Numerical Solution of Linear and Nonlinear Swinging Pendulum

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Abstract A pendulum is a weight suspended from a pivot so that it can swing freely. When a pendulum is displaced sideways from its resting, equilibrium position, it is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position. When released, the restoring force combined with the pendulum's mass causes it to oscillate about the equilibrium position, swinging back and forth. The time for one complete cycle, a left swing and a right swing, is called the period. The period depends on the length of the pendulum and also to a slight degree on the amplitude, the width of the pendulum's swing.

Keywords pendulum, numerical, vibration, displacement, velocity, Matlab, mechanical

1- Construction of Differential Equation

$$\sum F = -W\sin(\theta) = m * \text{acceleration} = \frac{W}{g} * \text{acceleration}$$

$$\alpha = \frac{\text{acceleration}}{\text{length}} \ .. \,(\text{angular acceleration})$$

$$\sum F = -W\sin(\theta) = \frac{Wl}{g}\alpha$$

$$-\cancel{W}\sin(\theta) = \frac{\cancel{W}l}{g}\alpha \xrightarrow{\text{yields}} -\sin(\theta) = \frac{l}{g}\alpha$$

$$\alpha + \frac{g}{l}\mathrm{Sin}(\theta) = 0$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0 \dots\dots (1)$$

This is the main equation to be used.

It is a second order differential equation, there is multiple methods to be used to solve this equation, either linear or nonlinear. Next we will be solving this equation and compare the results.
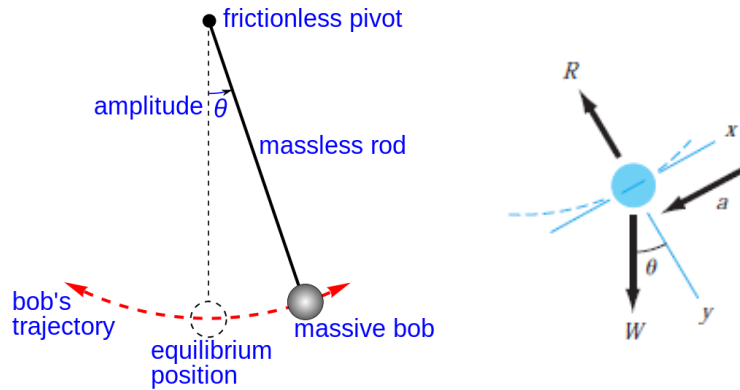
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Figure 1: A free-body diagram of the swinging pendulum showing the forces on the particle and the acceleration

2- Linear Analytical Solution

Series expansion for $\sin(\theta)$ can be expressed as:

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \cdots$$

For small angular displacements, $\sin(\theta)$ is approximately

equal to θ when expressed in radians

So $\sin(\theta) \approx \theta$ (rad) .... (2)

Recall equation 1 and replace $\sin(\theta)$ with equation 2

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \ ..... (3)$$

$$\theta = A \sin\left(\sqrt{\frac{g}{l}}\,t\right) + B \cos\left(\sqrt{\frac{g}{l}}\,t\right) ... (4)$$

substitute initial conditions $\theta(0) = \theta_o ; \frac{d\theta}{dt} = 0$

$$\theta = \theta_o \cos\left(\sqrt{\frac{g}{l}}\,t\right) .. (5)$$

Angular velocity $\omega = \frac{d\theta}{dt} = -\theta_o * \sqrt{\frac{g}{l}} \sin\left(\sqrt{\frac{g}{l}}\,t\right) ... (6)$

using equation 5 and 6 to obtain the data in the table

for $\theta_o = \frac{\pi}{3} \ and \ \theta_o = \frac{\pi}{10}$

Table 1: Linear Analytical Solution for θ and ω with initial condition $\theta_o = \pi/3$

	$\theta_o = \pi/3$	
T (sec)	θ(rad)	ω (rad/s)
0	1.0471	0

0.5	0.00497	-3.27988
1	-1.047	-0.03116
1.5	-0.014	3.27958
2	1.047	0.06232
2.5	0.0248	-3.27899
3	-1.046	-0.09347
3.5	-0.034	3.27810
4	1.0464	0.12461
4.5	0.0447	-3.27692
5	-1.046	-0.15575
g=9.81 m/s^2; length=1m		

Table 2: Linear Analytical Solution for θ and ω with initial condition
$\theta_o = \pi/10$

