Available online www.jsaer.com

Journal of Scientific and Engineering Research, 2025, 12(3):64-83

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Optimal Control of An Atmospheric Pollution Model

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Abstract: The level of air pollution caused by pollutants has reached significant levels in some countries. Most people in the world breathe polluted air. This causes health damage associated with exposure to air pollution. It is also the cause of climate change, which has an impact on ecosystems and environments.

Human activities play an important role. They are the source of emissions of gases and particles into the atmosphere. In order to combat this phenomenon, it is necessary to reduce or control emissions of pollutants into the atmosphere. This would contribute to an improvement in air quality and a less uncertain future for mankind. Air pollution can be represented by a mathematical model using a system of partial differential equations. In this article, we study an optimal pollution control problem governed by a system of partial differential equations describing the evolution of the concentration of a given pollutant in the atmosphere.

Keywords: Partial differential equations - Variational methods applied to PDEs - Atmosphere pollution - Numerical analysis - Optimal control - Algorithms with automatic result verification

MSC: 35A15 - 65G20 - 65M06 - 65N30 - 65N30

1. Introduction

The optimal control problem, the objective of our work, is a distributed control problem. Control theory allows us to study the possibility of acting on a time-dependent dynamic system in such a way that we can bring the state of this system to a given state (or target state) at a given time. Note here that we are trying to act at source in order to control pollutant emissions. The structure of our article is as follows:

In the first part, we present the model problem.

Then, in the second part, we proceed to the theoretical study: we prove the existence and the uniqueness of the problem on the functional spaces which are the natural functional frameworks of the system of partial differential equations.

In the third part, we determine the adjoint problem. Determining the latter leads to the resolution of two optimisation problems instead of one, which is more practical especially as the optimality condition obtained from the first problem cannot be exploited directly.

The fourth part is devoted to the numerical solution. Since we have to solve an optimisation problem, we use the gradient descent algorithm. However, calculating the direction of descent requires the primal and adjoint systems to be solved numerically. These two systems will be solved using the Lagrange P1 finite element method.

2. Presentation of the problem

Our optimal control problem is a distributed control problem given by:

$$\min_{v\in U} J(v) \tag{1}$$

with:

$$J_u(v) = \frac{1}{2}\int_0^T \int_C [Du(v) - z_{cib}]^2 drdt + \frac{\beta}{2}\int_0^T \int_C v^2 drdt \tag{2}$$

under constraints:

$$\begin{cases} \frac{\partial u}{\partial t} + div(\alpha u) + \sigma u - \frac{\partial}{\partial x_3}(\eta \frac{\partial u}{\partial x_3}) - \mu.\Delta_2 u = f + v, & on \quad [0;T] \times C, \\ u = u_S, & on \quad S, \\ \frac{\partial u}{\partial x_3} = d.u, & if \quad x_3 = 0, \\ \frac{\partial u}{\partial x_3} = 0, & if \quad x_3 = H \\ u = u_0, & if \quad t = 0 \end{cases} \tag{3}$$

where U, space of admissible controls is a sub - closed convex space of $L^2([0,T] \times C)$, D is a continuous linear observation operator.Z is the Hilbert space which is the space of observations with :$D \in L(L^2(0,T;V);Z)$. $z_{cib} \in Z$ is a given observation function. β is a given positive real v is the control used to act on the system.

3. Existence and uniqueness of control

Let's ask $f_1 = f + v. v \in L^2(0,T;C)$.

The functional framework is therefore the triplet (V, H, V*) such that:

$$V = H^2(C) \cap H^1_{0,\Gamma_S}(C) \text{ et } H = L^2(C)$$

Theorem 3.1 *Suppose that the function $v \mapsto J(v)$ is strictly convex, differentiable and coercive, then the problem* (1) *has a unique solution $u \in U$ and is characterised by :*

$$(J'(u), v - u) \geq 0 \quad \forall u \in U$$

3.1 Convexity results

(a) Let us call X the set of pairs (u,v) such that $v \in U$ and u a solution of the PDE satisfying the edge and initial conditions. These two pairs are therefore solutions of the optimal problem

Let $x = (u_1, v_1) \in X$, $y = (u_2, v_2) \in X$ et $t \in]0,1[$,

let $z = tx + (1-t)y = (tu_1 + (1-t)u_2, tv_1 + (1-t)v_2)$

The linearity of the gradient and the second-order differential operators means that if x and y are solutions of the PDE, then so is z.

Hence $z \in X$, which leads to the conclusion that X is a closed convex.

(b) Let's show that the functional J is convex:

$$J(z) = \frac{1}{2}\int_0^T \int_C |D(tu_1 + (1-t)u_2) - z_{cib}|^2 drdt + \frac{\beta}{2}\int_0^T \int_C (tv_1 + (1-t)v_2)^2 drdt$$

let say: $J(z) = \frac{1}{2}I_1 + \frac{\beta}{2}I_2$,

with $I_1 = \int_0^T \int_C |D(tu_1 + (1-t)u_2) - z_{cib}|^2 drdt$ et $I_2 = \int_0^T \int_C (tv_1 + (1-t)v_2)^2 drdt$

- $$I_2 = \int_0^T \int_C (tv_1 + (1-t)v_2)^2 drdt$$

we have: $\| tv_1 + (1-t)v_2 \|^2 = (tv_1 + (1-t)v_2, tv_1 + (1-t)v_2)$

$$= t^2 \| v_1 \|^2 + (1-t)^2 \| v_2 \|^2 + 2t(1-t)v_1 v_2$$

$$= [t + t(t-1)] \| v_1 \|^2 + [1 - t + t(t-1)] \| v_2 \|^2 + 2t(1-t)v_1 v_2$$

$$= t \| v_1 \|^2 + (1-t) \| v_2 \|^2 + t(t-1) \| v_1 - v_2 \|^2$$

hence

$$I_2 = \int_0^T \int_C [tv_1^2 + (1-t)v_2^2 + t(t-1)(v_1 - v_2)^2] drdt$$

$-t(1-t)<0$, then we deduce:

$$I_2 \leq t\int_0^T \int_C v_1^2 drdt + (1-t)\int_0^T \int_C v_2^2 drdt \qquad (4)$$

- $$I_1 = \int_0^T \int_C |D(tu_1 + (1-t)u_2) - z_{cib}|^2 drdt$$

we have: $D(tu_1 + (1-t)u_2) - z_{cib} = t(Du_1 - z_{cib}) + (1-t)(Du_2 - z_{cib})$
Using the same reasoning as above, we have:
$[D(tu_1 + (1-t)u_2) - z_{cib}]^2 = t(Du_1 - z_{cib})^2 + (1-t)(Du_2 - z_{cib})^2 + t(t-1)[D(u_1-u_2), D(u_1-u_2)]$
or: $t(t-1)<0$ hence:

$$I_1 \leq t\int_0^T \int_C |Du_1 - z_{cib}|^2 drdt + (1-t)\int_0^T \int_C |Du_2 - z_{cib}|^2 drdt \qquad (5)$$

from (3) and (4), we deduce:

$$\frac{1}{2}I_1 + \frac{\beta}{2}I_2 \leq t[\frac{1}{2}\int_0^T \int_C |Du_1 - z_{cib}|^2 drdt + \frac{\beta}{2}\int_0^T \int_C v_1^2 drdt]$$
$$+(1-t)[\frac{1}{2}\int_0^T \int_C |Du_2 - z_{cib}|^2 drdt + \frac{\beta}{2}\int_0^T \int_C v_2^2 drdt]$$

hence:

$$J(tx + (1-t)y) \leq tJ(x) + (1-t)J(y)$$

We conclude that the functional J is convex.

(c) Let us prove that J is inferiorly semicontinuous:
Consider a sequence (u_n, v_n) which converges strongly to (u,v) in
$L^2(0,T;V) \times L^2(0,T;H)$. There then exists an extracted sub-sequence $(u_n(t), v_n(t)$ converging strongly to $(u(t), v(t))$ in $V \times H$.

$$K_n(u_n(t), v_n(t)) = \frac{1}{2}\int_C |Du_n(v) - z_{cib}|^2 dr + \frac{\beta}{2}\int_C v_n^2 dr$$
$$K_n(u_n(t), v_n(t)) \xrightarrow[n\to+\infty]{} K(u(t), v(t))$$

we have: $\lim_{n\to+\infty}\inf K_n(u_n(t), v_n(t)) = \lim_{n\to+\infty}\inf[\frac{1}{2}\int_C |Du_n(v) - z_{cib}|^2 dr + \frac{\beta}{2}\int_C v_n^2 dr]$

$$= \lim_{n\to+\infty}\inf\frac{1}{2}\int_C |Du_n(v) - z_{cib}|^2 drdt + \lim_{n\to+\infty}\inf\frac{\beta}{2}\int_C v_n^2 drdt$$
$$\geq \frac{1}{2}\int_C |Du(t) - z_{cib}|^2 dr + \frac{\beta}{2}\int_C v(t))^2 dr$$

hence

$$K(t) \leq \lim_{n\to+\infty}\inf K_n(u_n(t), v_n(t)) \qquad (6)$$
$$\Rightarrow J(u,v) = \int_0^T K(t)dt \leq \int_0^T \lim_{n\to+\infty}\inf K_n(u_n(t), v_n(t))dt$$

