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Research Article

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On the Coefficient and Fekete-Szegö Problem of the Pseudo-Starlike and Pseudo-Convex Bi-Univalent Function Class of Complex Order

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Abstract: In this paper, we defined a new subclass of starlike and convex bi-univalent functions and examine some geometric properties this function class. For this definition class, we gave some coefficient upper bound estimates and solve Fekete-Sezöge problem.

Keywords Starlike function, convex function, bi-univalent function, pseudo-starlike function, pseudo-convex function, complex order

1. Introduction

We will denote by $H(U)$ the class of analytic functions in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ of the complex plane $\mathbb{C}$. Let A be the class of the functions $f\in H(U)$ given by series expansions

$$f(z)=z+a_2z^2+a_3z^3+a_4z^4+\cdots+a_nz^n+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}\ . \quad (1.1)$$

The subclass of A, which are univalent functions in U is denoted by S in the literature. The class S was introduced by Köebe [1] first time and has become the core ingredient of advanced research in this field. After a short time, in 1916 Bieberbach [2] published a paper in which the coefficient hypothesis was proposed. This hypothesis states that if $f\in S$ and has the series form (1.1), then $|a_n|\le n$ for each $n\ge 2$. There are many articles in the literature regarding to this hypothesis (see [3-13, 15]).

Throughout the is paper, we always make use of the classical definition of quantum concepts as follows.

The q-numbers and q-factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q},\ q \neq 1,\ a \in \mathbb{C},$$

$$[0]_q! = 1,\ [n]_q! = [n-1]_q![n]_q.$$

In the standard approach to the q-calculus q-exponential function (see [14]) is defined as follow:

$$e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!},\ |q| \in (0,1),\ |z| < \frac{1}{|1-q|}.$$

It is known that the function f is called bi-univalent function, if itself and inverse is univalent in U and $f(U)$, respectively. The class of bi-univalent functions in U is denoted by Σ in the literature.

For the inverse $g(w) = f^{-1}(w)$ of the function $f \in \Sigma$, can written

$$g(w) = w + A_2 w^2 + A_3 w^3 + A_4 w^4 + \ldots = w + \sum_{n=2}^{\infty} A_n z^n,\ w \in f(U) = U_{r_0}, \qquad (1.2)$$

where

$$A_2 = -a_2,\ A_3 = 2a_2^2 - a_3,\ A_4 = -a_2^3 + 5a_2 a_3 - a_4, \ldots .$$

The bi-starlike and bi-convex function classes in the open unit disk U are defined analytically as follows and denoted by S_Σ^* and C_Σ, respectively

$$S_\Sigma^* = \left\{ f \in S:\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U \text{ and } \mathrm{Re}\left(\frac{wg'(w)}{g(w)}\right) > 0,\ w \in U_{r_0} \right\}$$

$$C_\Sigma \left\{ f \in S:\ \mathrm{Re}\left(\frac{(zf'(z))'}{f'(z)}\right) > 0,\ z \in U \text{ and } \mathrm{Re}\left(\frac{(wg'(w))'}{g'(w)}\right) > 0,\ w \in U_{r_0} \right\}.$$

Let's $f, g \in H(U)$, then it is said that f is subordinate to g and denoted by $f \prec g$, if there exists a Schwartz function ω, such that $f(z) = g(\omega(z))$.

In the past few years, numerous subclasses of the class S have been introduced as special choices of the class S_Σ^* and C_Σ (see for example [3, 8-13, 15-22]).

2. Materials and Methods

Now, let's we define new subclass of bi-univalent functions in the open unit disk U.

Definition 2.1. For $\tau \in \mathbb{C}-\{0\}, \beta \in [0,1],\ q \in (0,1)$ and $\lambda > \frac{1}{2}$ the function $f \in \Sigma$ is said to be in the class $\chi_\Sigma\left(\tau, \beta, \lambda; e_q^z\right)$, if the following conditions are satisfied

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}-1\right]\right\} \prec e_q^z,\ z \in U \text{ and}$$

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}-1\right]\right\} \prec e_q^w,\ w \in U_{r_0}.$$

In the cases $\tau=1$, $\beta=0$, $\beta=1$ and $\lambda=1$ from the Definition 2.1, we have the following classes of bi-univalent functions.

