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Analytical Solution of Two Dimensional Diffusion Equation using Hankel Transform

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Abstract Hankel transformation is used to solve the advection-diffusion equation in two-dimensional to get normalized cross-wind integrated concentration of pollutants at the surface of the earth with variable Eddy diffusivity. The pollutants are assumed to be totally reflected on the ground. The results of predicted model were compared with measuring observed data from the Research Reactor at Inshas, Egypt in unstable conditions.

Keywords Diffusion Equation, Hankel Transformation, Wind Speed, Eddy Diffusivity, friction velocity

Introduction

There are several integral transforms which are frequently used as a tool for solving numerous scientific problems. In many real applications, Fourier transforms as well as Mellin transforms and Hankel transforms are all very useful. The case for functions of two variables primarily to understand an origin for a Hankel transform is studied by Cornacchio, *et al* [1] and Hanna *et al* [2]. Air pollutants are transported, dispersed or deposited by meteorological and topographical conditions**.** The atmospheric advection-diffusion equation was used to describe the transport of pollutant in a turbulent atmosphere that was investigated by Seinfeld [3]. An analytical solution with physical phenomena was studied by Pasquill *et al* [4], Runca *et al* [5], Liu *et al* [6] and Runca [7], investigated an analytical solution to calculate the accuracy and performance of the numerical solutions. Pimentel *et al* [8] used the performance of a unified formal analytical solution for the simulation of atmospheric diffusion problems under stable conditions. Simple fractional differential equation models for the steady state spatial distribution of concentration of a non-reactive pollutant in Planetary Boundary Layer (PBL) was studied by Goulart *et al* [9]. They found that fractional derivatives models perform better than the traditional Gaussian model.

Essa *et al* [10] used mathematical model for dispersion of air pollutants in moderated winds taking the eddy diffusivity and wind speed were constant. Kumar *et al* [11], obtained an analytical dispersion Model for sources in the atmospheric surface layer with dry deposition to the ground Surface. Also studying the variation of eddy diffusivity on the behavior of advection-diffusion equation was studied by Essa *et al* [12]. Dispersion from two dimensions time-dependent and three dimensions of diffusion equation in different stability conditions is studied by Essa *et al* [13].

Hankel transform is used to solve the advection-diffusion equation in two-dimensional to get normalized cross-wind integrated concentration of pollutants at the surface of the earth with variable Eddy diffusivity. The pollutants are assumed to be totally reflected on the ground. The results of predicted model were compared with measuring observed data of Iodine-135from the Research Reactor at Inshas, Egypt in unstable conditions.

Mathematical Treatments

The advection diffusion equation can be written as follows:

$$u\frac{\partial C_y(x,z)}{\partial x}=\frac{\partial}{\partial z}\left(k_z\frac{\partial C_y(x,z)}{\partial z}\right) \tag{1}$$

where $C_y(x,z)$ is the crosswind integrated concentration of pollutants, u is the wind speed in m/s and k_z is vertical eddydiffusivity that is taken as power law in vertical distance "z". Eq. (1) is solved under the boundary conditions as follows:

The null flux condition of contaminants on the ground surface and the top at the vertical height are used:

$$\frac{\partial C_y}{\partial z}=0 \qquad \text{at} \quad z=0,h \tag{2a}$$

Where "h" is the height of the planetary boundary layer (PBL). In addition, the mass continuity of the source with emission rate "Q" at the height of the source "h_s".

$$uC_y(0,z)=Q\delta(z-h_s) \tag{2b}$$

where δ is Dirac delta function.

Assuming the vertical eddy diffusivity as follows:

$$k_z=\alpha z^n \tag{3}$$

where α is a parameter which depends on vertical scale velocity and wind speed.