According to Fatou's lemma we have:

$$J(u,v) \leq \lim_{n\to+\infty}\inf[\frac{1}{2}\int_0^T \int_C |Du_n(t) - z_{cib}|^2 drdt + \frac{\beta}{2}\int_0^T \int_C v_n(t))^2 drdt]$$

We deduce:

$$J(u,v) \leq \lim_{n\to+\infty}\inf J(u_n, v_n) \qquad (7)$$

3.2 Différentiabilité du critère:

the application: $v \to u(v)$ is affine and therefore $l(v) = u(v) + u(0)$ is linear. Let's ask $J(u,v) = J_u(v)$. Let's calculate:

$$\lim_{h\to 0}\frac{J_u(v+h\theta) - J_u(v)}{h}$$

We have:

$$J_u(v+h\theta)=\frac{1}{2}\int_0^T\int_C[Du(v+h\theta)-z_{cib}]^2drdt+\frac{\beta}{2}\int_0^T\int_C(v+h\theta)^2drdt$$
$$Du(v+h\theta)-z_{cib}=u(v)+hl(\theta)$$

hence:

$$\|Du(v+h\theta)-z_{cib}\|^2=(Du(v)+Dhl(\theta)-z_{cib},Du(v)+Dhl(\theta)-z_{cib})$$
$$=\|Du(v)-z_{cib}\|^2+2h(D(u(v)-z_{cib},Dl(\theta))+h^2\|Dl(\theta\|^2$$

hence:

$$J(v+h\theta)=\frac{1}{2}\|Du(v)-z_{cib}\|^2+\frac{h^2}{2}\|Dl(\theta\|^2+h(D(u(v)-z_{cib},Dl(\theta))+\frac{\beta}{2}[\|v\|^2+2h(v,\theta)+h^2\|\theta\|^2]$$

$$\Leftrightarrow J(v+h\theta)-J(v)=\frac{h^2}{2}\|Dl(\theta\|^2+h(D(u(v)-z_{cib},Dl(\theta))+\beta(v,\theta)+\frac{\beta h^2}{2}\|\theta\|^2$$

Passing to the limit, we have:

$$\lim_{h\to 0}\frac{J(v+h\theta)-J(v)}{h}=(D(u(v)-z_{cib},Dl(\theta))+\beta(v,\theta)$$

with $l(\theta)=u(\theta)-u(0))$ We conclude that:

$$(J_{u\prime}(v),\theta)=\int_0^T\int_C[(Du(v)-z_{cib})]D[u(\theta)-u(0)]drdt+\beta\int_0^T\int_C v\theta drdt \quad (8)$$

and therefore, J is Gateau - differentiable.

let's ask: $\theta=v-u\Rightarrow l(\theta)=l(v-u)=u(v)-u(u)$. We can then deduce:

$$(J_{u\prime}(v),v-u)=\int_0^T\int_C[(Du(v)-z_{cib})]D[u(v)-u(u)]drdt+\beta\int_0^T\int_C v(v-u)drdt \quad (9)$$

and so J is Gateau - differentiable.

3.3 Coercivity

$$J_u(v)=\frac{1}{2}\int_0^T\int_C[Du(v)-z_{cib}]^2drdt+\frac{\beta}{2}\int_0^T\int_C v^2drdt$$
$$\Leftrightarrow J_u(v)\geq\frac{\beta}{2}\int_0^T\int_C v^2drdt\Leftrightarrow J_u(v)\geq\frac{\beta}{2}\|v\|^2_{L^2([0,T]\times C)}$$

Hence J is coercive.

4. Adjoint problem and optimality condition

Theorem 4.1 *We assume that $H=L^2(0,T,V)$ and $H'=L^2(0,T,V')$ with $V=H^2(C)\cap H^1(C)$, then the solution v of our optimal control problem is characterised by the following optimality system:*

$$\begin{cases}\frac{\partial u}{\partial t}-div(\alpha u)+\sigma u-\frac{\partial}{\partial x_3}(\eta\frac{\partial u}{\partial x_3})-\mu.\Delta_2 u=f+redv, & on\ ]0;T[\times C,\\ u=u_S, & on\ \Gamma_S,\\ \frac{\partial u}{\partial x_3}=du, & on\ ]0;T[\times\Gamma_0,\\ \frac{\partial u}{\partial x_3}=0, & on\ ]0;T[\times\Gamma_H,\\ u(0,.)=u_0\end{cases} \quad (10)$$

$$\begin{cases}-\frac{\partial p}{\partial t}-div(\alpha p)+\sigma p-\frac{\partial}{\partial x_3}(\eta\frac{\partial p}{\partial x_3})-\mu.\Delta_2 p=D^*\Lambda(Du-z_{cib}), & on\ ]0;T[\times C,\\ p=0, & on\ ]0;T[\times\partial C,\\ p(T,.)=0,\\ J(v)=\frac{1}{2}\int_0^T\int_C|Du(v)-z_{cib}|^2drdt+\frac{\beta}{2}\int_0^T\int_C|v|^2drdt,\\ \nabla J(v)=p+\beta v\end{cases} \quad (11)$$

Theorem 4.2 *The optimal control $\overline{u}$ is given by:*

$$\overline{u}=-p\beta^{-1}$$

If D is an injective operator of $L^2(0,T,V)\to L^2(0,T,H)$, Λ is the identity operator. The adjoint problem is then defined by:

$$\begin{cases} -\frac{\partial p}{\partial t} - div(\alpha p) + \sigma p - \frac{\partial}{\partial x_3}(\eta \frac{\partial p}{\partial x_3}) - \mu.\Delta_2 p = u - z_{cib}, & on \quad]0;T[\times C, \\ p = 0, & on \quad]0;T[\times \partial C, \\ p(T,.) = 0 \end{cases} \tag{12}$$

If D is an injective operator of $L^2(0,T,V) \rightarrow L^2(0,T,V)$, we have:

$$\Lambda = (-\Delta + I)$$

The adjoint problem is then defined by:

$$\begin{cases} -\frac{\partial p}{\partial t} - div(\alpha p) + \sigma p - \frac{\partial}{\partial x_3}(\eta \frac{\partial p}{\partial x_3}) - \mu.\Delta_2 p = (-\Delta + I)(u - z_{cib}), & on \quad]0;T[\times C, \\ p = 0, & on \quad]0;T[\times \partial C, \\ p(T,.) = 0 \end{cases} \tag{13}$$

Proof *Reformulation of the optimality condition:*

If $\overline{u}$ is the solution to the control problem, then according to the theorem :

$$\forall v \in U_{ad}, (\nabla J(\overline{u}), v - \overline{u}) \geq 0$$

From (8), we can deduce:

$$\Leftrightarrow (J_{u'}(\overline{u}), v - \overline{u}) = \int_0^T \int_C [(Du(\overline{u}) - z_{cib})] D[u(v) - u(\overline{u})] drdt + \beta \int_0^T \int_C \overline{u}(v - \overline{u}) drdt$$

And using the theorem, we can deduce the optimality condition:

$$\int_0^T \int_C [(Du(\overline{u}) - z_{cib})] D[u(v) - u(\overline{u})] drdt + \beta \int_0^T \int_C \overline{u}(v - \overline{u}) drdt \geq 0 \tag{14}$$

Using (14), posing $u(v) = y(v)$ and $w = v - \overline{u}$, we find :

$$(\nabla J(\overline{u}), w) = \int_0^T \int_C [Dy(\overline{u}) - z_{cib}] Dy(w) drdt + \beta \int_0^T \int_C \overline{u}(w) drdt$$

where $y(w)$ check:

$$\begin{cases} \frac{\partial y(w)}{\partial t} + div(\alpha y(w)) + \sigma y(w) - \frac{\partial}{\partial x_3}(\eta \frac{\partial y(w)}{\partial x_3}) - \mu.\Delta_2 y(w) = w, & on \quad [0;T] \times C, \\ y(w) = 0, & on \quad \partial C, \end{cases} \tag{15}$$

and $y(w, 0) = 0$ By multiplying (15) by p and integrating by parts we get:

$$\int_0^T \int_C [-\frac{\partial p}{\partial t} - div(\alpha p) + \sigma p - \frac{\partial}{\partial x_3}(\eta \frac{\partial p}{\partial x_3}) - \mu.\Delta_2 p] y(w) drdt = \int T \int_C wpdrdt$$

hence:

$$\int_0^T \int_C D^* \Lambda (Du - z_{cib}) y(w) drdt = \int T \int_C wpdrdt$$

$$\Leftrightarrow D^* \Lambda (Du - z_{cib}) y(w) = wp$$

or :

$$((D^* \Lambda Du - z_{cib}), y(w)) = ((\Lambda Du - z_{cib}), Dy(w)) = ((Du - z_{cib}), Dy(w))$$

hence:

$$\int_0^T \int_C D^* \Lambda (Du - z_{cib}) y(w) drdt = \int T \int_C wpdrdt = \int_0^T \int_C (Du - z_{cib}) Dy(w) drdt$$

$$\Leftrightarrow (Du - z_{cib}) Dy(w) = wp$$

and therefore::

$$(\nabla J(\overline{u}), w) = \int_0^T \int_C wpdrdt + \beta \int_0^T \int_C \overline{u} wdrdt = \int_0^T \int_C (p + \beta \overline{u}) wdrdt$$

we deduce the optimality condition in accordance with the theorem:

$$(p + \beta \overline{u})(v - \overline{u}) \geq 0 \tag{16}$$

In addition:

$$\nabla J(\overline{u}) = 0 \Leftrightarrow p + \beta \overline{u} = 0 \Leftrightarrow \overline{u} = -p\beta^{-1} \tag{17}$$

Determining the adjoint problem:

The optimality condition (14) is not currently exploitable. To get around this problem we need to use the adjoint state, which will allow us to obtain an explicit expression for this condition.

