becomes

$$e^{R_3}e^{R_4} f(x,y,z) = \left(1-\frac{xy}{z^2}\right)\exp\left(\frac{\beta y}{z^2}\right) f\left(\frac{xy}{(1-xy/z^2)(y-z^2)}, \frac{y-z^2}{(1-xy/z^2)^2}, \frac{z}{1-(xy/z^2)}\right)$$

(4.13)

Hence, if we take $f(x,y,z) = Y_n^{(\alpha+n)}(x)y^n z^{\alpha}$ then it will be result into following relation

$$e^{R_3}e^{R_4} f(x,y,z) = \left(1-\frac{xy}{z^2}\right)\exp\left(\frac{\beta y}{z^2}\right)\left(\frac{y-z^2}{(1-xy/z^2)^2}\right)^n \left(\frac{z}{1-(xy/z^2)}\right)^{\alpha}$$

$$\times Y_n^{(\alpha+n)}\left(\frac{xy}{(1-xy/z^2)(y-z^2)}\right) \quad (4.14)$$

Separately, we consider the left hand side of (4.14) and we write exponential operators in series form so that we have the following relation.

$$e^{R_3}e^{R_4}[Y_n^{(\alpha+n)}(x)y^n z^\alpha] = \sum_{p,k=0}^{\infty} \frac{(\beta)^p(-n)_k}{p!k!} Y_{n+p-k}^{(\alpha+n+k-p)}(x)\, y^{n+p-k} z^{\alpha+2k-2p} \tag{4.15}$$

Equating (4.14) and (4.15) and then putting $\frac{y}{z^2} = w$, we get

$$(1-wx)^{(1-2n-\alpha)}(w-1)^n \exp(\beta w) Y_n^{(\alpha+n)}\left(\frac{wx}{(1-wx)(w-1)}\right)$$

$$= \sum_{p,k=0}^{\infty} \frac{(\beta)^p(-n)_k}{p!k!} Y_{n+p-k}^{(\alpha+n+k-p)}(x)\, w^{n+p-k} \tag{4.16}$$

Which is believed to be new generating function.

5. **Conclusion**

In this section, we conclude the findings of this paper. Basically we adopt the Weisner method to determine the new generating functions for modified Bessel polynomial. In the beginning of this paper, we defined the linear operators R_1, R_2, R_3, R_4 *and* L. In section 2, we discuss the commutative properties of these operators. We extend these operators in exponential form and we used them to determine the new and known generating functions such as (4.5), (4.6), (4.8), (4.9), (4.12), and (4.16). We also believe that the operators $\{1, R_i; i = 1,2,3,4)\}$, where 1 stands for the banality operator form a Lie group.

References

[1]. Chatterjea, S. K. (1975), Unification of a class of bilateral generating relations for certain special functions, Bull. Cal. Math. Soc., 67, pp. 115-127.

[2]. Chatterjea, S. K. (1977), An extension of a class of bilateral generating functions for certain special functions, Bull. Inst. Math. Acad. Sinica 5(2).

[3]. Chongdar, A. K. (1987), On the unification of a class of bilateral generating functions for certain special functions, Tamkang Jour. Math., 18 (3), pp. 53-59.

[4]. Chongdar, A. K. and Guhe Thak, B. K. (1986), Some generating functions of modified Bessel polynomials by Lie-algebraic method, Bull. Inst. Math. Acad. Sinica 14(2), 215-221.

[5]. Chongdar, A. K. (1987), Lie-algebra and modified Bessel polynomials, Bull. Inst. Math. Acad. Sinica 15(4), 417-431.

[6]. Krall, H. L. and Frink, O. (1949), A new class of orthogonal polynomials; the Bessel polynomials, Trans. Amer. Math. Soc., 65, 100-115.

[7]. McBride, E. B. (1971), Obtaining generating function, Springer Veriag, New York, Heidelberg and Berlin.

[8]. Mukherjee, M. C. (1998), Some generating functions of Bessel polynomials by group theoretic method, Indian. Acad. Math., Vol 20 No.2. 199-206.

[9]. Rainville, E. D. (1960), Special Functions, Chelsea Publication Company, New York.

[10]. Singhal J. P. and Srivastava, H. M. (1972), A class of bilateral generating functions for certain classical polynomials, Paci_c J. Math., 42, No.3, pp. 755-762.

[11]. Srivastava, H. M. and Manocha, H. L. (1984), A Treatise on Generating Functions, Halsted Press, John Wiley & Sons, New York.

[12]. Srivastava, H. M. and Karlsson, P. W. (1985), Multiple Gaussian hypergeometric series, Halsted Press, John Wiley & Sons, New York.

[13]. Weisner, L. (1955), Group theoretic origin of certain generating functions, Paci_c J. Math. 5, pp.1033-1039.

[14]. Weisner, L. (1959 a), Generating function for Hermite functions, Canad. J. Math. 11, pp. 141-147.

[15]. Weisner, L. (1959 b), Generating function for Bessel functions, Canadian J. Math. 11, pp. 148-155.