Definition 2.2. For $\beta \in [0,1],\ q \in (0,1)$ and $\lambda > \frac{1}{2}$ the function $f \in \Sigma$ is said to be in the class $\chi_\Sigma\left(\beta, \lambda; e_q^z\right)$, if the following conditions are satisfied

$$(1-\beta)\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}+\beta\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)} \prec e_q^z,\ z \in U,$$

$$(1-\beta)\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}+\beta\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)} \prec e_q^w,\ w \in U_{r_0}.$$

Definition 2.3. For $\tau \in \mathbb{C}-\{0\}, q \in (0,1)$ and $\lambda > \frac{1}{2}$ the function $f \in \Sigma$ is said to be in the class $S_\Sigma^*\left(\tau, \lambda; e_q^z\right)$, if the following conditions are satisfied

$$1+\frac{1}{\tau}\left[\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right] \prec e_q^z,\ z \in U \text{ and } 1+\frac{1}{\tau}\left[\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}-1\right] \prec e_q^w,\ w \in U_{r_0}.$$

Definition 2.4. For $\tau \in \mathbb{C}-\{0\}, q \in (0,1)$ and $\lambda > \frac{1}{2}$ the function $f \in \Sigma$ is said to be in the class $C_\Sigma\left(\tau, \lambda; e_q^z\right)$, if the following conditions are satisfied

$$1+\frac{1}{\tau}\left[\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}-1\right]\prec e_q^z,\ z\in U \text{ and } 1+\frac{1}{\tau}\left[\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}-1\right]\prec e_q^w,\ w\in U_{r_0}.$$

Definition 2.5. For $\tau\in\mathbb{C}-\{0\}$, $\beta\in[0,1]$ and $q\in(0,1)$ the function $f\in\Sigma$ is said to be in the class $\chi_\Sigma\left(\tau,\beta;e_q^z\right)$, if the following conditions are satisfied

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)}-1\right]\right\}\prec e_q^z,\ z\in U \text{ and}$$

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{wg'(w)}{g(w)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left(wg'(w)\right)'}{g'(w)}-1\right]\right\}\prec e_q^w,\ w\in U_{r_0}.$$

Let P be the class of analytic functions in U satisfied the conditions $p(0)=1$ and $\mathrm{Re}\left(p(z)\right)>0,\ z\in U.$ It is clear that the functions that satisfy these conditions have the following series expansion

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U. \tag{2.1}$$

The class P defined above is known as the class Caratheodory functions in the literature [23].

Now, let us give some necessary lemmas for the proof of our main results.

Lemma 2.1 ([24]). Let the function p belong to the class P. Then,

$$\left|p_n\right|\le 2 \text{ for each } n\in\mathbb{N},\ \left|p_n-\nu p_kp_{n-k}\right|\le 2 \text{ for } n,\ k\in\mathbb{N},\ n>k \text{ and } \nu\in[0,1].$$

The equalities hold for the function

$$p(z)=\frac{1+z}{1-z}.$$

Lemma 2.2 ([24]) Let the an analytic function p be of the form (2.1), then

$$2p_2=p_1^2+\left(4-p_1^2\right)x,$$

$$4p_3=p_1^3+2\left(4-p_1^2\right)p_1x-\left(4-p_1^2\right)p_1x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)y$$

for some $x,y\in\mathbb{C}$ with $|x|\le 1$ and $|y|\le 1$.

In this paper, we give some coefficient estimates and solve Fekete-Szegö problem for the class $\chi_\Sigma\left(\tau,\beta,\lambda;e_q^z\right)$. Additionally, the results obtained for specific values of the parameters in our study are compared with the results obtained in the literature.

3. Results & Discussion

In this section, we give some coefficient estimates for the functions belonging to the class $\chi_\Sigma\left(\tau,\beta,\lambda;e_q^z\right)$ and solve Fekete-Szegö problem for this class.