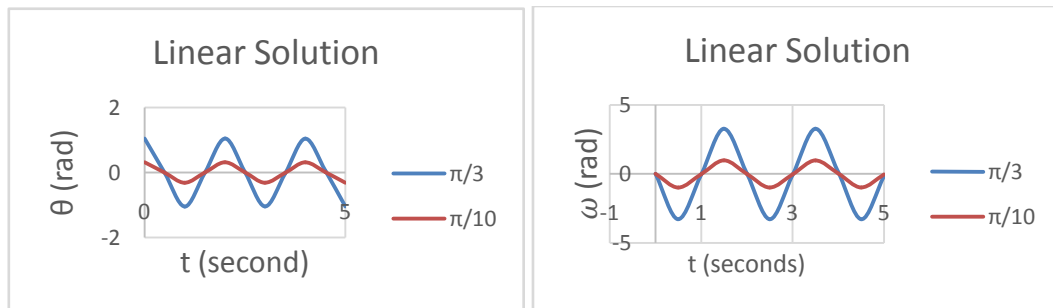|	$\theta_o = \pi/10$	
T (sec)	Θ (rad)	ω(rad/s)
0	0.3141	0
0.5	0.001492	-0.9839
1	-0.314	-0.0093
1.5	-0.00447	0.9838
2	0.3141	0.00187
2.5	0.007461	-0.98369
3	-0.314	-0.02804
3.5	-0.01044	0.98343
4	0.3139	0.03738
4.5	0.0134	-0.98307
5	-0.313	-0.04672
g=9.81 m/s^2; length=1m		

Figure 2: θ and ω relationship with time for two initial conditions $\theta_o = \pi/3$, $\theta_o = \pi/10$

3- Second Order Differential Equation Solution Using Runge-Kutta 4th Order Methods

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0 \dots (1) \quad ; \quad \omega = \frac{d\theta}{dt} \dots (6)$$

Rearrange equations

$$\frac{d\omega}{dt} = -9.81\sin(\theta) \dots (7) \ (g = 9.81 \frac{m}{s^2}; l = 1m)$$

$$\frac{d\theta}{dt} = \omega \dots (8)$$

$$\binom{\omega}{\theta}_{i+1} = \binom{\omega}{\theta}_i + \frac{h}{6}\left[\binom{k11}{k21} + 2\binom{k12}{k22} + 2\binom{k13}{k23} + \binom{k14}{k24}\right] \dots (9)$$

Taking h=0.001

$$\theta_o = \frac{\pi}{3}, \frac{\pi}{10}; \ \omega_o = 0; t_o = 0$$

Coefficients related to ω	**Coefficients related to θ**
$k11 = f(t_o, \omega_o, \theta_o)$	$k21 = f(t_o, \omega_o, \theta_o)$
$k12 = f\left(t_o + \dfrac{h}{2}, \omega_o + \dfrac{h*k11}{2}, \theta_o + \dfrac{h}{2}k21\right)$	$k22 = f\left(t_o + \dfrac{h}{2}, \omega_o + \dfrac{h*k11}{2}, \theta_o + \dfrac{h}{2}k21\right)$
$k13 = f\left(t_o + \dfrac{h}{2}, \omega_o + \dfrac{h*k12}{2}, \theta_o + \dfrac{h}{2}k22\right)$	$k23 = f\left(t_o + \dfrac{h}{2}, \omega_o + \dfrac{h*k12}{2}, \theta_o + \dfrac{h}{2}k22\right)$
$k14 = f(t_o + h, \omega_o + hk13, \theta_o + hk23)$	$k24 = f(t_o + h, \omega_o + hk13, \theta_o + hk23)$

This is very time consuming process but very efficient, instead of solving it manually, a Matlab program will be used to evaluate the results for θ and ω for a step of 0.001s and then substitute in equation 9

For example, $\theta_o = \dfrac{\pi}{3}; \omega_o = 0; t_o = 0$

$$\theta_{i+1} = \theta_i + \frac{h}{6}(k21 + 2k22 + 2k23 + k24)$$

$$\theta_{i+1} = \frac{\pi}{3} + \frac{0.001}{6}(-8.420863017 * 10^{-3})$$

$$\theta_{i+1} = 1.047196148 \; rad$$

$$\omega_{i+1} = \omega_i + \frac{h}{6}(k11 + 2k12 + 2k13 + k14)$$

$$\omega_{i+1} = \frac{0.001}{6}(-42.39518331) = -7.065863886 * 10^{-3} rad/s$$

K11	-8.495709211
K21	0
K12	-8.495709211
K22	-0.004247854606
K13	-8.495698793
K23	-0.000004247854606
K14	0.08334190469
K24	0.00008334190469

t	$\theta(RK4)$	$\theta(linear)$	$\omega(RK4)$	$\omega(linear)$
0	$\dfrac{\pi}{3}$	$\dfrac{\pi}{3}$	0	0
0.001	1.047194723	0.003279914	-0.007065863	-0.010273
g=9.81 m/s^2; length=1m				

Since h=0.001, for 5 seconds we need 5000 solutions! So we must use Matlab for solution.

