# Upper Bound Estimate for the Second Hankel Determinant for Certain Subclass of Univalent Functions 

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Abstract In this paper, we introduce and investigate a unification of starlike and convex functions of order $\alpha \in[0,1)$ in the unit disk in complex plane. We obtain upper bound estimate for the second Hankel determinant of the functions belonging to this class. Some consequences of the results obtained here are also discussed.

Keywords Analytic functions, Starlike functions, Convex functions, Hankel determinant

1. Introduction and Preliminaries

Let $U=\{z \in \mathbb{C}:|z|<1\}$ and $A$ be the class of analytic functions in $U$ normalized by $f(0)=f^{\prime}(0)-1=0$, in the form

It is well-known that a function $f: U \rightarrow \mathbb{C}$ is said to be univalent in $U$ if the following condition is satisfied: $z_{1}=z_{2}$ if $f\left(z_{1}\right)=f\left(z_{2}\right)$ or $f\left(z_{1}\right) \neq f\left(z_{2}\right)$ if $z_{1} \neq z_{2}$. We define by $S$ the subclass of $A$ which is also univalent.
Some of the important subclasses of $S$ are $S^{*}(\alpha)$ and $C(\alpha)$, respectively, starlike and convex functions of order $\alpha \geq 0$. By definition (see for details, [2, 4], also [12])

$$
\begin{equation*}
S^{*}(\alpha)=\left\{f \in S: \operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right)>\alpha, z \in U\right\} \tag{1.2}
\end{equation*}
$$

and

$$
\begin{equation*}
C(\alpha)=\left\{f \in S: \operatorname{Re}\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)>\alpha, z \in U\right\} . \tag{1.3}
\end{equation*}
$$

For $\alpha=0$ the subclasses $S^{*}(0)=S^{*}$ and $C(0)=C$ are, respectively, well known starlike and convex functions in $U$. It is easy to verify that $C \subset S^{*} \subset S$. For details on these classes, one could refer to the monograph by Goodman [4].
In 1976, Noonan and Thomas [10] defined the $q$ th Hankel determinant of $f$ for $q \in \mathbb{N}$ by

$$
H_{q}(n)=\left|\begin{array}{ccc}
a_{n} & \cdots & a_{n+q-1} \\
\cdot & \cdots & \cdot \\
a_{n+q-1} & \cdots & a_{n+2 q-2}
\end{array}\right|, a_{1}=1
$$

For $q=2$ and $n=1$ Fekete and Szegö [3] considered the Hankel determinant of $f$ as $H_{2}(1)=\left|\begin{array}{ll}a_{1} & a_{2} \\ a_{2} & a_{3}\end{array}\right|=a_{1} a_{3}-a_{2}^{2}$. They made an earlier study for the estimates of $\left|a_{3}-\mu a_{2}^{2}\right|$ when $a_{1}=1$ with real $\mu \in \mathbb{R}$. The well-known result due to them states that if $f \in A$, then

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq \begin{cases}3-4 \mu & \text { if } \mu \in(-\infty, 0] \\ 1+2 \exp \left(\frac{-2 \mu}{1-\mu}\right) & \text { if } \mu \in[0,1) \\ 4 \mu-3 & \text { if } \mu \in[1,+\infty)\end{cases}
$$

Furthermore, Hummel [6, 7] obtained sharp estimates for $\left|a_{3}-\mu a_{2}^{2}\right|$ when $f$ is a convex function and also Keogh and Merkes [8] obtained sharp estimates for $\left|a_{3}-\mu a_{2}^{2}\right|$ when $f$ is a close-to-convex function, starlike and convex function in $U$.
The second Hankel determinant $H_{2}(2)$ is given by $H_{2}(2)=a_{2} a_{4}-a_{3}^{2}$. One of the important tools in the theory of analytic functions is the functional $H_{2}(2)=a_{2} a_{4}-a_{3}^{2}$ which is known as the second Hankel determinant. The bounds for the second Hankel determinant obtained for the classes starlike and convex functions in [13].
Motivated by the aforementioned works, we define a subclass of univalent functions $S$ as follows.
Definition 1.1. A function $f \in S$ given by (1.1) is said to be in the class $M_{\beta}(\alpha), \alpha \in[0,1), \beta \geq 0$ if the following condition is satisfied

$$
\operatorname{Re}\left\{(1-\beta) \frac{z f^{\prime}(z)}{f(z)}+\beta \frac{\left(z f^{\prime}(z)\right)^{\prime}}{f^{\prime}(z)}\right\}>\alpha, z \in U
$$