Theorem 3.1. Let the function f given by series expansions (1.1) belong to the class $\chi_\Sigma\left(\tau,\beta,\lambda;e_q^z\right)$. Then, we have the following inequalities

$$|a_2|\le\frac{|\tau|}{(2\lambda-1)(1+\beta)} \text{ and}$$

$$|a_3|\le|\tau|\begin{cases}\dfrac{1}{(3\lambda-1)(1+2\beta)} & if \ \ |\tau|(3\lambda-1)(1+2\beta)\le(2\lambda-1)^2(1+\beta)^2,\\[2ex] \dfrac{|\tau|}{(2\lambda-1)^2(1+\beta)^2} & if \ \ |\tau|(3\lambda-1)(1+2\beta)\ge(2\lambda-1)^2(1+\beta)^2.\end{cases} \tag{3.1}$$

Proof. Let $f\in\chi_\Sigma\left(\beta,\lambda;e^z\right)$, then exists Schwartz functions $\omega:U\to U,\varpi:U_{r_0}\to U_{r_0}$, such that

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{z\left(f'(z)\right)^\lambda}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(zf'(z)\right)'\right]^\lambda}{f'(z)}-1\right]\right\}\prec e_q^{\omega(z)},\ z\in U\ , z\in U \text{ and}$$

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{w\left(g'(w)\right)^\lambda}{g(w)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(wg'(w)\right)'\right]^\lambda}{g'(w)}-1\right]\right\}\prec e_q^{\varpi(w)},\ w\in U_{r_0}. \tag{3.2}$$

Let's the functions $p,q\in P$ defined as follows:

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_2z^2+p_3z^3+\ldots=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U\ ,$$

$$q(w)=\frac{1+\varpi(w)}{1-\varpi(w)}=1+q_1w+q_2w^2+q_3w^3+\ldots=1+\sum_{n=1}^{\infty}q_nw^n,\ w\in U_{r_0}. \tag{3.3}$$

It follows from that

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{p_1}{2}z+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)z^2+\frac{1}{2}\left(p_3-p_1p_2-\frac{p_1^3}{4}\right)z^3\ldots,\ z\in U\ ,$$

$$\varpi(w)=\frac{q(w)-1}{q(w)+1}=\frac{q_1}{2}w+\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)w^2+\frac{1}{2}\left(q_3-q_1q_2-\frac{q_1^3}{4}\right)w^3\ldots,w\in U_{r_0}\ . \quad (3.4)$$

From the (3.2) and (3.4) can written

$$(1+\beta)(2\lambda-1)a_2z+\left[(3\lambda-1)(1+2\beta)a_3+(1+3\beta)\left(2\lambda^2-4\lambda+1\right)a_2^2\right]z^2+\cdots$$
$$=\tau\left\{\frac{p_1}{2}z+\frac{1}{4}\left[2p_2-\left(1-\frac{1}{[2]_q!}\right)p_1^2\right]z^2+\cdots\right\},\ z\in U,$$
$$(1+\beta)(2\lambda-1)A_2w+\left[(3\lambda-1)(1+2\beta)A_3+(1+3\beta)\left(2\lambda^2-4\lambda+1\right)A_2^2\right]w^2+\cdots$$
$$=\tau\left\{\frac{q_1}{2}w+\frac{1}{4}\left[2q_2-\left(1-\frac{1}{[2]_q!}\right)q_1^2\right]w^2+\cdots\right\},\ w\in U_{r_0}. \quad (3.5)$$

If we consider that $A_2=-a_2$ and $A_3=2a_2^2-a_3$, we write the equations (3.5) as follows:

$$(1+\beta)(2\lambda-1)a_2z+\left[(3\lambda-1)(1+2\beta)a_3+(1+3\beta)\left(2\lambda^2-4\lambda+1\right)a_2^2\right]z^2+\cdots$$
$$=\tau\left\{\frac{p_1}{2}z+\frac{1}{4}\left[2p_2-\left(1-\frac{1}{[2]_q!}\right)p_1^2\right]z^2+\cdots\right\},\ z\in U,$$
$$-(1+\beta)(2\lambda-1)a_2w+\left[(3\lambda-1)(1+2\beta)\left(2a_2^2-a_3\right)+(1+3\beta)\left(2\lambda^2-4\lambda+1\right)a_2^2\right]w^2+\cdots$$
$$=\tau\left\{\frac{q_1}{2}w+\frac{1}{4}\left[2q_2-\left(1-\frac{1}{[2]_q!}\right)q_1^2\right]w^2+\cdots\right\},\ w\in U_{r_0}. \quad (3.6)$$

