Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2018, 5(10):213-221**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

A General Method for Generating Univariate Continuous Distributions

Mohamed I. Riffi, Hanin S. Al-Masri

Department of Mathematics, Islamic University of Gaza, Gaza, Palestine

Abstract In this paper, a new method for generating new families of continuous distributions is proposed. The proposed method is easy to use and implement, more flexible, and can be used with a wide range of distributions. It generalizes many methods that are used to generate new families of continuous distributions. Moreover, it can be used to generate wrapped distributions as well as circular distributions. Some important examples are given in this paper to demonstrate the application of the proposed method on various standard distributions, including the Normal, Beta, Weibull, exponential, beta, Lindley, Rayleigh, and gamma distributions.

Keywords Generating distributions; beta-generated distributions; exponentiated distributions, transmuted distributions, wrapped distributions

1. Introduction

Looking for methods and techniques to generate new flexible probability distributions from the classical distributions is motivated by the need to adequately fit real lifetime data and describe its complexity. Recently, many methods and techniques have been proposed by many statisticians to generate and extend new families of univariate continuous probability models [1]. These methods include exponentiation, beta-generation, transmutation, etc.

For example, Eugene et al., [2] proposed a method to generate a family of distributions called the beta-generated family of distributions. The authors used the beta pdf as a generator. The cdf of a beta-generated random variable X is defined as

$$G(x) = \int_0^{F(x)} b(t)\,dt, \tag{1.1}$$

where $F(x)$ is the cdf of some baseline distribution and $b(t)$ is the pdf of a beta distribution with parameters $\alpha > 0$ and $\beta > 0$. This is, $b(t)$ is given by

$$b_{\alpha,\beta}(t) = \frac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1}, \quad t \in (0,1), \alpha > 0, \beta > 0. \tag{1.2}$$

If X is continuous, the corresponding pdf is given by

$$g_{\alpha,\beta}(x) = \frac{f(x)}{B(\alpha,\beta)} F^{\alpha-1}(x)(1-F(x))^{\beta-1}, \tag{1.3}$$

where $f(x)$ is the pdf of the baseline distribution.

Jones, [3] and Cordeiro and deCastro [4] extended the beta-generated family of distributions by using the Kumaraswamy distribution. In other words, they used the pdf

$$k_{\alpha,\beta}(t) = \alpha\beta t^{\alpha-1}(1-t^{\alpha})^{\beta-1}, \quad t \in (0,1), \alpha > 0, \beta > 0 \tag{1.4}$$

instead of the beta pdf to generate new distributions.

The pdf of the Kumaraswamy-generated family of distributions is given by

$$g_{\alpha,\beta}(x) = \alpha\beta f(x) F^{\alpha-1}(x)(1-F^{\alpha}(x))^{\beta-1}, \quad \alpha > 0, \beta > 0, \tag{1.5}$$

where F is the cdf of the baseline distribution.

Alzaatreh et al.[5] proposed a method for generating new families of distributions by replacing the beta pdf with another pdf of some continuous random variable. His method includes the use of a function $W(F(x))$ that satisfies the following conditions.

$$\left.\begin{array}{l}(1)\quad W(F(x)) \in [a,b], \\ (2) W \text{ isdifferentiableandmonotonicallynonecreasing}, \\ (3) W\big(F(x)\big) \to a \text{ as } x \to -\infty \text{ and } W\big(F(x)\big) \to b \text{ as} x \to \infty,\end{array}\right\} \tag{1.6}$$

where $[a,b]$ is the support of the random variable T for $-\infty \le a < b \le \infty$.

This method aims at developing what the authors called the $T\text{–}X$ family of distributions. The cdf of the $T\text{–}X$ family of distributions is defined as

$$G(x) = \int_a^{W(F(x))} r(t)\, dt = R\{W(F(x))\}, \tag{1.7}$$

where R is the cdf of the generated random variable T.

The corresponding pdf is

$$g(x) = \left\{\frac{d}{dx} W(F(x))\right\} r\{W(F(x))\}, \tag{1.8}$$

where r is the pdf of the T.

The conditions (1.6) imposed on the method limits its use and applicability. Our proposed method in this paper widens the generated family of distributions by relieving the conditions imposed on the function W. It is described in details in the next section.

2. The Proposed Method

The proposed method for generating continuous distributions is described in the following theorem.