As time increases the difference in results increases.

Other numerical methods for solving ODEs was not used because RK4 gives very good results so we do not need to use other methods.

4- Matlab Solution for θ

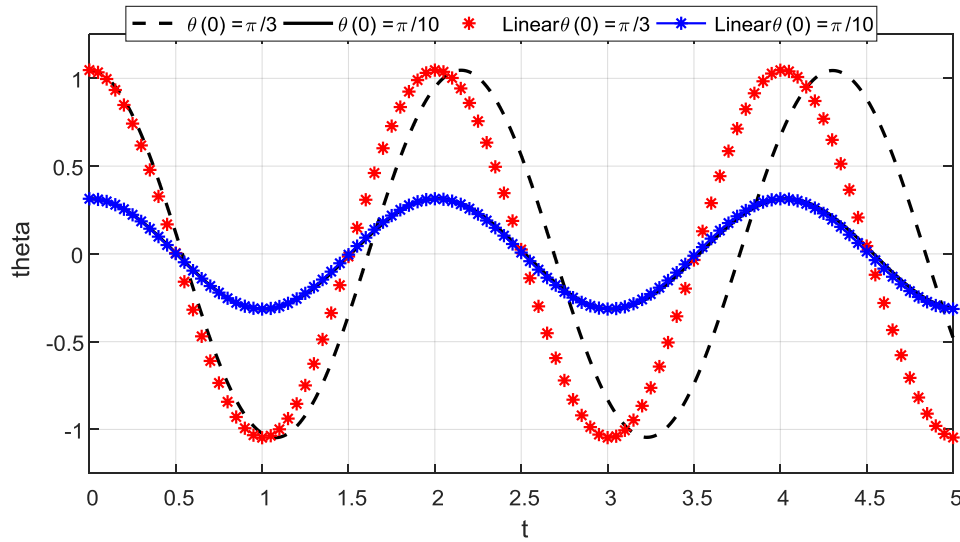*Matlab code for θ Solution is available at the end of the paper (Code #1)*

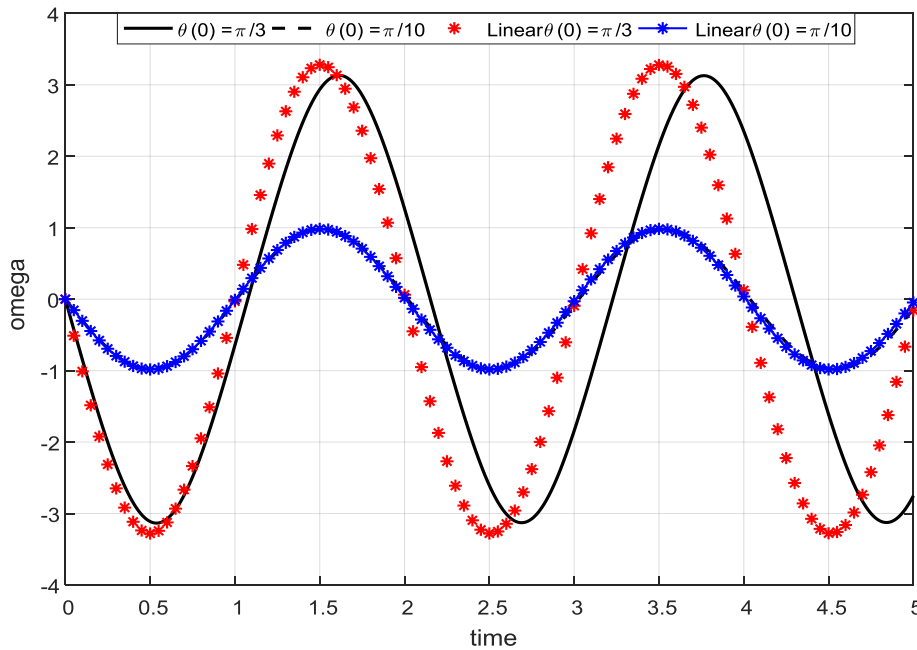*Figure 3: Graph of $\theta - time$ for two initial conditions $\theta_o = \pi/3$, $\theta_o = \pi/10$*

Figure 4: Graph of $\omega - time$ for two initial conditions $\theta_o = \pi/3$, $\theta_o = \pi/10$

5- Matlab Solution for ω

*Matlab code for ω Solution is available at the end of the paper (Code #2),

for the previous section a second order differential equation for theta was defined and solution for θ was obtained.

for ω an approach mentioned in section 3 was used to find $\omega(t)$, keep it in mind that the error of the linearization for ω will be higher than θ because the linear function of θ (equation 5) that has approximation error was derived to find ω (equation 6) which results in higher error.