Definition 1.2. A function $f \in S$ given by (1.1) is said to be in the class $M_{\beta}$, if the following condition is satisfied

$$
\operatorname{Re}\left\{(1-\beta) \frac{z f^{\prime}(z)}{f(z)}+\beta \frac{\left(z f^{\prime}(z)\right)^{\prime}}{f^{\prime}(z)}\right\}>0, z \in U
$$

Remark 1.1. Choose $\beta=0$ in Definition 1.1, we have function class $M_{0}(\alpha)=S^{*}(\alpha), \alpha \in[0,1)$.
Remark 1.2. Choose $\beta=1$ in Definition 1.1, we have function class $M_{1}(\alpha)=C(\alpha), \alpha \in[0,1)$.
The main object of the present paper is to find upper bound estimates for the second Hankel determinant of the functions belonging to the class $M_{\beta}(\alpha)$ and its special cases.
To prove our main results, we shall need the following lemmas concerning functions with positive real part (see e. g. $[1,5,9,11])$.

We denote by P , the class of the functions $p$ analytic in $U$ with expansion series

$$
p(z)=1+\sum_{n=1}^{\infty} p_{n} z^{n}
$$

and satisfying $p(0)=1, \operatorname{Re} p(z)>0$ for each $z \in U$
Lemma 1.1. If $p \in \mathrm{P}$, then provided estimates $\left|p_{n}\right| \leq 2, n=1,2,3, \ldots$. These estimates are sharp for the function $p(z)=\frac{1+z}{1-z}$.
Lemma 1.2. If the function $p \in \mathrm{P}$, then

$$
\left|p_{2}-\frac{v}{2} p_{1}^{2}\right| \leq 2 \max \{1,|v-1|\}=2 \cdot \begin{cases}1, & v \in[0,2] \\ |v-1|, & \text { elsewhere } .\end{cases}
$$

Lemma 1.3. If the function $p \in \mathrm{P}$ and $B \in[0,1], B(2 B-1) \leq D \leq B$, then

$$
\left|p_{3}-2 B p_{1} p_{2}+D p_{1}^{3}\right| \leq 2 .
$$

## 2. Upper bound for the second Hankel determinant of the class $M_{\beta}(\alpha)$

In this section, we prove the following theorem on upper bound estimate for the second Hankel determinant of the function class $M_{\beta}(\alpha)$.
Theorem 2.1. Let the function $f(z)$ given by (1.1) be in the class $M_{\beta}(\alpha), \alpha \in[0,1), \beta \geq 0$. Then,

$$
\begin{aligned}
& \left|a_{2} a_{4}-a_{3}^{2}\right| \leq \frac{(1-\alpha)^{2}}{3(1+\beta)^{4}(1+2 \beta)^{2}(1+3 \beta)} \\
& \times\left\{\begin{array}{l}
(1+\beta)^{3}\left(7 \beta^{2}+4 \beta+5\right)+3(1-\alpha)\left[\begin{array}{l}
(1+\beta)^{2}(1+2 \beta)(1+5 \beta) \\
+4(1+3 \beta)^{3}(1-\alpha)
\end{array}\right], \alpha \in\left[0, \beta_{2}\right], \\
4(1+\beta)^{3}(1+2 \beta)^{2}+3(1+\beta)^{4}(1+3 \beta)+12(1-\alpha)^{2}(1+3 \beta)^{3}, \alpha \in\left(\beta_{2}, 1\right)
\end{array}\right.
\end{aligned}
$$

for $0 \leq \beta \leq 4.9527$, where $\beta_{2}=1-\frac{(1+\beta)^{2}(1+3 \beta)}{2(1+5 \beta)(1+2 \beta)}$ and