Theorem 2.1 *Let X be a continuous random variable with cdf $F(x)$, pdf $f(x)$, and support (a,b), where $-\infty \le a < b \le \infty$. Let $\phi(x)$ be a non-negative differentiable monotone function with support (c,d), where $-\infty \le c < d \le \infty$. Assume that $\ell_c := lim_{x\to c}\phi(x)$ and $\ell_d := lim_{x\to d}\phi(x)$ exist. If $F(\ell_c) \neq F(\ell_d)$, then*

$$g(x) = \frac{\phi'(x) f(\phi(x))}{F(\ell_d) - F(\ell_c)}, \quad c < x < d \tag{2.1}$$

is a pdf.

Proof. Clearly, the function $g(x)$ defined by (9) integrates to 1 over the interval (c,d). Also, it is clear that $g(x) \ge 0$ on (c,d). To see this, note that if $\phi(x)$ is increasing on (c,d), then for all $x \in (c,d)$, $\phi'(x) > 0$ and $F(\ell_d) - F(\ell_c) > 0$, implying that $g(x) > 0$ on (c,d). Similarly, If $\phi(x)$ is decreasing on (c,d), then for all $x \in (c,d)$, $\phi'(x) < 0$ and $F(\ell_d) - F(\ell_c) < 0$, implying that $g(x) > 0$ on (c,d). This shows that $g(x) > 0$ on (c,d).

Theorem 2.2 *Let X be a continuous random variable with cdf $F(x)$, pdf $f(x)$, and support (a,b), where $-\infty \le a < b \le \infty$. Let $\phi(x)$ be a non-negative differentiable monotone function with support (c,d), where $-\infty \le c < d \le \infty$. Assume that $\ell_c := lim_{x\to c}\phi(x)$ and $\ell_d := lim_{x\to d}\phi(x)$ exist. If $F(\ell_c) \neq F(\ell_d)$, then the function G defined by*

$$G(x) = \frac{F(\phi(x)) - F(\ell_c)}{F(\ell_d) - F(\ell_c)}, \quad c < x < d, \tag{2.2}$$

is a cdf of some distribution.

Proof. Note that G inherits the cdf properties from F. It only remains to show that $0 \le G(x) \le 1$ for $c < x < d$, because the support of ϕ may be different from the support of X. Consider the following two cases.

Case 1: ϕ is increasing.

suppose that there exists a point $e_1 \in (c,d)$ such that $\phi(e_1) \ge b$. Since ϕ is increasing, $\phi(c) \le \phi(e_1) \le \phi(d)$. In particular, $\ell_c \le \phi(e_1) \le \ell_d$.Hence, $F(\ell_c) \le F(\phi(e_1)) = 1 \le F(\ell_d)$. Then $F(\ell_d) = 1$ and

$$G(e_1) = \frac{F(\phi(e_1)) - F(\phi(c))}{F(\phi(d)) - F(\ell_c)} = \frac{1 - F(\ell_c)}{1 - F(\ell_c)} = 1. \tag{2.3}$$

Note that $F(\ell_c) < 1$, since $F(\ell_c) \ngeq F(\ell_d) = 1$.
Now, suppose that $\phi(e_2) \leq a$ for some $e_2 \in (c, d)$. Then $\phi(c) \leq \phi(e_2) \leq \phi(d)$. In particular, $\ell_c \leq \phi(e_2) \leq \ell_d$.Hence, $F(\ell_c) \leq F(\phi(e_2)) = 0 \leq F(\ell_d)$. Then $F(\ell_c) = 0$ and

$$G(e_2) = \frac{F(\phi(e_2)) - F(\ell_c)}{F(\phi(d)) - F(\ell_c)} = \frac{0-0}{F(\phi(d))-0} = 0. \tag{2.4}$$

Case 2: ϕ is increasing.
This case is similar to Case 1.
This proves that $0 \leq G(x) \leq 1$ for $c < x < d$.