6- Applications of Swinging Pendulum

Students learn what a pendulum is and how it works in the context of amusement park rides. While exploring the physics of pendulums, they are also introduced to Newton's first law of motion — about continuous motion and inertia. Pendulums are used in many engineered objects, such as clocks, metronomes, amusement park rides

and earthquake seismometers. In addition, engineers know that understanding the physics of how pendulums behave is an important step towards understanding motion, gravity, inertia and centripetal force. Engineers apply their understanding of these motion concepts to determine the force needed to propel an object into outer space, the braking power required to stop a vehicle at high speeds, and the optimal curve of a highway ramp. Teams of engineers work on a wide range of projects and solve problems that are important to society.

The pendulum-slosh problem

Suspending a rectangular vessel which is partially filled with fluid from a single rigid pivoting pole produces an interesting theoretical model with which to investigate the dynamic coupling between fluid motion and vessel rotation. The exact equations for this coupled system are derived with the fluid motion governed by the Euler equations relative to the moving frame of the vessel, and the vessel motion governed by a modified forced pendulum equation. The nonlinear equations of motion for the fluid are solved numerically via a time-dependent conformal mapping, which maps the physical domain to a rectangle in the computational domain with a time dependent conformal modulus.

The numerical scheme expresses the implicit free-surface boundary conditions as two explicit partial differential equations which are then solved via a pseudo-spectral method in space. The coupled system is integrated in time with a fourth-order Runge-Kutta method.
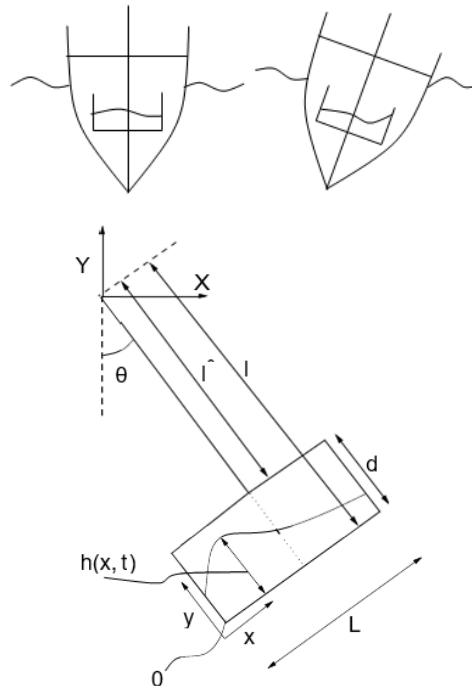

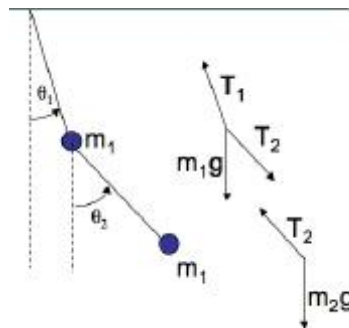*Figure 5: The Pendulum-Slosh Problem*

Figure 6: Double Pendulum

Double Pendulum

In physics and mathematics, in the area of dynamical systems, a double pendulum is a pendulum with another pendulum attached to its end, and is a simple physical system that exhibits rich dynamic behavior with a strong sensitivity to initial conditions. The motion of a double pendulum is governed by a set of coupled ordinary differential equations and is chaotic. Lagrange's equation for θ_1

$$0 = (m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2l_1l_2\ddot{\theta}_2\cos(\theta 2 - \theta 1) - m_2l_1l_2\dot{\theta}_2^2\sin(\theta 2 - \theta 1) + (m_1 + m_2)gl_1\sin(\theta_1)$$

Lagrange's equation for θ_2

$$0 = m_2l_2^2\ddot{\theta}_2 + m_2l_1l_2\ddot{\theta}_1\cos(\theta 2 - \theta 1) + m_2l_1l_2\dot{\theta}_1^2\sin(\theta 2 - \theta 1) + m_2gl_2\sin(\theta_2)$$

We collect the two Lagrange equations of motion, which are, of course, the same ones we got from Newton's law:

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2\cos(\theta 2 - \theta 1) = m_2l_2\dot{\theta}_2^2\sin(\theta 2 - \theta 1) - (m_1 + m_2)g\sin(\theta_1) \dots (A)$$

$$l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta 2 - \theta 1) = -l_1\dot{\theta}_1^2\sin(\theta 2 - \theta 1) - g\sin(\theta_2) \dots (B)$$

7- Conclusion

Pendulum is a great tool to express the motion of a body, an important application in chaos theory which is a branch in mathematics focused on the behavior of dynamical systems that are highly sensitive to initial conditions. So it was necessary to explain its equation and find solutions methods to expect its motion related to time and depending on the initial condition. It was seen that for small initial angular displacement the linearization gives fairly good results and less time consuming, while large initial angular displacement cannot be solved linearly, and the differential equation is in complex form, so we used numerical methods to solve the equation of motion using Matlab because it is a time consuming process, increasing the time interval (h value) increases accuracy of solution, for 5 seconds a time interval of 0.001 second was used, so a 5000 process was done using Matlab, without Computer Softwares it would be impossible to solve the problem. Graphs has already been shown and it will be available in large pixels in the next page. Double Pendulum expands the ability to represent more complex motions, but more complex equations, and more time needed to solved the problem. More pendulums can be used as much as needed.

8- References and Computer Applications

A) References

[1]. Theory of Vibration 5th edition, William T Thomson
[2]. Numerical Methods for Engineering 6th edition, Steven C. Chapra, Raymond P. Canale
[3]. Thomas' Calculus 13th edition, George B. Thomas Jr., Maurice D. Weir, Joel R. Hass
[4]. The pendulum-slosh problem: simulation using a time-dependent conformal mapping, by M.R. Turner, T.J. Bridges & H. Alemi Ardakani
[5]. Single and Double Plane Pendulum, by Gabriela Gonz´alez
[6]. wikipedia.com
[7]. mathworks.com

B) Computer Applications

[1]. Matlab R2016b
[2]. Microsoft Excel 2016

Code #1	Code #2

<pre>function main
%define g and L
g=9.81;L=1;
K=g/L;
%nonlinear solution, define theta and time (initial conditions)
%case 1
%define time change for 0.001
xo=[pi/3,0];
[t,x]=ode45(@DE2,[0:0.001:5],xo);
plot(t,x(:,1),'k--','Linewidth',1.5)
xlabel('t'),ylabel('theta'),grid on
axis([0,5,-2,2])
hold on
%case 2
%define time change for 0.001
xo=[pi/10,0];
[t,x]=ode45(@DE2,[0:0.001:5],xo);
plot(t,x(:,1),'k-','Linewidth',1.5)
%linear solution
%case 1
B=pi/3;
T=0:0.05:5;
z=B*cos(sqrt(K)*T);
plot(T,z,'r*','Linewidth',1)
%case 2
B=pi/10;
T=0:0.05:5;
z=B*cos(sqrt(K)*T);
plot(T,z,'b-*','Linewidth',1)
%define legends
legend('\theta(0) = \pi/3','\theta(0) = \pi/10'...
,'Linear \theta(0) = \pi/3','Linear \theta(0) = \pi/10')
%define ode function
function ode=DE2(t,x)
g=9.81;L=1;
K=g/L;
ode=[x(2);-K*sin(x(1))];</pre>
	<pre>%save this as pendulumcats.m
	function dy = pendulumcats(t,y)
	dy = zeros(2,1);
	omega = 9.81*(4-5);
	dy(1) = y(2);
	dy(2) = omega*sin(y(1));
	%make a new m-file for the next code
	[t,y] = ode45('pendulumcats',[0:0.001:5], [pi/3 0]);
	plot(t,y(:,2),'k-','Linewidth',1.5);
	hold on
	[t,y] = ode45('pendulumcats',[0:0.001:5], [pi/10 0]);
	plot(t,y(:,2),'k--','Linewidth',1.5);
	hold on
	%linear solution
	%case 1
	B=pi/3;
	T=0:0.05:5;
	z=-B*sqrt(9.81)*sin(sqrt(9.81)*T);
	plot(T,z,'r*','Linewidth',1)
	hold on
	%case 2
	B=pi/10;
	T=0:0.05:5;
	z=-B*sqrt(9.81)*sin(sqrt(9.81)*T);
	plot(T,z,'b-*','Linewidth',1)
	xlabel('time');
	ylabel('omega');
	legend('\theta(0) = \pi/3','\theta(0) = \pi/10'...
	,'Linear \theta(0) = \pi/3','Linear \theta(0) = \pi/10')</pre>