$$
\left|a_{2} a_{4}-a_{3}^{2}\right| \leq \frac{\left[4(1+\beta)^{3}(1+2 \beta)^{2}+3(1+\beta)^{4}(1+3 \beta)+12(1-\alpha)^{2}(1+3 \beta)^{3}\right](1-\alpha)^{2}}{3(1+\beta)^{4}(1+2 \beta)^{2}(1+3 \beta)}
$$

for each $\alpha \in[0,1)$ and $\beta \geq 4.9527$.
Proof. Let $f \in M_{\beta}(\alpha), \alpha \in[0,1), \beta \geq 0$. Then,

$$
\begin{equation*}
(1-\beta) \frac{z f^{\prime}(z)}{f(z)}+\beta \frac{\left(z f^{\prime}(z)\right)^{\prime}}{f^{\prime}(z)}=\alpha+(1-\alpha) p(z), z \in U \tag{2.1}
\end{equation*}
$$

where $p \in \mathrm{P}$.
By simple computation from (2.1) for the coefficients $a_{2}, a_{3}$ and $a_{4}$, we obtain

$$
\begin{align*}
& a_{2}=\frac{1-\alpha}{1+\beta} p_{1},  \tag{2.2}\\
a_{3}= & \frac{1-\alpha}{2(1+2 \beta)} p_{2}+\frac{(1+3 \beta)(1-\alpha)^{2}}{2(1+\beta)^{2}(1+2 \beta)} p_{1}^{2},  \tag{2.3}\\
a_{4}= & \frac{1-\alpha}{3(1+3 \beta)} p_{3}+\frac{(1+5 \beta)(1-\alpha)^{2}}{2(1+\beta)(1+2 \beta)(1+3 \beta)} p_{1} p_{2} \\
& +\frac{\left(17 \beta^{2}+6 \beta+1\right)(1-\alpha)^{3}}{6(1+\beta)^{3}(1+2 \beta)(1+3 \beta)} p_{1}^{3} . \tag{2.4}
\end{align*}
$$

From (2.2) - (2.4) for the second Hankel determinant $a_{2} a_{4}-a_{3}^{2}$ we can easily establish

$$
a_{2} a_{4}-a_{3}^{2}=(1-\alpha)^{2}\left\{\frac{p_{1}}{3(1+\beta)(1+3 \beta)} I_{1}-\frac{p_{2}}{4(1+2 \beta)^{2}} I_{2}-\frac{(1-\alpha)^{2}(1+3 \beta)^{2}}{4(1+\beta)^{4}(1+2 \beta)^{2}} p_{1}^{4}\right\}
$$

where

$$
\begin{gathered}
I_{1}=p_{3}-\frac{3(1-\alpha)(1+3 \beta)^{2}}{2(1+\beta)(1+2 \beta)^{2}} p_{1} p_{2}+\frac{\left(17 \beta^{2}+6 \beta+1\right)(1-\alpha)^{2}}{2(1+\beta)^{3}(1+2 \beta)} p_{1}^{3} \\
I_{2}=p_{2}-\frac{2(1+5 \beta)(1+2 \beta)(1-\alpha)}{(1+\beta)^{2}(1+3 \beta)} p_{1}^{2}
\end{gathered}
$$

Thus, for $\left|a_{2} a_{4}-a_{3}^{2}\right|$ we can write the following inequality

$$
\begin{equation*}
\left|a_{2} a_{4}-a_{3}^{2}\right| \leq(1-\alpha)^{2}\left\{\frac{\left|p_{1}\right|}{3(1+\beta)(1+3 \beta)}\left|I_{1}\right|+\frac{\left|p_{2}\right|}{4(1+2 \beta)^{2}}\left|I_{2}\right|+\frac{(1-\alpha)^{2}(1+3 \beta)^{2}}{4(1+\beta)^{4}(1+2 \beta)^{2}}\left|p_{1}\right|^{4}\right\} \tag{2.5}
\end{equation*}
$$