Remark 2.1 *Note that $\phi(x)$ is not necessarily a function of the cdf of any random variable. This makes the proposed method more flexible and easy to use and implement.*
In the sequel of this paper, we will call the proposed method "the $X\phi$ method," where X is the baseline distribution that is used to generated the new distribution and ϕ is the generator.
The method used to produce exponentiated distribution is a special case of our proposed method. To see that, let X have a power distribution with parameter $\alpha > 0$ on the interval $(0,1]$. That is, the pdf of X is given as

$$f_\alpha(x) = \alpha x^{\alpha-1}, \quad 0 < x \leq 1.$$

Now, let $\phi(x)$ be the cdf of the distribution needed to be exponentiated.

Example 2.1 *To produce the exponentiated inverse Weibull distribution, let $\phi(x) = e^{-\left(\frac{1}{\beta x}\right)^\gamma}$, where $\beta > 0$, $\gamma > 0$, and $x > 0$. Then the generated distribution has a pdf given by*

$$g_{\alpha,\beta,\gamma}(x) = \beta^{-\gamma}(\alpha\gamma)x^{-\gamma-1}e^{-\alpha\left(\frac{1}{\beta x}\right)^\gamma}, \quad x > 0.$$

It is the pdf of the exponentiated inverse Weibull distribution [6].
We can also use our proposed method to produce quadratic map transmuted distribution. For the information about the method of generating transmuted distributions, we refer the reader to [7]. Let Y be a random variable with pdf

$$t_\lambda(y) = (1+\lambda) - 2\lambda y, \quad 0 \leq y \leq 1, -1 \leq \lambda \leq 1.$$

Clearly, $0 \leq t_\lambda(y) \leq 2$ for $0 \leq y \leq 1$ and $-1 \leq \lambda \leq 1$. Also,

$$\int_0^1 t_\lambda(y)\, dy = 1.$$

The cdf of Y is given by

$$T_\lambda(y) = \begin{cases} 0 & y < 0, \\ y(1+\lambda-\lambda y) & 0 \leq y < 1, \\ 1 & y \geq 1. \end{cases}$$

Then, the transmutation of a random variable X with cdf $F(x)$ is performed by setting $\phi(x) = F(x)$.

Example 2.2 *Let Y be as above. Let $\phi(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^\eta}$, where $\eta > 0, \sigma > 0$, and $x > 0$. Let $a = c = 0$ and $b = d = \infty$. Then, the Y-generated distribution has a pdf given by*

$$g_{\eta,\sigma,\lambda}(x) = \frac{\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1} e^{-\left(\frac{x}{\sigma}\right)^\eta}\left(1 - \lambda + 2\lambda e^{-\left(\frac{x}{\sigma}\right)^\eta}\right), \qquad x > 0,$$

where $\eta > 0, \sigma > 0$, and $-1 \leq \lambda \leq 1$.
This is the pdf of the transmuted Weibull distribution [8].
Also, the $T\!-\!X$ method is a special case of our method, since we can jut let $\phi(x) = W(F(x))$, $-\infty < x < \infty$, where the function W satisfies the conditions (1.6).

3. Examples
3.1. Beta-generated distributions
3.1.1. Modified beta distributions
Let X have a beta distribution with parameters $\alpha > 0$ and $\beta > 0$. That is, the pdf of X is

$$f_{\alpha,\beta}(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 < x < 1, \alpha > 0, \beta > 0. \tag{3.1}$$

Let $\phi(x) = 1 - x^2$, $0 < x < 1$. Then the generated distribution has a pdf given by

$$g_{\alpha,\beta}(x) = \frac{2(1-x^2)^{\alpha-1}x^{2\beta-1}}{B(\alpha,\beta)}, \quad 0 < x < 1. \tag{3.2}$$

Note that

$$\int_0^1 g(x)\,dx = 1.$$

Now, let $\phi(x) = e^{-x}$, $0 < x < \infty$. Then

$$g_{\alpha,\beta}(x) = \frac{e^{-\alpha x}(1-e^{-x})^{\beta-1}}{B(\alpha,\beta)}, \quad 0 < x < \infty. \tag{3.3}$$

Now, let $\phi(x) = 1 - e^{-x}$, $0 < x < \infty$. Then

$$g_{\alpha,\beta}(x) = \frac{e^{-\beta x}(1-e^{-x})^{\alpha-1}}{B(\alpha,\beta)}, \quad 0 < x < \infty. \tag{3.4}$$