By equate the same coefficients of the parameters z and w, we obtain the following equalities

$$a_2=\frac{\tau p_1}{2(2\lambda-1)(1+\beta)},$$

$$(3\lambda-1)(1+2\beta)a_3+(1+3\beta)\left(2\lambda^2-4\lambda+1\right)a_2^2=\frac{\tau}{4}\left[2p_2-\left(1-\frac{1}{[2]_q!}\right)p_1^2\right], \quad (3.7)$$

$$a_2=-\frac{\tau q_1}{2(2\lambda-1)(1+\beta)},$$

$$(3\lambda-1)(1+2\beta)\left(2a_2^2-a_3\right)+(1+3\beta)\left(2\lambda^2-4\lambda+1\right)a_2^2=\frac{\tau}{4}\left[2q_2-\left(1-\frac{1}{[2]_q!}\right)q_1^2\right]. \quad (3.8)$$

Then,

$$\frac{\tau p_1}{2(2\lambda-1)(1+\beta)}=a_2=-\frac{\tau q_1}{2(2\lambda-1)(1+\beta)}\text{; that is, } p_1=-q_1. \quad (3.9)$$

Using Lemma 2.1 to the equality (3.9), we have first result of theorem.

If we subtract (3.8) from the equality (3.7), we get the following equality for a_3:

$$a_3=\frac{\tau^2 p_1^2}{4(2\lambda-1)^2(1+\beta)^2}+\frac{\tau(p_2-q_2)}{4(3\lambda-1)(1+2\beta)}. \quad (3.10)$$

Since $p_2-q_2=\frac{4-p_1^2}{2}(x-y)$ (see Lemma 2.2), we can write

$$a_3=\frac{\tau^2 p_1^2}{4(2\lambda-1)^2(1+\beta)^2}+\frac{\tau\left(4-p_1^2\right)}{8(3\lambda-1)(1+2\beta)}(x-y)$$

for some $x,y\in\mathbb{C}$ with $|x|\le 1$ and $|y|\le 1$.

Then, applying triangle inequality to the last equality, we have

$$|a_3|\le\frac{|\tau|^2 t^2}{4(2\lambda-1)^2(1+\beta)^2}+\frac{|\tau|\left(4-t^2\right)}{8(3\lambda-1)(1+2\beta)}(\zeta+\varsigma),\ \zeta,\varsigma\in[0,1], \quad (3.11)$$

where $\zeta=|x|$, $\varsigma=|y|$ and $t=|p_1|$. From the inequality (3.11), we can write

$$|a_3|\le|\tau|\left[\frac{a(\lambda,\beta)}{4}t^2+\frac{1}{(3\lambda-1)(1+2\beta)}\right],\ t\in[0,2],$$

where

$$a(\lambda,\beta)=\frac{|\tau|}{(2\lambda-1)^2(1+\beta)^2}-\frac{1}{(3\lambda-1)(1+2\beta)}.$$

Then, maximizing the function

$$\varphi(t)=\frac{a(\lambda,\beta)}{4}t^2+\frac{1}{(3\lambda-1)(1+2\beta)},\ t\in[0,2]$$

it can easily be seen that $\varphi(t)\le\frac{1}{(3\lambda-1)(1+2\beta)}$ if $a(\lambda,\beta)\le 0$ and $\varphi(t)\le\frac{|\tau|}{(2\lambda-1)^2(1+\beta)^2}$ if $a(\lambda,\beta)\ge 0.$

With this, the proof of second inequality of (3.1) is provided.

Thus, the proof of theorem is completed.

In the case $\tau=1,\ \beta=0,\ \beta=1$ and $\lambda=1$ from the Theorem 3.1, we obtain the following results, respectively.

Corollary 3.1. If $f\in\chi_\Sigma\left(\beta,\lambda;e_q^z\right)$, then

$$|a_2|\le\frac{1}{(2\lambda-1)(1+\beta)}\ \text{and}\ |a_3|\le\begin{cases}\frac{1}{(3\lambda-1)(1+2\beta)} & if\ \frac{3\lambda-1}{(2\lambda-1)^2}\le\frac{(1+\beta)^2}{(1+2\beta)},\\ \frac{1}{(2\lambda-1)^2(1+\beta)^2} & if\ \frac{3\lambda-1}{(2\lambda-1)^2}\ge\frac{(1+\beta)^2}{(1+2\beta)}.\end{cases}$$