To do this, we will introduce the Lagrangian defined as the sum of $J_u(v)$ and the equation of state multiplied by p:

$$L(v,u,p) = \frac{1}{2}\int_0^T \int_C [Du(v) - z_{cib}]^2 drdt + \frac{\beta}{2}\int_0^T \int_C v^2 drdt + \int_0^T \int_C p.[-\frac{\partial u}{\partial t} - \nabla(\alpha u) - \sigma u + \frac{\partial}{\partial x_3}(\eta\frac{\partial u}{\partial x_3}) + \mu\Delta_2 u + f + v]drdt \quad (18)$$

either:

$$L(v,u,p,) = \frac{1}{2}\int_0^T \int_C [Du(v) - z_{cib}]^2 drdt + \frac{\beta}{2}\int_0^T \int_C v^2 drdt + < -\frac{\partial u}{\partial t} - \nabla(\alpha u) - \sigma u + \frac{\partial}{\partial x_3}(\eta\frac{\partial u}{\partial x_3}) + \mu\Delta_2 u + f + v, p >$$

Considering the equation of state and p the adjoint state, we have:

$$< \frac{\partial u}{\partial t}, p > + \alpha < \nabla u, p > + \sigma < u, p > - < \mu\Delta_2 u + \frac{\partial}{\partial x_3}(\eta\frac{\partial u}{\partial x_3}), p > = < f, p > + < v, p >$$

Let's say :

$$I_1 = < \frac{\partial u}{\partial t}, p > = \int_C \int_0^T \frac{\partial u}{\partial t} pdrdt$$

$$I_2 = \alpha < \nabla u, p > = \alpha \int_C \int_0^T \nabla updrdt$$

$$I_3 = < \mu\Delta_2 u + \frac{\partial}{\partial x_3}(\eta\frac{\partial u}{\partial x_3}), p > = \mu \int_C \int_0^T \Delta_2 updrdt + \int_C \int_0^T \frac{\partial}{\partial x_3}(\eta\frac{\partial u}{\partial x_3})pdrdt$$
$$= J_1 + J_2$$

1. For the 1^{st} *term: using Green's formula:*

$$I_1 = \int_C \int_0^T \frac{\partial u}{\partial t} pdrdt = -\int_C \int_0^T u\frac{\partial p}{\partial t} drdt + \int_C [u(T)p(T) - u(0)p(0)]drdt \quad (19)$$

2. For the convection term:

$$I_2 = \alpha \int_C \int_0^T \nabla updrdt = -\alpha \int_C \int_0^T \nabla pudrdt + \alpha \int_{\Gamma_B \cup \Gamma_S} upd\sigma$$

we pose $p = 0$ on Γ_S, hence:

$$I_2 = -\alpha \int_C \int_0^T \nabla pudrdt + \alpha \int_{\Gamma_B} upd\sigma \quad (20)$$

3. For diffusion terms:

- $$J_1 = \mu \int_C \int_0^T \Delta_2 pdrdt = \mu \int_{\Gamma_B \cup \Gamma_S} \nabla_2 upn_{x_1x_2} d\sigma_{x_1x_2} - \mu \int_C \int_0^T \nabla_2 u \nabla_2 pdrdt$$

on Γ_B the normal vectors have a zero projection in a plane parallel to the plane $(\vec{\imath},\vec{\jmath})$ because they are orthogonal to the plane

$$\Rightarrow \int_{Gamma_B} \nabla_2 upn_{x_1x_2} d\sigma_{x_1x_2} = 0$$

so:

$$J_1 = \mu \int_{\Gamma_S} \nabla_2 upn_{x_1x_2} d\sigma_{x_1x_2} - \mu \int_C \int_0^T \nabla_2 u \nabla_2 pdrdt$$

or $p = 0$ sur $\Gamma_S \Rightarrow \int_{\Gamma_B \cup \Gamma_S} \nabla_2 upn_{x_1x_2} d\sigma_{x_1x_2} = 0$ et $J_1 = -\mu \int_C \int_0^T \nabla_2 u \nabla_2 pdrdt$

by reusing Green's formula:

$$J_1 = -\mu[\int_{\Gamma_B \cup \Gamma_S} \nabla_2 u \nabla_2 pn_{x_1x_2} d\sigma_{x_1x_2} - \int_C \int_0^T u\Delta_2 pdrdt]$$

hence:

$$J_1 = \mu \int_C \int_0^T u\Delta_2 pdrdt \quad (21)$$

- for the term $J_2 = \int_C \int_0^T \frac{\partial}{\partial x_3}(\eta\frac{\partial u}{\partial x_3})pdrdt$

$$J_2 = \int_{\Gamma_B} \eta \frac{\partial u}{\partial x_3} pnd\sigma - \int_C \int_0^T \eta \frac{\partial u}{\partial x_3} \frac{\partial p}{\partial x_3} drdt$$

or: $\int_C \int_0^T \eta \frac{\partial u}{\partial x_3} \frac{\partial p}{\partial x_3} drdt = \int_{\Gamma_B} \eta u \frac{\partial p}{\partial x_3} nd\sigma - \int_C \int_0^T u \frac{\partial}{\partial x_3} (\eta \frac{\partial p}{\partial x_3}) drdt$

d'où

$$J_2 = \int_{\Gamma B} \eta (\frac{\partial u}{\partial x_3} p - u \frac{\partial p}{\partial x_3}) nd\sigma + \int_C \int_0^T u \frac{\partial}{\partial x_3} (\eta \frac{\partial p}{\partial x_3}) drdt$$

and therefore:

$$J_2 = \int_{\Gamma 0} \eta (dup - u \frac{\partial p}{\partial x_3}) nd\sigma + \int_C \int_0^T u \frac{\partial}{\partial x_3} (\eta \frac{\partial p}{\partial x_3}) drdt \quad (22)$$

- Putting together (21) and (22), we deduce:

$$I_3 = J_1 + J_2 = \int_C \int_0^T (\mu \Delta_2 p + \frac{\partial}{\partial x_3} (\eta \frac{\partial p}{\partial x_3})) udrdt + \int_{\Gamma_0 \cup \Gamma_H} \eta u (dp - \frac{\partial p}{\partial x_3}) nd\sigma \quad (23)$$

4. using (19), (20) *and* (23), *we obtain:*

$$-I_1 - I_2 + I_3 = -\int_C \int_0^T [-\frac{\partial p}{\partial t} - \alpha \nabla p + (\mu \Delta_2 p + \frac{\partial}{\partial x_3} (\eta \frac{\partial p}{\partial x_3}))] udrdt - \int_C [u(T)p(T) - u(0)p(0)] dr$$
$$+ \int_{\Gamma_0 \cup \Gamma_H} [\alpha p - \eta (dp - \frac{\partial p}{\partial x_3})] und\sigma$$