In the previous two cases of this example,

$$\int_0^\infty g_{\alpha,\beta}(x)\,dx = 1.$$

In the first case,

$$\int_0^1 g_{\alpha,\beta}(x)\,dx = 1.$$

3.1.2. The Beta Sarhan-Zaindin modified Weibull distribution

Let X be a beta distributed random variable with parameters $\alpha > 0$ and $\beta > 0$. Let $\phi(x) = e^{-\beta x^k - \lambda x}$, where $k > 0$ and $\lambda > 0$. Then, the pdf of the generated distribution is

$$g_{\alpha,\beta,\lambda,k}(x) = \frac{1}{B(\alpha,\beta)}(\lambda + \beta k x^{k-1})\left(1 - e^{-\beta x^k - \lambda x}\right)^{\beta-1} e^{-\alpha\beta x^k - \alpha\lambda x}. \tag{3.5}$$

It is the pdf of the Beta Sarhan-Zaindin modified Weibull distribution introduced by Saboor et al. [9].

3.1.3. The Generalized Rayleigh distribution

Let X be a beta distributed random variable with parameters 1 and $\alpha > 0$. That is, the pdf of X is given by

$$f_\alpha(x) = \alpha x^{\alpha-1}, \quad 0 < x < 1.$$

Let $\phi(x) = e^{-\lambda^2 x^2}$, $x > 0$. Then the generated distribution has a pdf given by

$$g_{\alpha,\lambda}(x) = 2\alpha\lambda^2 x e^{-\lambda^2 x^2}\left(1 - e^{-\lambda^2 x^2}\right)^{\alpha-1}, \quad 0 < x < \infty. \tag{3.6}$$

This is the pdf of the generalized Rayleigh distribution introduced by Raqab [10].

3.1.4. The Kumaraswamy distribution

Let X be a beta distributed random variable with parameters $\beta > 0$ and $beta = 1$ (see (13)). Let $c = 0, d = 1$, and $\phi(x) = x^\alpha$, where $\alpha > 0$. Then the generated distribution has a pdf given by

$$g_{\alpha,\beta}(x) = \alpha\beta x^{\alpha-1}(1 - x^\alpha)^{\beta-1}, \quad 0 < x < 1. \tag{3.7}$$

This is the pdf of the Kumaraswamy distribution [11].

3.2. Gamma-Generated Distributions

3.2.1. Modified gamma distributions

If $\phi(x) = e^x$, $x > 0$, and X has a gamma distribution with parameters $\alpha > 0$ and $1/\beta$, where $\beta > 0$; that is, the pdf of X is

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, \quad x > 0.$$

Then

$$g_{\alpha,\beta}(x) = \frac{\beta^\alpha e^{\alpha x - \beta e^x}}{\Gamma(\alpha,\beta)}, \quad 0 < x < \infty. \tag{3.8}$$

Similarly, If $\phi(x) = x^\gamma$, $x > 0$ and $\gamma > 0$, and X has a gamma distribution with parameters $\alpha > 0$ and $\beta > 0$, then

$$g_{\alpha,\beta,\gamma}(x) = \frac{\gamma e^{-\beta x^\gamma}(\beta x^\gamma)^\alpha}{x\Gamma(\alpha)}, \quad 0 < x < \infty. \tag{3.9}$$

3.2.2. A trigonometric gamma distribution

Let $\phi(x) = \sin x$, $0 \le x \le \pi/2$, and X be a gamma distributed random variable with parameters $\alpha > 0$ and $\beta > 0$. Then

$$g_{\alpha,\beta}(x) = \frac{\beta^{\alpha} \cot x e^{-\beta \sin x} \sin^{\alpha} x}{\Gamma(\alpha) - \Gamma(\alpha,\beta)}, \quad 0 < x < \pi/2. \tag{3.10}$$

Now, clearly,

$$\int_0^{\pi/2} g_{\alpha,\beta}(x)\, dx = 1.$$

3.2.3. The Generalized gamma distribution

Let X have a gamma distribution with parameters $\frac{\gamma+1}{\delta}$ and $1/a$, where $\gamma > -1$, $\delta > 0$, and $a > 0$. Let $\phi(x) = x^{\delta}$, $x > 0$. Then the resulting distribution has a pdf given by

$$g_{a,\delta,\gamma}(x) = \frac{\delta a^{\frac{\gamma+1}{\delta}}}{\Gamma\left(\frac{\gamma+1}{\delta}\right)} x^{\gamma} e^{-a x^{\delta}}, \quad x > 0. \tag{3.11}$$

This is the pdf of the generalized gamma distribution introduced in Stacy [12].