Then Eq. (1) can be written as

$$\alpha z^n\frac{\partial^2 C_y}{\partial z^2}+\alpha n z^{n-1}\frac{\partial C_y}{\partial z}-u\frac{\partial C_y}{\partial x}=0) \tag{4}$$

Changing the independent variable z to ξ by the substitution $\xi=z^{\frac{2-n}{2}}$ then Eq. (4) becomes:

$$\xi^2\frac{\partial^2 C_y}{\partial \xi^2}+\frac{n}{2-n}\xi\frac{\partial C_y}{\partial \xi}-\frac{u}{\alpha}\left(\frac{2}{2-n}\right)^2\xi^2\frac{\partial C_y}{\partial x}=0 \tag{5}$$

Eq. (5) can be further simplified by the substitution $C_y(x,z)=\xi^m\psi(x,\xi)$, where $m=\frac{1-n}{2-n}$, $n\neq 2$, then we have

$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial\psi}{\partial\xi}-\frac{m^2}{\xi^2}\psi-\frac{4u}{\alpha}(1-m)^2\frac{\partial\psi}{\partial x}=0 \tag{6}$$

Eq. (6) can be solved for $\psi(x,\xi)$ by using Hankel transform which is defined as

$$\mathcal{H}_m\{f(z)\}=\tilde{f}(s)\equiv\int_0^\infty f(z)J_m(sz)z\,dz$$

where the Bessel differential operator is defined as

$$\Delta_m f(z)\equiv\frac{d^2f(z)}{dz^2}+\frac{1}{z}\frac{df(z)}{dz}-\left(\frac{m}{z}\right)^2 f(z)$$

which has the Hankel transform given by

$$\mathcal{H}_m\{\Delta_m f(z)\}\equiv -s^2\tilde{f}(s)$$

Applying the Hankel transform to Eq. (6)

$$\mathcal{H}_m\left\{\Delta_m\psi=\frac{4u}{\alpha}(1-m)^2\frac{\partial\psi}{\partial x}\right\} \tag{7}$$

One gets:

$$-s^2\tilde{\psi}=\frac{4u}{\alpha}(1-m)^2\frac{\partial\tilde{\psi}}{\partial x} \tag{8}$$

Eq. (8) has the solution,

$$\tilde{\psi}(x,s)=A\exp\left[-\frac{\alpha}{4u(1-m)^2}xs^2\right] \tag{9}$$

Using the boundary condition Eq. (2a) then Eq. (9) becomes:

$$\tilde{\psi}(x,s)=\frac{Qh_s^{\frac{m}{2(1-m)}}}{2u(1-m)}J_m\left(sh_s^{\frac{1}{2(1-m)}}\right)\exp\left[-\frac{\alpha}{4u(1-m)^2}xs^2\right] \tag{10}$$

Now assuming the inverse of Hankel Transformation

$$\mathcal{H}_m^{-1}\{\tilde{\psi}(x,s)\}=\psi(x,\xi)\equiv\int_0^\infty\tilde{\psi}(x,s)J_m(s\xi)s\,ds \tag{11}$$

Then, we will have

$$\psi(x,\xi)=\frac{2Qh_s^{\frac{m}{2(1-m)}}(1-m)}{\alpha x}exp\left(-\frac{u(1-m)^2\left(h_s^{\frac{1}{(1-m)}}+\xi^2\right)}{\alpha x}\right)I_m\left(\frac{2u\xi(1-m)^2h_s^{\frac{1}{2(1-m)}}}{\alpha x}\right) \tag{12}$$

By using the inverse substitution $\xi=z^{\frac{2-n}{2}}$

$$C_y(x,\xi) = \xi^m \psi(x,\xi) = \frac{2Q h_s^{\frac{m}{2(1-m)}} \xi^m (1-m)}{\alpha x} exp\left(-\frac{u(1-m)^2\left(h_s^{\frac{1}{(1-m)}}+\xi^2\right)}{\alpha x}\right) I_m\left(\frac{2u\xi(1-m)^2 h_s^{\frac{1}{2(1-m)}}}{\alpha x}\right) \quad (13)$$