Now we will use Lemma 1.3 to find a upper bound estimate for $\left|I_{1}\right|$.
Let's write the expression $\left|I_{1}\right|$ as follows:

$$
\left|I_{1}\right|=\left|p_{3}-2 B p_{1} p_{2}+D p_{1}^{3}\right|
$$

where

$$
B=\frac{3(1+3 \beta)^{2}(1-\alpha)}{4(1+\beta)(1+2 \beta)^{2}} \text { and } D=\frac{\left(17 \beta^{2}+6 \beta+1\right)(1-\alpha)^{2}}{2(1+\beta)^{3}(1+2 \beta)}
$$

It is clear that $B>0$ for each $\alpha \in[0,1)$ and $\beta \geq 0$. Also, it can be easily shown that $B \leq 1$ if $\alpha \geq \beta_{0}$, where

$$
\beta_{0}=1-\frac{4(1+\beta)(1+2 \beta)^{2}}{3(1+3 \beta)^{2}}
$$

Since $\beta_{0}<0$ for each $\beta \geq 0$ and $\alpha \in[0,1)$, condition $\alpha \geq \beta_{0}$ evidently satisfied. Thus, $B \in[0,1]$ for each $\alpha \in[0,1)$ and $\beta \geq 0$.

Also, it is easily shown that $D \leq B$ when $\alpha \geq \beta_{1}$, where

$$
\beta_{1}=1-\frac{3(1+\beta)^{2}(1+3 \beta)^{2}}{2(1+2 \beta)\left(17 \beta^{2}+6 \beta+1\right)}
$$

We can easily show that $\beta_{1}<0$ for each $\beta \geq 0$. Since $\alpha \in[0,1)$ so that $\alpha \geq \beta_{1}$ for each $\beta \geq 0$. Thus, $D \leq B$ for each $\alpha \in[0,1)$ and $\beta \geq 0$.
The inequality $B(2 B-1) \leq D$ is likewise can be proved for each $\beta \geq 0$.
Thus, in view of Lemma 1.3, we can write $\left|I_{1}\right| \leq 2$ for each $\beta \geq 0$.
For $\left|I_{2}\right|$ we write

$$
\left|I_{2}\right|=\left|p_{2}-\frac{v}{2} p_{1}^{2}\right|
$$

where

$$
v=\frac{4(1+5 \beta)(1+2 \beta)(1-\alpha)}{(1+\beta)^{2}(1+3 \beta)}
$$

Now, we use Lemma 1.2 to find a upper bound estimate for $\left|I_{2}\right|$. It is clear that $0<v$ for each $\alpha \in[0,1)$ and $\beta \geq 0$. Also, $v \leq 2$ if $\alpha \geq \beta_{2}$, where

$$
\beta_{2}=1-\frac{(1+\beta)^{2}(1+3 \beta)}{2(1+5 \beta)(1+2 \beta)}
$$

We can easily show that $\beta_{2}<0$ when $\beta \geq 4.9527$. Thus, since $\alpha \in[0,1)$, condition $\alpha \geq \beta_{0}$ evidently satisfied for each $\beta \geq 4.9527$. On the other hand $\beta_{2} \in[0,1]$ when $0 \leq \beta \leq 4.9527$.
Thus, in view of Lemma 1.2, we obtain the following inequality for $\left|I_{2}\right|$

$$
\left|I_{2}\right| \leq 2 \begin{cases}v-1, & \alpha \in\left[0, \beta_{2}\right] \\ 1, & \alpha \in\left(\beta_{2}, 1\right)\end{cases}
$$

for $0 \leq \beta \leq 4.9527$ and $\left|I_{2}\right| \leq 2$ for $\beta \geq 4.9527$.
Thus, applying Lemma 1.3 and Lemma 1.2 for $\left|I_{1}\right|$ and $\left|I_{2}\right|$ in (2.5), respectively, and again using the inequalities $\left|p_{n}\right| \leq 2, n=1,2,3$ from Lemma 1.1 for $\left|a_{2} a_{4}-a_{3}^{2}\right|$, we get