The Lagrangian can then be given in the form:

$$L(v,u,p) = \int_0^T \int_C [\frac{1}{2} (Du - z_{cib})^2 + \frac{\beta}{2} v^2 + [\frac{\partial p}{\partial t} + \alpha \nabla p - \sigma p + \mu \Delta_2 p + \frac{\partial}{\partial x_3} (\eta \frac{\partial p}{\partial x_3})] u + (f + v) p] drdt$$
$$+ \int_0^T \int_{\Gamma_B} [\alpha up - \eta (\frac{\partial u}{\partial x_3} p - u \frac{\partial p}{\partial x_3})] ndtd\sigma - \int_C [u(T)p(T) - u(0)p(0)] dr \quad (24)$$

with $p = 0$ on Γ_S

$\frac{\partial L}{\partial u}$ gives the adjoint state; We have:

$$L(v, u + \epsilon w, p) - L(v,u,p) = \frac{1}{2} [\int_0^T \int_C [(Du - z_{cib})^2 + 2\epsilon (Du - z_d, Dw) + \epsilon^2 (Dw, Dw)] drdt -$$
$$\int_0^T \int_C (Du - z_{cib})^2 drdt + \int_0^T \int_C [\frac{\partial p}{\partial t} + \alpha \nabla p - \sigma p + \mu \Delta_2 p + \frac{\partial}{\partial x_3} (\eta \frac{\partial p}{\partial x_3})] \epsilon w +$$
$$\epsilon \int_0^T \int_{\Gamma_B} [\alpha wp - \eta (\frac{\partial w}{\partial x_3} p - w \frac{\partial p}{\partial x_3})] ndtd\sigma - \epsilon \int_C [w(T)p(T) - w(0)p(0)] dr \quad (25)$$

hence:

$$\lim_{\epsilon \to 0} \frac{L(v,u+\epsilon w,p) - L(v,u,p)}{\epsilon} = \frac{\partial L(u,v,p)}{\partial u}.w = \int_0^T \int_C D^* \Lambda (Du - z_{cib}) wdrdt - \int_C [w(T)p(T) - w(0)p(0)] dr$$
$$\int_0^T \int_C [\frac{\partial p}{\partial t} + \alpha \nabla p - \sigma p + \mu \Delta_2 p + \frac{\partial}{\partial x_3} (\eta \frac{\partial p}{\partial x_3})] w +$$
$$\int_0^T \int_{\Gamma_B} [\alpha wp - \eta (\frac{\partial w}{\partial x_3} p - w \frac{\partial p}{\partial x_3})] ndtd\sigma \quad (26)$$

Since the Lagrangian is stationary, this derivative must be zero in any direction w. This gives us the 3 conditions:

$$\begin{cases} \int_0^T \int_C D^* \Lambda (Du - z_{cib}) drdt + \int_0^T \int_C [\frac{\partial p}{\partial t} + \alpha \nabla p - \sigma p + \mu \Delta_2 p + \frac{\partial}{\partial x_3} (\eta \frac{\partial p}{\partial x_3})] = 0 \\ \int_0^T \int_{\Gamma_B} [\alpha p - \eta (\frac{\partial w}{\partial x_3} p - w \frac{\partial p}{\partial x_3})] ndtd\sigma = 0 \\ p(T;.) = 0 \end{cases} \quad (27)$$

we deduce the adjoint problem:

$$\begin{cases} -\frac{\partial p}{\partial t} - div(\alpha p) + \sigma p - \frac{\partial}{\partial x_3}(\eta \frac{\partial p}{\partial x_3}) - \mu.\Delta_2 p = D^*\Lambda(Du - z_{cib}), & on \quad [0;T]\times C, \\ p = 0, & on \quad \Gamma_S, \\ (\alpha - \eta d)p + \frac{\partial p}{\partial x_3} = 0, & on \quad \Gamma_0, \\ \alpha p + \frac{\partial p}{\partial x_3} = 0, & on \quad \Gamma_H, \end{cases} \tag{28}$$

we can then put $p = 0$ on $\Gamma_0 \cup \Gamma_H$, the problem (29) becomes:

$$\begin{cases} -\frac{\partial p}{\partial t} - div(\alpha p) + \sigma p - \frac{\partial}{\partial x_3}(\eta \frac{\partial p}{\partial x_3}) - \mu.\Delta_2 p = D^*\Lambda(Du - z_{cib}), & on \quad [0;T]\times C, \\ p = 0, & on \quad \partial C, \end{cases} \tag{29}$$

avec $p(T,.) = 0$

5. Solving and numerically simulating the optimal control problem

In this section, D is an injective operator from $L^2(0,T,V)$ into $L^2(0,T,H)$ and Λ is the identity operator.

5.1 Optimal system approximation

The domain will be subdivided essentially into tetrahedrons $(T_k)_{k\geq 1}$. The mesh thus obtained will be defined as in Chapter (2). The chosen method is the Lagrangian finite element method $P1$.

In this section, we will give a discretisation in space, then complete it with a discretisation in time of the direct problem first and then of the adjoint problem.

5.1.1 Approximation of the direct problem

We give here the results of discretisations in time and space. We solve the system:

$$(M + \Delta t S)U^{n+1} = MU^n + \Delta t(B^{n+1} + V^{n+1}) \tag{30}$$

$S = N + \sigma M + A + D$, where N, M, A, B are the matrices given by:

$$M_{ij} = \int_{T_k} \lambda_i\lambda_j dr,\ N_{ij} = \int_{T_k} \nabla\lambda_i\lambda_j dr,\ A_{ij} \int_{T_k} (A\nabla\lambda_i)\nabla\lambda_j dr,$$

$$D_{ij} = d\eta_0 \int_{\Gamma_0 \cap T_k} \lambda_i\lambda_j d\epsilon$$

where

$$V^{n+1}_{T_k,j} = \int_{T_k} v^{n+1}\lambda_j dr \quad et \quad V^{n+1}_{T_k} = (V^{n+1}_{T_k,j})^t_{j=1,\dots 4}$$

with:

$$V^{n+1} = \sum_{k=1}^{nt} V^{n+1}_{e,T_k}$$

V^{n+1}_{e,T_k} being the extended vectors of the vectors $V^{n+1}_{T_k}$ on the mesh and nt the total number of tetrahedra.

Where I is the set of indices of the nodes of the mesh, J the set of indices of the nodes belonging to Γ_S and K the set of indices of the nodes belonging to Γ_0 .

In the same way

$$B^{n+1}_{T_k,j} = \int_{T_k} f^{n+1}\lambda_j dr$$

5.1.2 Approximation of the adjoint problem

Using the same procedure, we will determine an approximation to the adjoint problem.

$\underline{Semi - discretizationinspaceoftheadjointproblem}$

$\Omega = \cup_{k=1}^{nt} T_k$. On a tetrahedron T_k, we pose:

$$p = \sum_{i=1}^{4} p_i\lambda_i$$

We then obtain the following approximation to the adjoint problem:

$$\begin{aligned} &-\textstyle\sum_{i\in I} p_{i'}(t) \int \lambda_i\lambda_j dr + \sum_{i\in I} - (\alpha \int \nabla\lambda_i\lambda_j dr - \sigma \int \lambda_i\lambda_j dr + \int A\nabla\lambda_i\nabla\lambda_j dr)p_i(t) \\ &= \textstyle\sum_{i\in I} (\int \lambda_i\lambda_j dr)u_i(t) - \sum_{i\in I} \int z_{cib}\lambda_j dr \end{aligned} \tag{31}$$

Recall here that I is the set of indices of the nodes in the mesh. This approximation leads us to the following system:

$$\begin{cases} -Mp'(t) + S_2 p(t) = B_2(t) \\ p(T) = 0 \end{cases} \quad (32)$$

where: $S_2 = -N + \sigma M - A$, with $M_{ij} = \int_C \lambda_i \lambda_j dr$, $N_{ij} = \int_C \nabla\lambda_i \lambda_j dr$, $A_{ij} \int_C (A\nabla\lambda_i)\nabla\lambda_j dr$,

$B_2 i = M_{ij} u_j - \int_C z_{cib} \lambda_j dr.$

Completetimeandspacediscretizationoftheadjointproblem

An implicit time scheme is also used. The time derivatives are then approached using an implicit finite difference scheme. The result is:

$$(-M + \Delta t S_2) P^{n+1} = -MP^n + \Delta t(MU^{n+1} - Z_{cib}^{n+1}) \quad (33)$$

$$Z_{cib,T_k,j}^{n+1} = \int_{T_k} z_{cib}^{n+1} \lambda_j dr, \quad Z_{cib,T_k}^{n+1} = (Z_{cib,T_k,j}^{n+1})_{j=1,\dots 4}^t$$

d'où:

$$Z_{cib}^{n+1} = \sum_{k=1}^{nt} Z_{cib,T_k,e}^{n+1}$$

$Z_{cib,T_k,e}^{n+1}$ being the extended vectors of the z_{cib,T_k}^{n+1} vectors on the mesh.

the algebraic systems resulting from the resolution of the two problems, i.e. the direct problem and the adjoint problem, gives us:

$$\begin{pmatrix} -M + \Delta t S_2 & -\Delta t M \\ 0 & M + \Delta t S \end{pmatrix} . \begin{pmatrix} P^{n+1} \\ U^{n+1} \end{pmatrix} = \begin{pmatrix} -M & 0 \\ 0 & M \end{pmatrix} . \begin{pmatrix} P^n \\ U^n \end{pmatrix} + \Delta t \begin{pmatrix} -Z_{cib}^{n+1} \\ B^{n+1} + V^{n+1} \end{pmatrix} \quad (34)$$

Evaluationofsecondlimbvectors

These vectors are defined by:

$$Z_{cib,T_k,j}^{n+1} = \int_{T_k} z_{cib}^{n+1} \lambda_j dr \quad (35)$$

$$V_{T_k,j}^{n+1} = \int_{T_k} v^{n+1} \lambda_j dr \quad (36)$$

$$B_{T_k,j}^{n+1} = \int_{T_k} f^{n+1} \lambda_j dr \quad (37)$$

En décomposant z_{cib}^{n+1},v^{n+1} et f^{n+1} comme suit:

$$z_{cib}^{n+1} = z_{cib,1}^{n+1} . z_{cib,2}(r) \quad (38)$$

$$v^{n+1} = v_1^{n+1} v_2(r) \quad (39)$$

$$f^{n+1} = f_1^{n+1} f_2(r) \quad (40)$$

and assuming that the values of z_{cib}, v and f are known at all points in our mesh, a quadrature gives us the values of the vectors as follows:

$$Z_{cib,T_k}^{n+1} = \frac{z_{cib,1}^{n+1} . |T_k|}{4} . z_{cib,2}(1,1,1,1) \quad (41)$$

$$V_{T_k}^{n+1} = \frac{v_1^{n+1} . |T_k|}{4} . v_2(1,1,1,1) \quad (42)$$

$$f_{T_k}^{n+1} = \frac{f_1^{n+1} . |T_k|}{4} . f_2(1,1,1,1) \quad (43)$$

5.2 Optimisation algorithm

Let's say $Y_k = (u_k, p_k, v_k)$. We are therefore looking for the vector $\overline{Y} = (\overline{u}, \overline{p}, \overline{v})^T$ solution of the optimal system consisting of the primal problem (P_p) and the adjoint problem (P_a).