3.2.4. The gamma-Pareto distribution

Let X be a gamma distributed random variable with parameters α and β/k, where $\alpha > 0, \beta > 0$, and $k > 0$. Let $\phi(x) = \log\left(\frac{x}{\theta}\right)$, where $\theta > 0$. Let $c = \theta$ and $d = \infty$ in (9). Then the generated distribution has a pdf given by

$$g_{\alpha,\beta,k,\theta}(x) = \frac{k^{\alpha}}{x\beta^{\alpha}\Gamma(\alpha)} \left(\frac{\theta}{x}\right)^{k/\beta} \log^{\alpha-1}\left(\frac{x}{\theta}\right), \quad x > \theta. \tag{3.12}$$

Setting $\beta/k = c$, we get the pdf of the gamma-Pareto distribution [13] with pdf

$$g_{\alpha,\beta,c}(x) = \frac{1}{x\Gamma(\alpha)c^{\alpha}} \left(\frac{\theta}{x}\right)^{1/c} \log^{\alpha-1}\left(\frac{x}{\theta}\right), \quad x > \theta. \tag{3.13}$$

3.3. Exponential-generated distributions

3.3.1. The Beta Weibull Poisson distribution

Let $\phi(x) = 1 - e^{-\beta x^{\alpha}}$, $0 < x < \infty, \alpha > 0$ and $\beta > 0$. Let X have an exponential distribution with parameter $\lambda > 0$. That is, the cdf of X is given as

$$F_{\lambda}(x) = 1 - e^{-\lambda x}, \quad x > 0.$$

Note that

$$\lim_{x\to\infty}\left[1 - e^{-\lambda\left(1-e^{-\beta x^{\alpha}}\right)}\right] = 1 - e^{-\lambda} \quad \text{and} \quad \lim_{x\to 0}\left[1 - e^{-\lambda\left(1-e^{-\beta x^{\alpha}}\right)}\right] = 0.$$

Therefore,

$$g_{\alpha,\beta,\lambda}(x) = \frac{\alpha\beta\lambda\, x^{\alpha-1} e^{-\lambda\left(1-e^{-\beta x^{\alpha}}\right)-\beta x^{\alpha}}}{1-e^{-\lambda}}, \quad x > 0. \tag{3.14}$$

The pdf in (3.14) is that of a random variable called beta Weibull Poisson random variable [13]. The corresponding cdf is given by

$$G_{\alpha,\beta,\lambda}(x) = \frac{e^{\lambda e^{-\beta x^{\alpha}}} - e^{\lambda}}{1-e^{\lambda}}, \quad x > 0. \tag{3.15}$$

This result can also be obtained in a similar way if we choose $\phi(x) = -e^{-\beta x^{\alpha}}, x > 0, \alpha > 0$ and $\beta > 0$.

3.3.2. The Modified Weibull distribution

Let X be an exponentially distributed random variable with parameter $\alpha > 0$ and let $\phi(x) = x^{\gamma} e^{x\lambda}$, where $\lambda > 0, \gamma > 0$ and $x > 0$.

Then the generated pdf

$$g_{\alpha,\gamma,\lambda}(x) = \alpha x^{\gamma-1}(\gamma + \lambda x)e^{\lambda x - \alpha x^{\gamma} e^{\lambda x}}, \quad x > 0, \tag{3.16}$$

is the pdf of a modified Weibull distribution introduced by Lai et al. [14].
The corresponding cdf has the form

$$G_{\alpha,\gamma,\lambda}(x) = 1 - e^{-\alpha x^{\gamma} e^{\lambda x}}, \quad x > 0. \tag{3.17}$$

3.3.3. The Modified Weibull distribution

Let X be an exponentially distributed random variable with parameter $\alpha > 0$ and let $\phi(x) = x^{\gamma} e^{\lambda(-x)}$, $\gamma > 0, \lambda > 0$, and $x > 0$.
Then the generated pdf

$$g_{\alpha,\gamma,\lambda}(x) = \alpha x^{\gamma-1}(\gamma + \lambda x)e^{\lambda x - \alpha x^{\gamma} e^{\lambda x}}, \quad x > 0. \tag{3.18}$$

This is the pdf of the beta modified Weibull distribution studied by Lai et al. [14].