Corollary 3.2. If $f\in S_\Sigma^*\left(\tau,\lambda;e_q^z\right)$, then

$$|a_2|\le\frac{|\tau|}{2\lambda-1}\ \text{and}\ |a_3|\le|\tau|\begin{cases}\frac{1}{3\lambda-1} & if\ \ |\tau|\le\frac{(2\lambda-1)^2}{(3\lambda-1)},\\ \frac{|\tau|}{(2\lambda-1)^2} & if\ \ |\tau|\ge\frac{(2\lambda-1)^2}{(3\lambda-1)}.\end{cases}$$

Corollary 3.3. If $f\in C_\Sigma\left(\tau,\lambda;e_q^z\right)$, then

$$|a_2|\le\frac{|\tau|}{2(2\lambda-1)}\ \text{and}\ |a_3|\le|\tau|\begin{cases}\frac{1}{3(3\lambda-1)} & if\ \ |\tau|\le\frac{4(2\lambda-1)^2}{3(3\lambda-1)},\\ \frac{|\tau|}{4(2\lambda-1)^2} & if\ \ |\tau|\ge\frac{4(2\lambda-1)^2}{3(3\lambda-1)}.\end{cases}.$$

Corollary 3.4. If $f\in\chi_\Sigma\left(\tau,\beta;e_q^z\right)$, then

$$|a_2| \le \frac{|\tau|}{1+\beta} \text{ and } |a_3| \le |\tau| \begin{cases} \dfrac{1}{2(1+2\beta)} & if \ \ |\tau| \le \dfrac{(1+\beta)^2}{2(1+2\beta)}, \\ \dfrac{|\tau|}{(1+\beta)^2} & if \ \ |\tau| \ge \dfrac{(1+\beta)^2}{2(1+2\beta)}. \end{cases}$$

Now, we give the following theorem on the Fekete-Szegö problem for the class $\chi_\Sigma\left(\beta, \lambda; e^z\right)$.

Theorem 3.2. Let $f \in \chi_\Sigma\left(\tau, \beta, \lambda; e_q^z\right)$ and $\mu \in \mathbb{C}$, then

$$\left|a_3 - \mu a_2^2\right| \le |\tau| \begin{cases} \dfrac{1}{(3\lambda-1)(1+2\beta)} & if \ \ |1-\mu||\tau| \le \dfrac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}, \\ \dfrac{|1-\mu||\tau|}{(2\lambda-1)^2(1+\beta)^2} & if \ \ |1-\mu||\tau| \ge \dfrac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}. \end{cases} \tag{3.12}$$

Proof. Let $f \in \chi_\Sigma\left(\tau, \beta, \lambda; e_q^z\right)$, then from the equalities (3.9) and (3.10), we can write

$$a_3 - \mu a_2^2 = \frac{(1-\mu)\tau^2 p_1^2}{4(2\lambda-1)^2(1+\beta)^2} + \frac{\tau(p_2 - q_2)}{4(3\lambda-1)(1+2\beta)}. \tag{3.13}$$

Then, applying Lemma 2.2 the expression $a_3 - \mu a_2^2$ can written as follows

$$a_3 - \mu a_2^2 = \frac{(1-\mu)\tau^2 p_1^2}{4(2\lambda-1)^2(1+\beta)^2} + \frac{\tau\left(4-p_1^2\right)}{8(3\lambda-1)(1+2\beta)}(x-y) \tag{3.14}$$

for some $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$.

Applying triangle inequality to this equality, we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{|1-\mu||\tau|^2 t^2}{4(2\lambda-1)^2(1+\beta)^2} + \frac{|\tau|\left(4-t^2\right)}{8(3\lambda-1)(1+2\beta)}(\xi+\eta), \ \xi, \eta \in [0,1], \tag{3.15}$$

where $\xi = |x|$, $\eta = |y|$ and $t = |p_1|$.