To solve our optimisation problem, we will construct an algorithm that calculates the optimal control. This will be a descent algorithm using the non-linear conjugate gradient method.

Step 1: set v_0, then calculate $J_0 = J(v_0)$ and $\nabla J_0 = \nabla J(v_0)$; then we have $d_0 = \nabla J_0$

Step 2: Solve the optimal system $P1$ using the finite element method:

- find u_k solutiuon of (P_p).
- find p_k solutiuon of (P_a)
- put $d_k = p_k + \beta u_k$.

NB: $\nabla J_k = d_k$

As long as a convergence criterion is not met:

Step 3: Determination of the α_k step by a one-dimensional optimisation or a linear search of our choice. Compute a new iterate:

$$v_{k+1} = v_k - \alpha_k d_k;$$

Step 4: Evaluation of the new gradient ∇J_{k+1};

Step 5: Incrementation: $k = k + 1$

6. Numerical simulations

For the optimal control problem, the synthetic solution we use is the one obtained from the one used to solve the direct problem by changing the variable so that it is zero on the $Gamma_S$ edge. It is given by:

$$u(t;x;y;z) = -\cos\left(\frac{\pi y^2}{2(R^2 - x^2)}\right)\sin\left(\frac{\pi(z-H+3)}{2}\right)e^{-\frac{3(\mu+\eta_0+\sigma)\pi^2 t}{4H^2}} \quad (44)$$

with:

$$\begin{aligned} &\frac{\partial u}{\partial z} = \frac{\pi}{2}\cot\left(\frac{\pi(z-H+3)}{2}\right).u = d.u \quad if \quad z = 0 \\ &\frac{\partial u}{\partial z} = 0 \quad if \quad z = H \\ &u(x,y,z,t) = 0 \quad on \quad \Gamma_S \\ &u(0;x;y;z) = u_0 \end{aligned} \quad (45)$$

The target function or observation function is given by:

$$z_{cib} = \cos(\pi x).\cos(\pi.y).\cos(\pi.(z-H+3)).e^{-\frac{3(\mu+\eta_0+\sigma)\pi^2 t}{4H^2}} \quad (46)$$

6.1 Numerical results

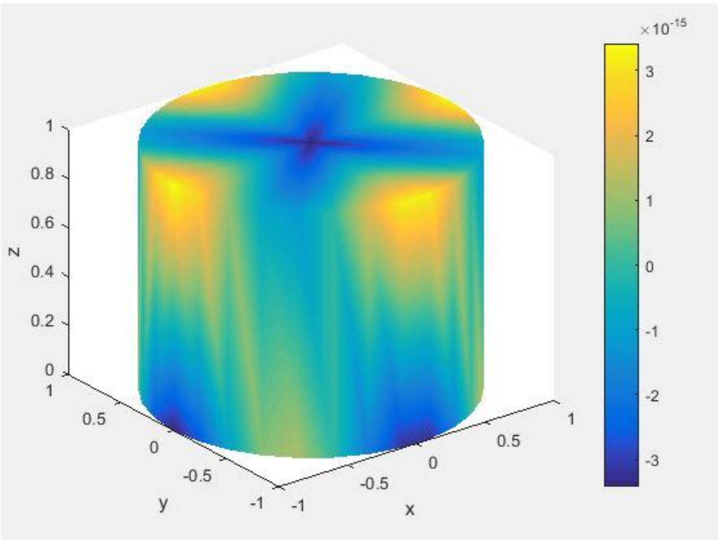

Figure 1: Target function with 64 nodes and $N = 50$.

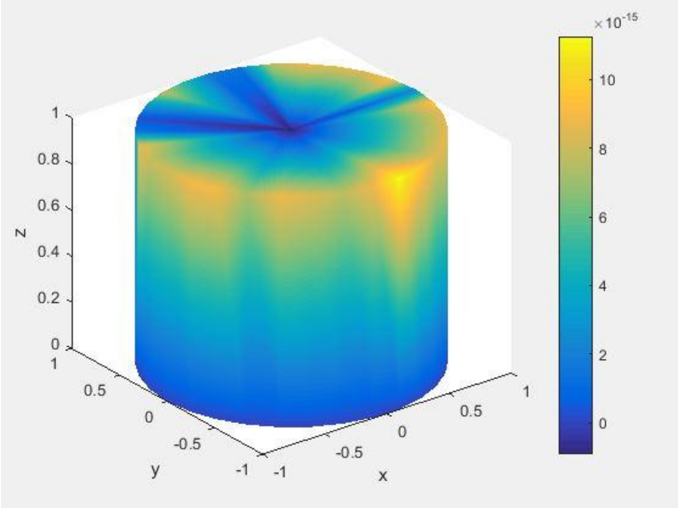

Figure 2: Controlled solution with 64 nodes and $N = 50$.

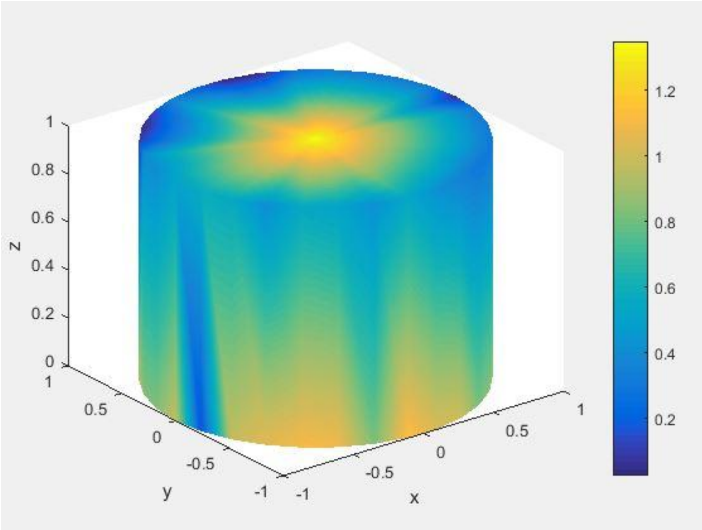

Figure 3: Optimal control with 64 nodes and $N = 50$.

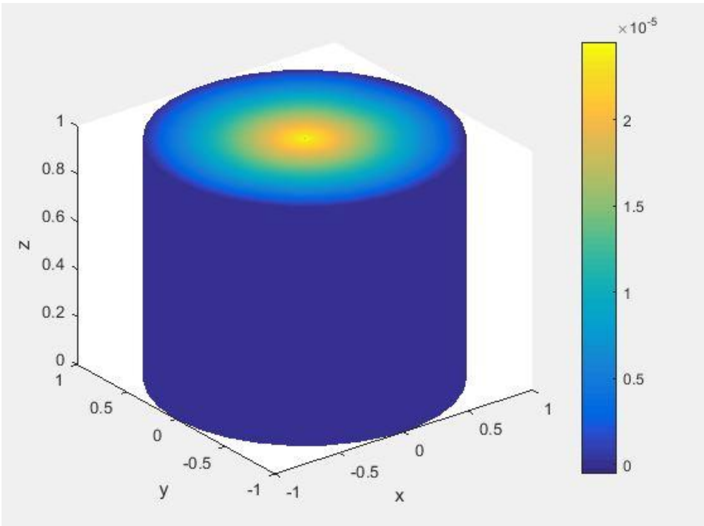

Figure 4: Uncontrolled solution-64 nodes and $N = 50$.