Substituting$m = \frac{1-n}{2-n}$, the final solution is in the form:

$$C_y(x,z) = \frac{2Q h_s^{\frac{1-n}{2}} z^{\frac{1-n}{2}}}{\alpha(2-n)x} exp\left[-\frac{u(h_s^{2-n}+z^{2-n})}{\alpha(2-n)^2 x}\right] I_{\frac{1-n}{2-n}}\left(\frac{2u z^{\frac{2-n}{2}} h_s^{\frac{2-n}{2}}}{\alpha(2-n)^2 x}\right) \quad (14)$$

then the concentration will be

$$C(x,y,z) = \frac{2Q h_s^{\frac{1-n}{2}} z^{\frac{1-n}{2}}}{\sqrt{2\pi}\sigma_y \alpha(2-n)x} exp\left[-\frac{u(h_s^{2-n}+z^{2-n})}{\alpha(2-n)^2 x}\right] I_{\frac{1-n}{2-n}}\left(\frac{2u z^{\frac{2-n}{2}} h_s^{\frac{2-n}{2}}}{\alpha(2-n)^2 x}\right) e^{-\frac{y^2}{2\sigma_y^2}} \quad (15)$$

Results and Discussion

The data used to calculate the concentration of I-135 isotope was obtained from dispersion experiments conducted in unstable conditions to collect air samples around the Research Reactor at Inshas. The samples were collected at a height of 0.7m above ground. The emissions were released from a stack of height 43m. The values of power-law exponent p and n and $\sigma_y$of wind speed and eddy diffusivity as a function of air stability and standard deviation of crosswind are taken from Hanna et. al. [14] and presented in Tables (1) and (2). The predicted concentrations by Eq. (14) below the plume center line are presented in Table (3). A comparison between predicted and observed concentrations of I-135 in unstable condition at Inshas are shown in Figures (1 and 2).

Table 1: Power-law exponent p and n of wind speed and eddy diffusivity as a function of air stability in urban area

	A	**B**	**C**	**D**	**E**	**F**
P	0.15	0.15	0.20	0.25	0.40	0.60
n	0.85	0.85	0.80	0.75	0.60	0.40

Table 2: The values of standard deviation in crosswind σ_y through different stabilities

Stability classes	**Values of σ_y**
A	$\sigma_y = 0.40x^{0.91}$
B	$\sigma_y = 0.40x^{0.91}$
C	$\sigma_y = 0.36x^{0.86}$
D	$\sigma_y = 0.32x^{0.78}$

Table 3: The comparison between the predicated and observed concentration at different downwind distance, wind speed and distance for the different runs.

Run	**Stability class**	**h (m)**	**Wind Direction (deg)**	**U_{10} m (m/s)**	**Q (Bq)**	**Distance (m)**	**Observed C (Bq/m^3)**	**Predicted C (Bq/m^3)**
1	A	600.85	301.1	4	1028571	100	0.025	0.059
2	A	801.13	278.7	4	1050000	98	0.037	0.053
3	B	973	190.2	6	42857.14	115	0.091	0.083
4	C	888	197.9	4	471428.6	135	0.197	0.234
5	A	921	181.5	4	492857.1	99	0.272	0.343
6	D	443	347.3	4	514285.7	184	0.188	0.211
7	C	1271	330.8	4	1007143	165	0.447	0.484
8	C	1842	187.6	4	1043571	134	0.123	0.232
9	A	1642	141.7	4	1033929	96	0.032	0.077