From the inequality (3.15), we can write

$$\left|a_3 - \mu a_2^2\right| \le |\tau| \left[\frac{b(\lambda, \beta; \mu)}{4} t^2 + c(\lambda, \beta)\right], \ t \in [0,2],$$

where

$$b(\lambda,\beta;\mu)=\frac{|1-\mu||\tau|}{(2\lambda-1)^2(1+\beta)^2}-\frac{1}{(3\lambda-1)(1+2\beta)} \text{ and } c(\lambda,\beta)=\frac{1}{(3\lambda-1)(1+2\beta)} .$$

Maximizing the function $\psi:[0,2]\to\mathbb{R}$ defined as follows

$$\psi(t)=\frac{b(\lambda,\beta;\mu)}{4}t^2+c(\lambda,\beta),\ t\in[0,2],$$

we can easily see that $\psi(t)\le c(\lambda,\beta)$ if $b(\lambda,\beta;\mu)\le 0$; that is if

$$(3\lambda-1)(1+2\beta)|1-\mu||\tau|\le(2\lambda-1)^2(1+\beta)^2$$

and

$$\psi(t)\le\frac{|1-\mu||\tau|}{(2\lambda-1)^2(1+\beta)^2}$$

if

$$(3\lambda-1)(1+2\beta)|1-\mu||\tau|\ge(2\lambda-1)^2(1+\beta)^2 .$$

Thus, the proof of theorem is completed.

In the case $\tau=1$, $\beta=0$, $\beta=1$ and $\lambda=1$ from the Theorem 3.2, we obtain the following results, respectively.

Corollary 3.5. If $f\in\chi_\Sigma(\beta,\lambda;e_q^z)$ and $\mu\in\mathbb{C}$, then

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{1}{(3\lambda-1)(1+2\beta)} & if\ |1-\mu|\le\dfrac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)},\\[2ex] \dfrac{|1-\mu|}{(2\lambda-1)^2(1+\beta)^2} & if\ |1-\mu|\ge\dfrac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}.\end{cases}$$

Corollary 3.6. If $f\in S_\Sigma^*(\tau,\lambda;e_q^z)$ and $\mu\in\mathbb{C}$, then

$$\left|a_3-\mu a_2^2\right| \leq |\tau| \begin{cases} \dfrac{1}{3\lambda-1} & if \ |1-\mu||\tau| \leq \dfrac{(2\lambda-1)^2}{3\lambda-1}, \\ \dfrac{|1-\mu||\tau|}{(2\lambda-1)^2} & if \ |1-\mu||\tau| \geq \dfrac{(2\lambda-1)^2}{3\lambda-1}. \end{cases}$$

Corollary 3.7. If $f \in C_{\Sigma}\left(\tau,\lambda;e_q^z\right)$ and $\mu \in \mathbb{C}$, then

$$\left|a_3-\mu a_2^2\right| \leq |\tau| \begin{cases} \dfrac{1}{3(3\lambda-1)} & if \ |1-\mu||\tau| \leq \dfrac{4(2\lambda-1)^2}{3(3\lambda-1)}, \\ \dfrac{|1-\mu||\tau|}{4(2\lambda-1)^2} & if \ |1-\mu||\tau| \geq \dfrac{4(2\lambda-1)^2}{3(3\lambda-1)}. \end{cases}$$

Corollary 3.8. $f \in \chi_{\Sigma}\left(\tau,\beta;e_q^z\right)$ and $\mu \in \mathbb{C}$, then

$$\left|a_3-\mu a_2^2\right| \leq |\tau| \begin{cases} \dfrac{1}{2(1+2\beta)} & if \ |1-\mu||\tau| \leq \dfrac{(1+\beta)^2}{2(1+2\beta)}, \\ \dfrac{|1-\mu||\tau|}{(1+\beta)^2} & if \ |1-\mu||\tau| \geq \dfrac{(1+\beta)^2}{2(1+2\beta)}. \end{cases}$$

Also, taking $\mu=0$ and $\mu=1$ in the Theorem 3.2, we obtain the following results, respectively.

Corollary 3.9. If $f \in \chi_{\Sigma}\left(\tau,\beta,\lambda;e_q^z\right)$, then

$$|a_3| \leq |\tau| \begin{cases} \dfrac{1}{(3\lambda-1)(1+2\beta)} & if \ |\tau| \leq \dfrac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}, \\ \dfrac{|1-\mu||\tau|}{(2\lambda-1)^2(1+\beta)^2} & if \ |\tau| \geq \dfrac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}. \end{cases}$$

Corollary 3.10. If $f \in \chi_{\Sigma}\left(\tau,\beta,\lambda;e_q^z\right)$, then

$$\left|a_3-a_2^2\right| \leq \frac{|\tau|}{(3\lambda-1)(1+2\beta)}.$$

Remark 3.1. We note that Corollary 3.9 confirms the second result of Theorem 3.1.

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