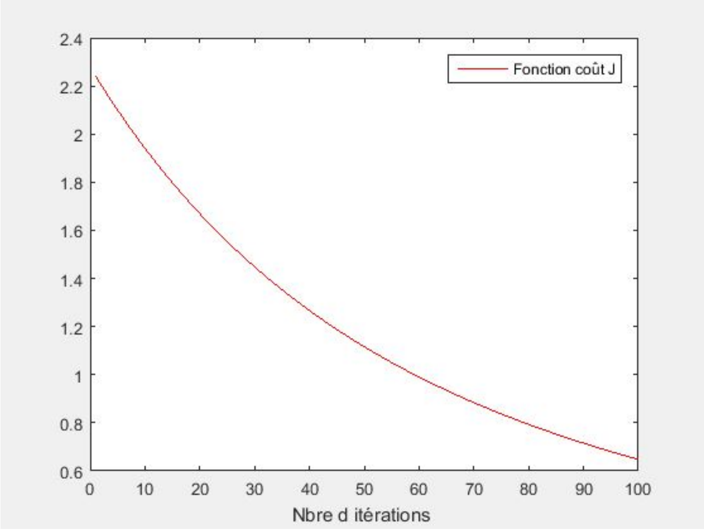

Figure 5: Evolution of the cost function as a function of the number of iterations for $N = 50$ and 64 nodes

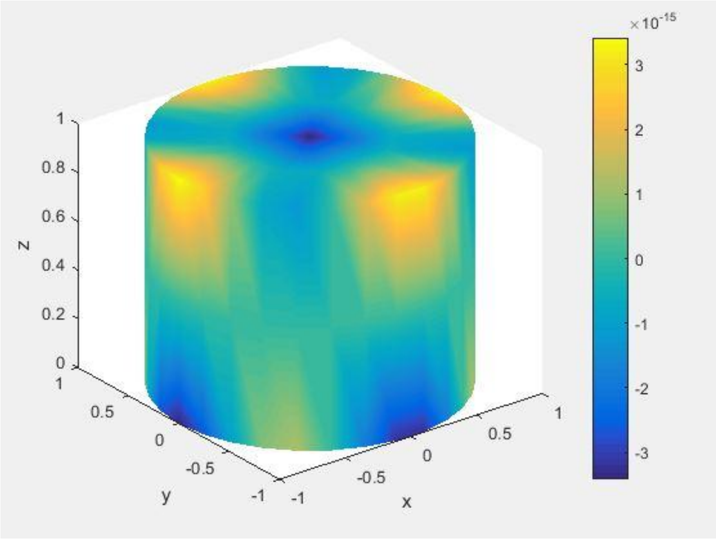

Figure 6: Target function with 189 nodes and $N = 100$.

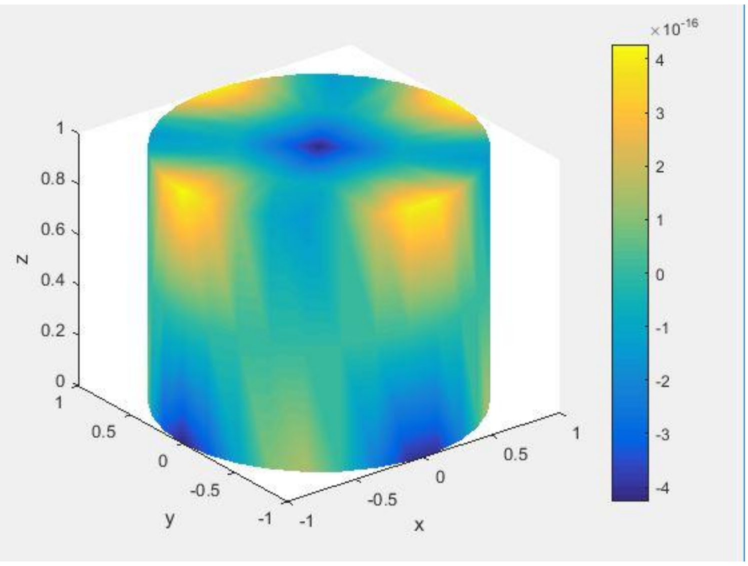

Figure 7: Controlled solution with 189 nodes and $N = 100$.

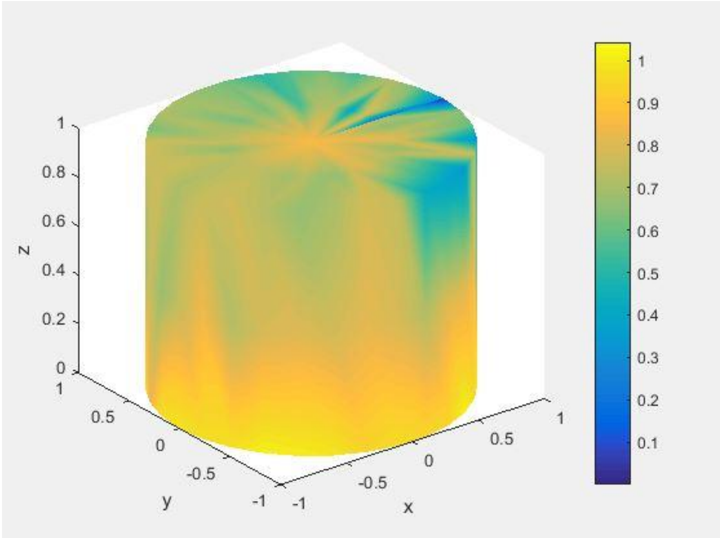

Figure 8: Optimal control with 189 nodes and $N = 100$.

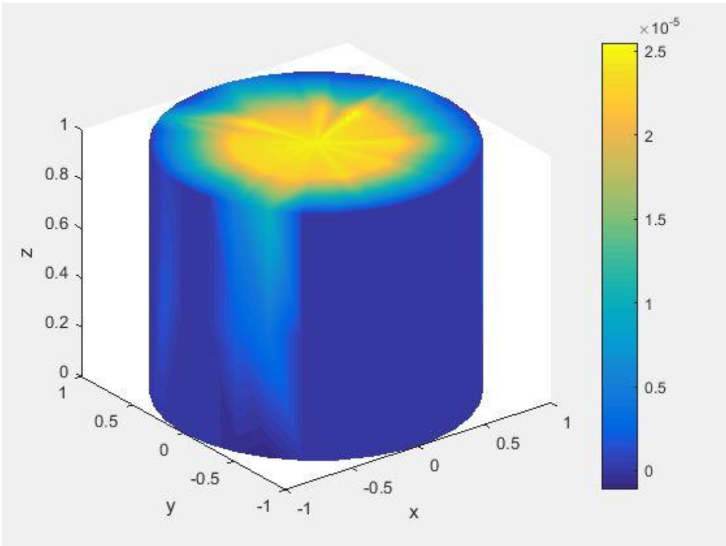

Figure 9: Uncontrolled solution-189 nodes and $N = 100$.

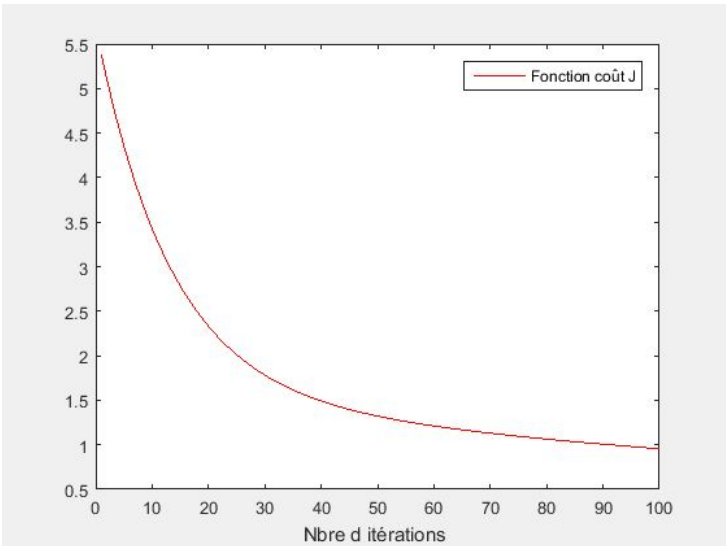

Figure 10: Evolution of the cost function as a function of the number of iterations for $N = 100$ and 189 nodes

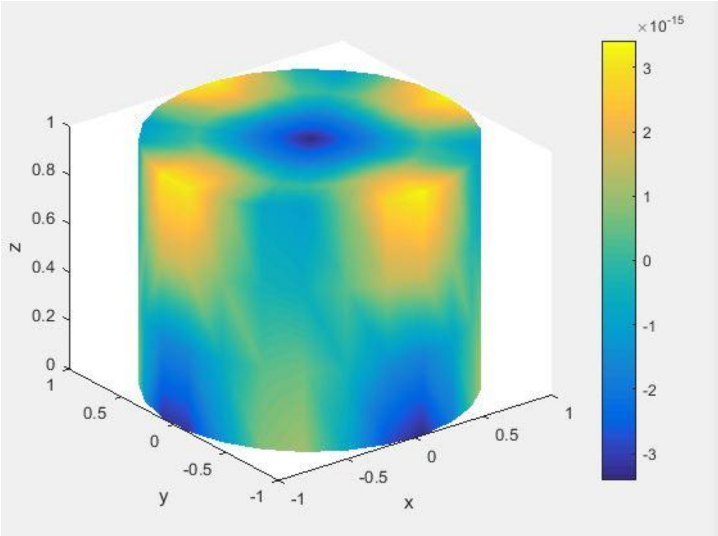

Figure 11: Target function with 315 nodes and $N = 150$.