3.3.4. The Beta modified Weibull distribution

Let X have a beta distribution with parameters $b > 0$ and $a > 0$. That is, the pdf of X is given by

$$f_{a,b}(x) = \frac{1}{B(a,b)} x^{b-1}(1-x)^{a-1}, \quad 0 < x < 1.$$

Let $\phi(x) = e^{-\alpha x^{\gamma} e^{\lambda x}}$, where $\alpha > 0, \gamma > 0, \lambda > 0$, and $x > 0$. Then the generated distribution has the beta modified Weibull distribution [15] with pdf

$$g_{a,b,\alpha,\beta,\gamma,\lambda}(x) = \frac{\alpha x^{\gamma-1} e^{\lambda x}(\gamma+\lambda x)}{B(a,b)} \left(1 - e^{-\alpha x^{\gamma} e^{\lambda x}}\right)^{a-1} e^{-b\alpha x^{\gamma} e^{\lambda x}}, \quad x > 0. \tag{3.19}$$

The beta modified Weibull distribution can also be generated using $\phi(x) = 1 - e^{-\alpha x^{\gamma} e^{\lambda x}}$ with X that has a beta distribution with parameters $a > 0$ and $b > 0$, respectively.

3.3.5. The wrapped exponential distribution

Let X be an exponentially distributed random variable with parameter $\lambda > 0$. Let $\phi(x) = x$, $c = 0$, and $d = 2\pi$.
Then the generated random variable has a pdf given as

$$g_{\lambda}(x) = \frac{\lambda e^{-\lambda x}}{1 - e^{-2\pi\lambda}}, \quad x \in [0, 2\pi). \tag{3.20}$$

This is the pdf of a wrapped exponential distribution introduced in Jammalamadaka and Kozubowski [16].

3.4. Normal-generated distributions

3.4.1. The Half-normal distribution

Let X have a standard normal distribution. Let $\phi(x) = (x/\sigma)^{\gamma}$, where $\sigma > 0$, $\gamma > 0$, and $x > 0$.
Then the generated pdf is given by

$$g_{\gamma,\sigma}(x) = \frac{2\gamma x^{\gamma-1}}{\sigma^{\gamma}\sqrt{2\pi}} e^{-x^{2\gamma}/2\sigma^{2\gamma}}, \quad x > 0. \tag{3.21}$$

If we let $\gamma = 1$ in (3.21), we get the pdf of the half-normal distribution with paprameter σ [17].

3.5. Lindley-generated distributions

3.5.1. The Lindley-exponential distribution

Let X have the Lindley distribution with parameter $\alpha > 0$. That is, the pdf of X is given by

$$f_{\alpha}(x) = \frac{\alpha^2 (x+1) e^{-\alpha x}}{\alpha+1}, \quad x > 0.$$

Let $\phi(x) = -\log\left(1 - e^{-\lambda x}\right)$, $\lambda > 0$ and $x > 0$. Let $c = 0$, $d = \infty$. Then the generated random variable has a Lindley-exponential distribution [18] with pdf

$$g_{\alpha,\lambda}(x) = \frac{\alpha^2 \lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}\left(1 - \log\left(1-e^{-\lambda x}\right)\right)}{\alpha+1}, \quad x > 0. \tag{3.22}$$

3.5.2. The Power Lindley distribution

X has a Lindley distribution with parameter $\beta > 0$. Let $\phi(x) = x^{\alpha}$, where $\alpha > 0$ and $x > 0$. Then, the generated random variable has a pdf given by

$$g_{\alpha,\beta}(x) = \frac{\alpha\beta^2 x^{\alpha-1}(x^{\alpha}+1)e^{-\beta x^{\alpha}}}{1+\beta}, \quad x > 0. \tag{3.23}$$

It is the pdf of the power Lindley distribution introduced in Ghitany et al. [19].