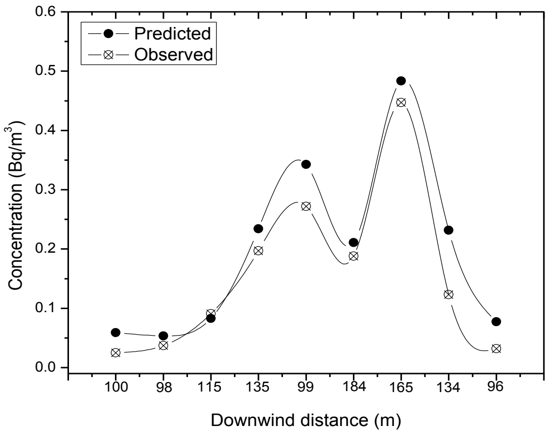

Figure 1: Observed and proposed concentration via downwind distance (m) of Iodine-135

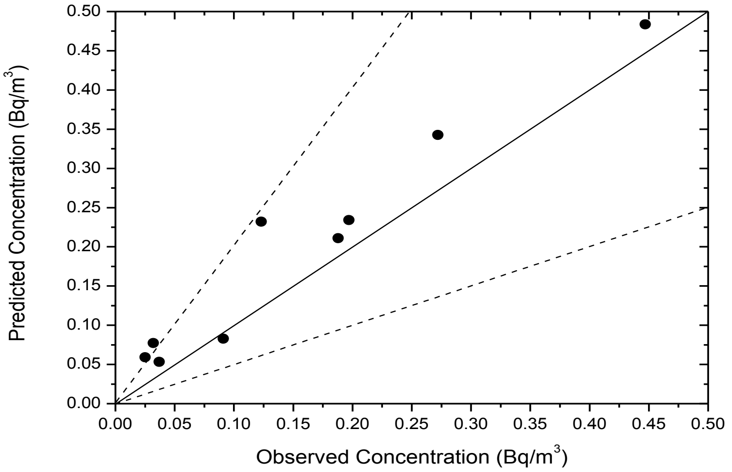

Figure 2: Observed and proposed concentration of Iodine-135

From the two figures (1) and (2), we can see that the predicted data inside a factor of two with the observed crosswind integrated concentration.

Model Evaluation Statistics

To evaluate the model accuracy we used the following statistical technique that characterize the agreement between the predicted and observed concentrations. These measures are discussed by Hanna [15] and defined as:

$$\text{Fraction Bias (FB)} = \frac{(\overline{C_o} - \overline{C_p})}{[0.5(\overline{C_o} + \overline{C_p})]}$$

$$\text{Normalized Mean Square Error (NMSE)} = \frac{\overline{(C_p - C_o)^2}}{(\overline{C_p}\,\overline{C_o})}$$

$$\text{Correlation Coefficient (COR)} = \frac{1}{N_m}\sum_{i=1}^{N_m}(C_{pi} - \overline{C_p}) \times \frac{(C_{oi} - \overline{C_o})}{(\sigma_p \sigma_o}$$

$$\text{Factor of Two (FAC2)} = 0.5 \le \frac{C_p}{C_o} \le 2.0$$

where σ_p and σ_o are the standard deviations of predicted ($Cp=C_{pred}$) and observed ($Co=C_{obs}$) concentration respectively. The over bar indicates the average value. The perfect model must have the following performances: NMSE = FB = 0 and COR= FAC2 = 1.0.

Table 4: The statistical evaluation of the present models against the Inshas experiments

NMSE	FB	COR	FAC2
0.08	0.12	0.97	0.89

Conclusion

We used Hankel transformation to solve the advection-diffusion equation in two-dimensional to get normalized cross-wind integrated concentration of pollutants at the surface of the earth with variable Eddy diffusivity. The pollutants are assumed to be totally reflected on the ground. The results of predicted model were compared with measuring observed data from Research Reactor at Inshas, Egypt in unstable conditions.

One finds that the present predicted model using Hankel transformation with variable vertical eddy diffusivity has good agreement with observed data in unstable condition.

From the statistical analysis, one finds that the analytical model is within factor of 2 (FAC2) with the observed data. The NMSR and FB are near to zero with Variable $K_z$in Enshas. Also, the COR and FAC2 are near to one.

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