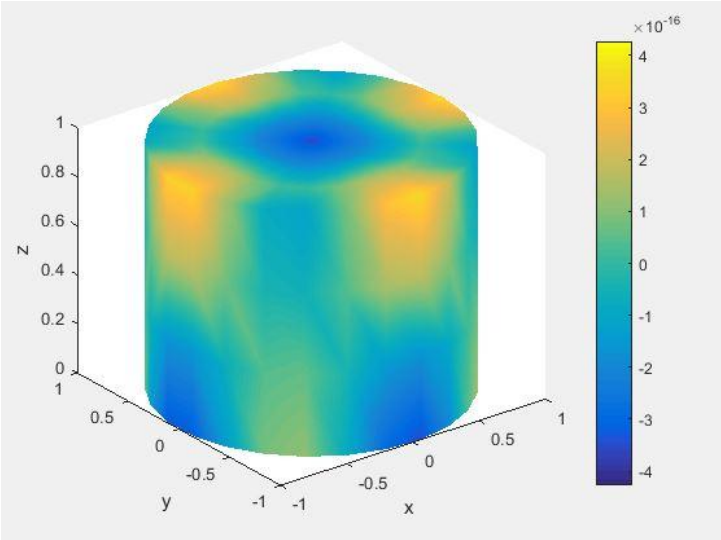

Figure 12: Controlled solution with 315 nodes and $N = 150$.

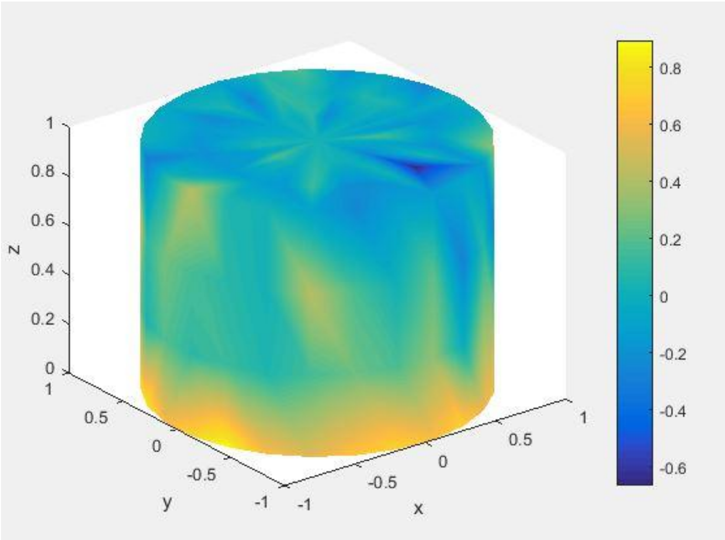

Figure 13: Optimal control with 315 nodes and $N = 150$.

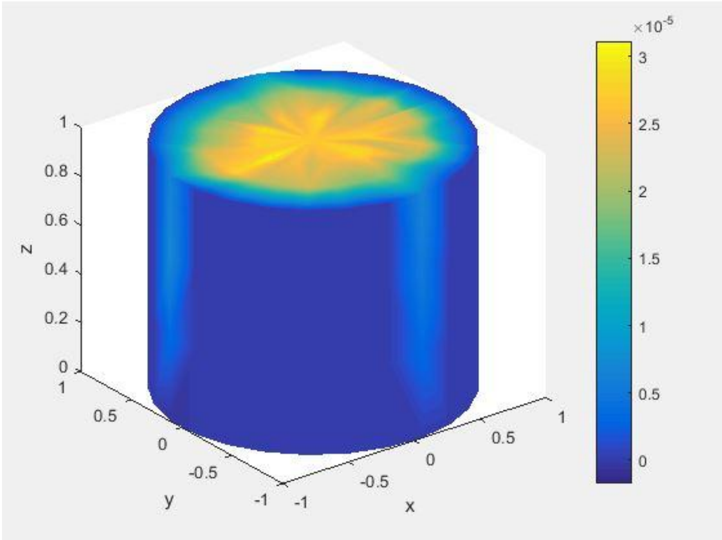

Figure 14: Uncontrolled solution-315 nodes and $N = 150$.

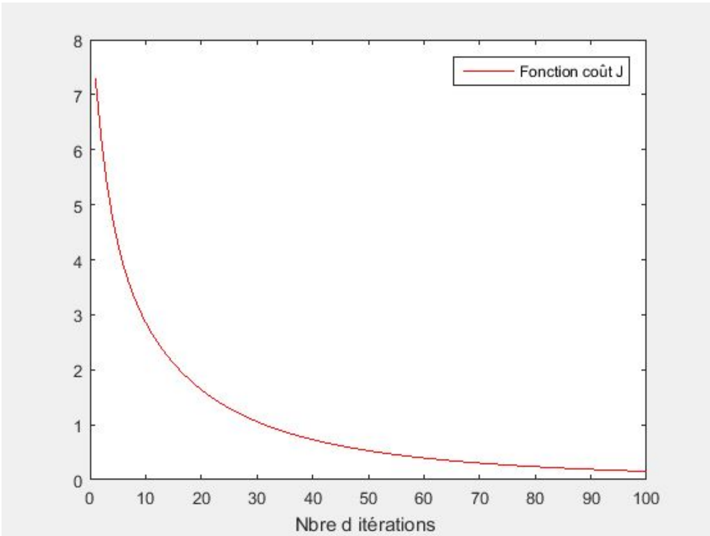

Figure 15: Evolution of the cost function as a function of the number of iterations for $N = 150$ and 315 nodes

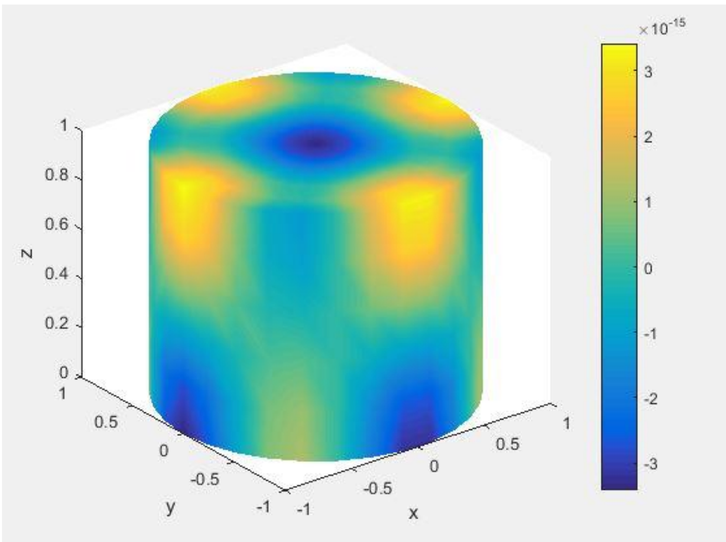

Figure 16: Target function with 625 nodes and $N = 200$.

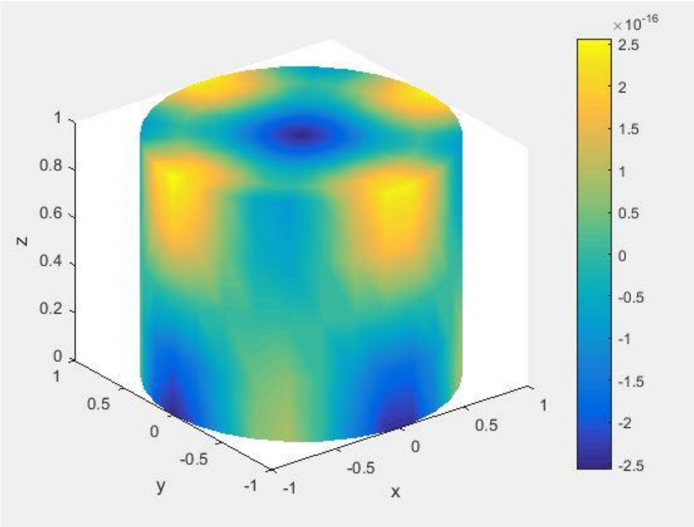

Figure 17: Controlled solution with 625 nodes and $N = 200$.