3.6. Uniform-generated distributions

3.6.1. The cardioid distribution

Let X be uniformly distributed on the interval $[0,2\pi)$. Let $\phi(x) = x + 2\rho\sin(x-\mu)$,where $0 \le \rho \le 1$, $c = 0$, and $d = 2\pi$. Then the generated distribution is called the cardioid distribution [20], where its pdf is given by

$$g_{\mu,\rho}(x) = \frac{1+2\rho\cos(x-\mu)}{2\pi}, \quad 0 \le x < 2\pi. \tag{3.24}$$

3.6.2. The exponentiated-exponential-logistic distribution

Let X have a uniform distribution on the unit interval $(0,1)$. Let $\phi(x) = \left(1-(1+e^{x})^{-\lambda}\right)^{\alpha}$, where $\alpha > 0$, $\lambda > 0$, $a = -\infty$, and $b = \infty$. Then the generated distribution has a pdf given by

$$g_{\alpha,\beta}(x) = \frac{\alpha\lambda e^{-\lambda x}}{(1+e^{-x})^{\lambda+1}}\left(1-(1+e^{x})^{-\lambda}\right)^{\alpha-1}, \quad -\infty < x < \infty. \tag{3.25}$$

Setting $\lambda = 1$, we get the exponentiated-exponential-logistic distribution (compare with Alzaatreh et al. [5]. Another way to generate the exponentiated-exponential-logistic distribution is to use the power distribution with parameter $\alpha > 0$ on the interval $(0,1]$ with $\phi(x) = 1-(e^{x}+1)^{-\lambda}$, $\lambda > 0$, $-\infty < x < \infty$.

3.7. Cauchy-generated distributions

3.7.1. The wrapped Cauchy distribution

Assume that X has a standard Cauchy distributed. That is, the pdf of X is

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Let $\phi(x) = \frac{(\rho+1)\tan\left(\frac{x-\mu}{2}\right)}{\rho-1}$, where $0 < \rho < 1$, $-\infty < \mu < \infty$, and $0 \le x < 2\pi$. Let $c = 0$ and $d = 2\pi$. Then the generated distribution is called a wrapped Cauchy distribution [16] whose pdf is given by

$$g_{\mu,\rho}(x) = \frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos(x-\mu)}, \quad 0 \le x < 2\pi. \tag{3.26}$$

A modified model of the wrapped Cauchy distribution has a pdf of the form

$$g_{\gamma,\mu}(x) = \frac{1}{2\pi}\frac{\sinh(\gamma)}{\cosh(\gamma)-\cos(x-\mu)}, \quad -\pi \le x < \pi, \tag{3.27}$$

where μ is real and $\gamma > 0$ [21].

This model can be generated as above with $\phi(x) = \coth\left(\frac{\gamma}{2}\right)\tan\left(\frac{x-\mu}{2}\right)$, $c = -\pi$, and $d = \pi$.

4. The Generalized Gamma Distribution

We have seen above how to generate the generalized gamma (GG) distribution from the gamma distribution using $\phi(x) = x^{\delta}$, where $\delta > 0$. The pdf of the GG distribution is given in (3.11). The cdf of the GG distribution is given by

$$F_{\alpha,\delta,\gamma}(x) = \frac{\Gamma\left(\frac{\gamma+1}{\delta}\right)-\Gamma\left(\frac{\gamma+1}{\delta},\alpha x^{\delta}\right)}{\Gamma\left(\frac{\gamma+1}{\delta}\right)}, \quad \alpha > 0, \delta > 0, \gamma > -1, x > 0. \tag{4.1}$$

Using the GG distribution as a baseline and $\phi(x) = \left(\frac{x-\mu}{\sigma}\right)^{-\delta}$ as a generator ($x > \mu, -\infty < \mu < \infty, \delta > 0, \sigma > 0$), we get a new distribution with pdf

$$g_{\alpha,\delta,\mu,\sigma}(x) = \frac{\delta\alpha^{\frac{\gamma+1}{\delta}}\left(\frac{x-\mu}{\sigma}\right)^{-(\gamma+1)}e^{-\alpha\left(\frac{x-\mu}{\sigma}\right)^{-\delta}}}{(x-\mu)\Gamma\left(\frac{\gamma+1}{\delta}\right)}, \quad x > \mu. \tag{4.2}$$

By setting $\gamma = \delta - 1$ and $\alpha = 1$ in (4.2), we get the Fréchet (or the inverse Weibull) distribution with pdf

$$w_{\delta,\sigma}(x) = \frac{\delta\left(\frac{x-\mu}{\sigma}\right)^{-\delta-1} e^{-\left(\frac{x-\mu}{\sigma}\right)^{-\delta}}}{\sigma}, \quad x > \mu.$$