$$
\left|a_{2} a_{4}-a_{3}^{2}\right| \leq(1-\alpha)^{2}\left\{\begin{array}{l}
\frac{4}{3(1+\beta)(1+3 \beta)}+\frac{v-1}{(1+2 \beta)^{2}}+\frac{4(1-\alpha)^{2}(1+3 \beta)^{2}}{(1+\beta)^{4}(1+2 \beta)^{2}}, \alpha \in\left[0, \beta_{2}\right] \\
\frac{4}{3(1+\beta)(1+3 \beta)}+\frac{1}{(1+2 \beta)^{2}}+\frac{4(1-\alpha)^{2}(1+3 \beta)^{2}}{(1+\beta)^{4}(1+2 \beta)^{2}}, \quad \alpha \in\left(\beta_{2}, 1\right)
\end{array}\right.
$$

for $0 \leq \beta \leq 4.9527$, where $v=\frac{4(1+5 \beta)(1+2 \beta)(1-\alpha)}{(1+\beta)^{2}(1+3 \beta)}$ and $\beta_{2}=1-\frac{(1+\beta)^{2}(1+3 \beta)}{2(1+5 \beta)(1+2 \beta)}$ and

$$
\left|a_{2} a_{4}-a_{3}^{2}\right| \leq(1-\alpha)^{2}\left\{\frac{4}{3(1+\beta)(1+3 \beta)}+\frac{1}{(1+2 \beta)^{2}}+\frac{4(1-\alpha)^{2}(1+3 \beta)^{2}}{(1+\beta)^{4}(1+2 \beta)^{2}}\right\}
$$

for $\beta \geq 4.9527$.
Thus the proof of Theorem 2.1 is competed.
The following theorem is direct result of Theorem 2.1.
Theorem 2.2. Let the function $f(z)$ given by (1.1) be in the class $M_{\beta}, \beta \geq 0$. Then,

$$
\left|a_{2} a_{4}-a_{3}^{2}\right| \leq \frac{(1+\beta)^{3}\left(7 \beta^{2}+4 \beta+5\right)+3\left[(1+\beta)^{2}(1+2 \beta)(1+5 \beta)+4(1+3 \beta)^{3}\right]}{3(1+\beta)^{4}(1+2 \beta)^{2}(1+3 \beta)}
$$

for $0 \leq \beta \leq 4.9527$, where $\beta_{2}=1-\frac{(1+\beta)^{2}(1+3 \beta)}{2(1+5 \beta)(1+2 \beta)}$ and

$$
\left|a_{2} a_{4}-a_{3}^{2}\right| \leq \frac{4(1+\beta)^{3}(1+2 \beta)^{2}+3(1+\beta)^{4}(1+3 \beta)+12(1+3 \beta)^{3}}{3(1+\beta)^{4}(1+2 \beta)^{2}(1+3 \beta)}
$$

for $\alpha \in[0,1)$ and $\beta \geq 4.9527$.
Taking $\beta=0$ and $\beta=1$ in Theorem 2.1, we obtain the following results.
Corollary 2.1. Let the function $f(z)$ given by (1.1) be in the class $S^{*}(\alpha), \alpha \in[0,1)$. Then,

$$
\left|a_{2} a_{4}-a_{3}^{2}\right| \leq \frac{(1-\alpha)^{2}}{3} \begin{cases}12 \alpha^{2}-15 \alpha+20, & \alpha \in\left[0, \frac{1}{2}\right] \\ 12 \alpha^{2}-24 \alpha+19, & \alpha \in\left(\frac{1}{2}, 1\right)\end{cases}
$$

Corollary 2.2. Let the function $f(z)$ given by (1.1) be in the class $C(\alpha), \alpha \in[0,1)$. Then,

$$
\left|a_{2} a_{4}-a_{3}^{2}\right| \leq \frac{(1-\alpha)^{2}}{216} \begin{cases}16+3(1-\alpha)(41-32 \alpha), & \alpha \in\left[0, \frac{5}{9}\right] \\ 60+96(1-\alpha)^{2}, & \alpha \in\left(\frac{5}{9}, 1\right)\end{cases}
$$

Corollary 2.3. Let the function $f(z)$ given by (1.1) be in the class $S^{*}$. Then,

$$
\left|a_{2} a_{4}-a_{3}^{2}\right| \leq \frac{20}{3}
$$

Corollary 2.4. Let the function $f(z)$ given by (1.1) be in the class $C$. Then,

$$
\left|a_{2} a_{4}-a_{3}^{2}\right| \leq \frac{139}{216}
$$

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