By setting $\gamma = -\frac{1}{2}, \alpha = \frac{1}{2}$, and $\delta = 1$ in (4.2), we get the Levy distribution which has a pdf given by

$$v_{\mu,\sigma}(x) = \frac{e^{-\frac{\sigma}{2(x-\mu)}}\left(\frac{\sigma}{x-\mu}\right)^{3/2}}{\sqrt{2\pi}\sigma}, \quad x > \mu. \tag{4.3}$$

Conclusion

The proposed method can efficiently be used to generate new distributions and generalize or extend standard distributions. The beta-generated distributions, exponentiated distributions, transmuted distributions can be produced by our proposed method in this paper. Our proposed method also generalizes the $T-X$ method of Alzaatreh et al. [5].

References

[1]. Lee, C., Famoye, F., and Alzaatreh, A. Y. (2013). Methods for generating families of univariate continuous distributions in the recent decades. *WIREs Comput Stat*, 5(3):219–238.

[2]. Eugene, N., Lee, C., and Famoye, F. (2002). The beta-normal distribution and its applications. *Commun. Stat. Theory Methods*, 31(4):497–512.

[3]. Jones, M. (2009). Kumaraswamyâ€™s distribution: A beta-type distribution with some tractability advantages. *Statistical Methodology*, 6:70–81.

[4]. Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81:883–898.

[5]. Alzaatreh, A., Lee, C., and Famoye, F. (2013). A new method for generating families of continuous distributions. *METRON*, 71:63–79.

[6]. Ali, M. M., Pal, M., and Woo), J. (2007). Some exponentiated distributions. *The Korean Communications in Statistics*, 14(1):93–109.

[7]. Shaw, W. T. and Buckley, I. R. (2009). The alchemy of probability distributions: beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *arXiv:0901.0434*.

[8]. Aryal, G. R. and Tsokos, C. P. (2011). Transmuted weibull distribution: A generalization of the weibull probability distribution. *European Journal of Pure and Applied Mathematics*, 4(2):89–102.

[9]. Saboor, A., Bakouch, H. S., and Khan, M. N. (2016). Beta sarhan-zaindin modified weibull distribution. *Applied Mathematical Modelling*, pages S0307904X1630021X–.

[10]. Raqab, D. K. M. Z. (2005). Generalized rayleigh distribution: different methods of estimations. *Computational Statistics & Data Analysis*, 49:187–200.

[11]. Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of Hydrology*, 46:0–88.

[12]. Stacy, E. W. (1962). A generalization of the gamma distribution. *The Annals of mathematical statistics*, 33(3):1187–1192.

[13]. Percontini, A., Blas, B., and Cordeiro, G. M. (2013). The beta weibull poisson distribution. *Chilean Journal of Statistics*, 4(2):3–26.

[14]. Lai, C., Xie, M., and Murthy, D. (2003). A modified weibull distribution. *Trans Reliab*, 52:33–37.

[15]. Silva, G. O., Ortega, E. M. M., and Cordeiro, G. M. (2010). The beta modified weibull distribution. *Lifetime Data Analysis*, 16:409–430.

[16]. Jammalamadaka, S. and Kozubowski, T. (2003). A new family of circular models: The wrapped laplace distributions. 3:77–103.

[17]. Olmos, N. M., Varela, H., G뎁mez, H. W., and Bolfarine, H. (2012). An extension of the half-normal distribution. *Statistical Papers*, 53:875–886.

[18]. Bhati, D., Malik, M. A., and Vaman, H. J. (2015). Lindley–exponential distribution: properties and applications. *METRON*, 73(3):335–357.

[19]. Ghitany, M., Al-Mutairi, D., Balakrishnan, N., and Al-Enezi, L. (2013). Power lindley distribution and associated inference. *Computational Statistics & Data Analysis*, 64:20–33.

[20]. Jammalamadaka, S. and SenGupta, A. (2001). *Topics in Circular Statistics*, volume 5 of *Multivariate Analysis*. World Scientific Publishing Pte. Ltd, Singapore.

[21]. Mardia, K. V. and Jupp, P. E. (1999). *Directional Statistics*. Wiley Series in Probability and Statistics. Wiley, Chichester.