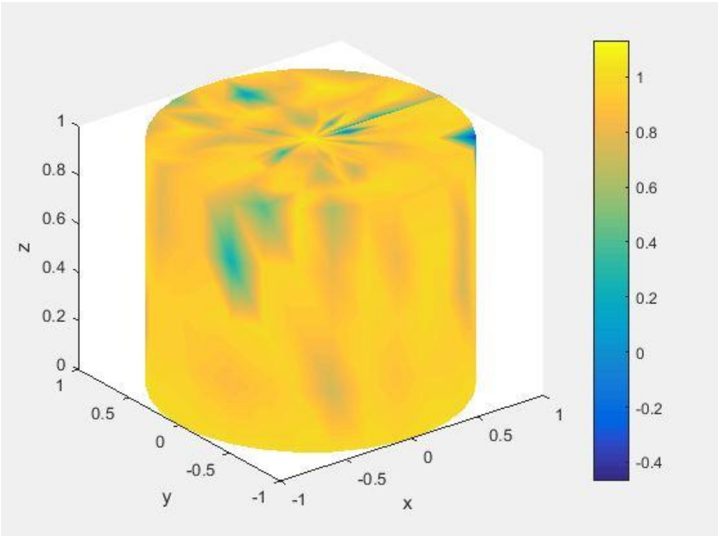

Figure 18: Optimal control with 625 nodes and $N = 200$.

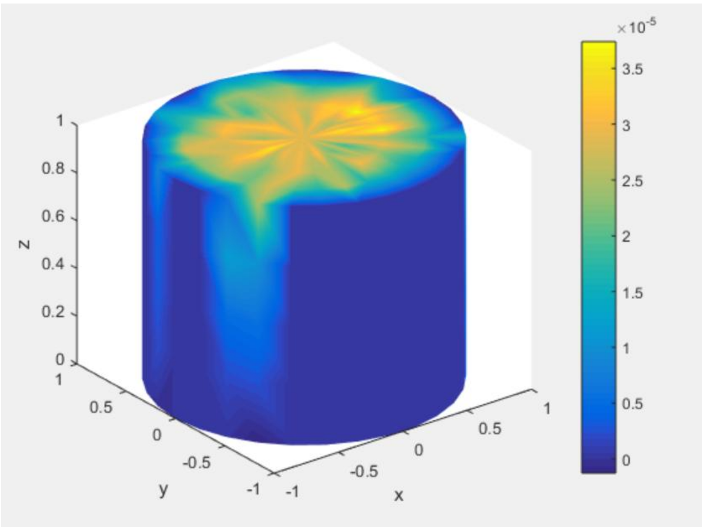

Figure 19: Uncontrolled solution-625 nodes and $N = 200$.

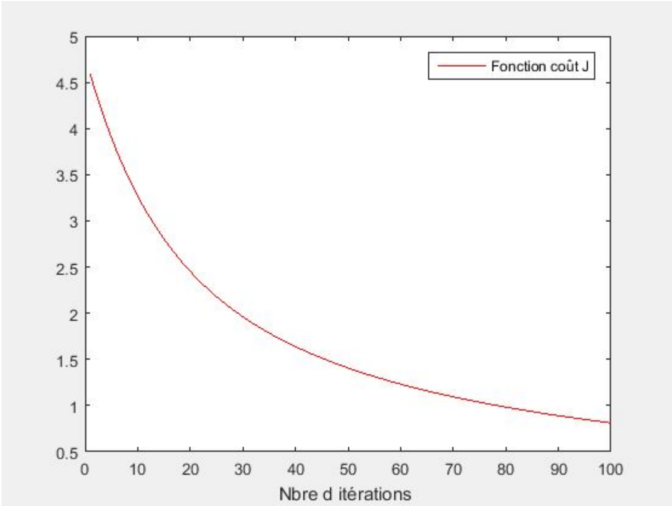

Figure 20: Evolution of the cost function as a function of the number of iterations for $N = 200$ and 625 nodes

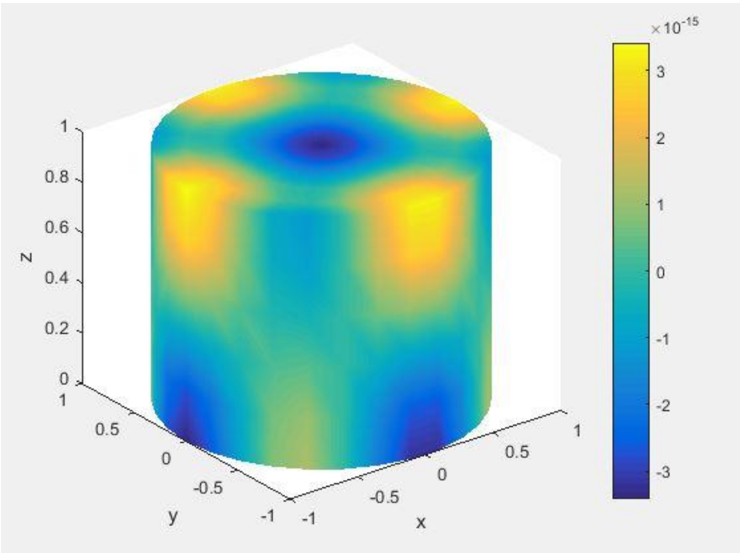

Figure 21: Target function with 936 nodes and $N = 250$.

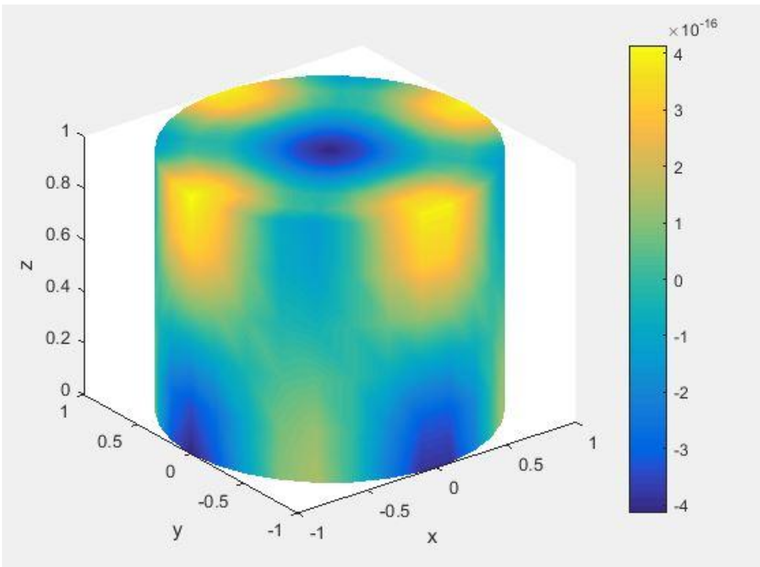

Figure 22: Controlled solution with 936 nodes and $N = 250$.

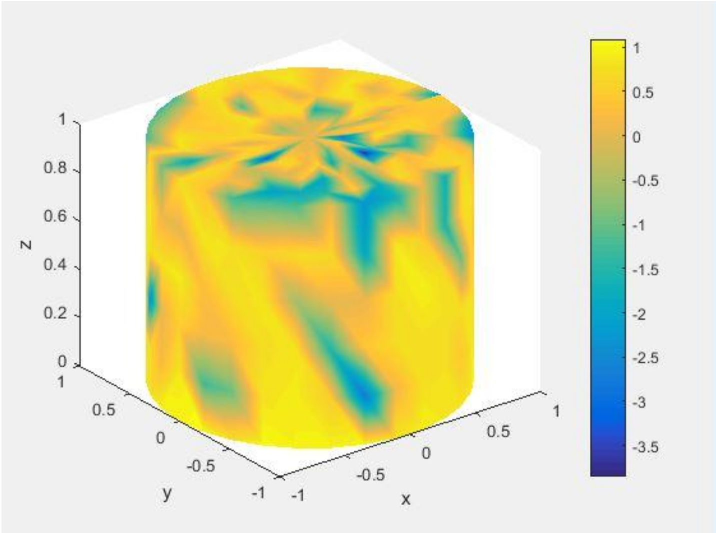

Figure 23: Optimal control with 936 nodes and $N = 250$.

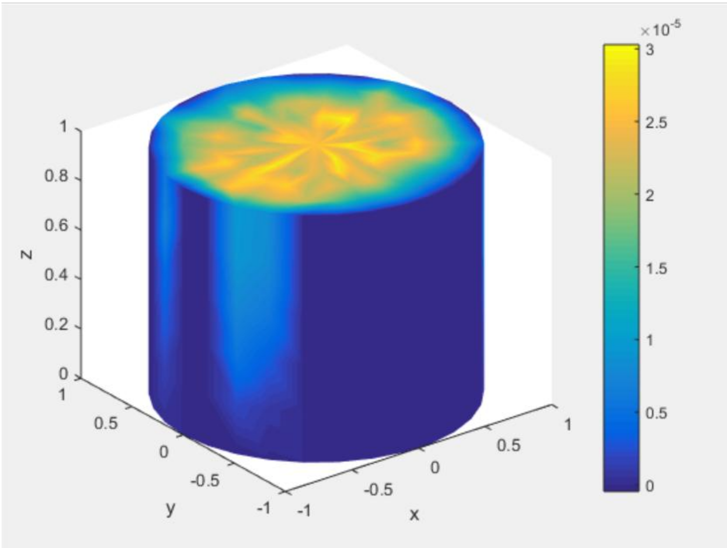

Figure 24: Uncontrolled solution-936 nodes and $N = 250$.

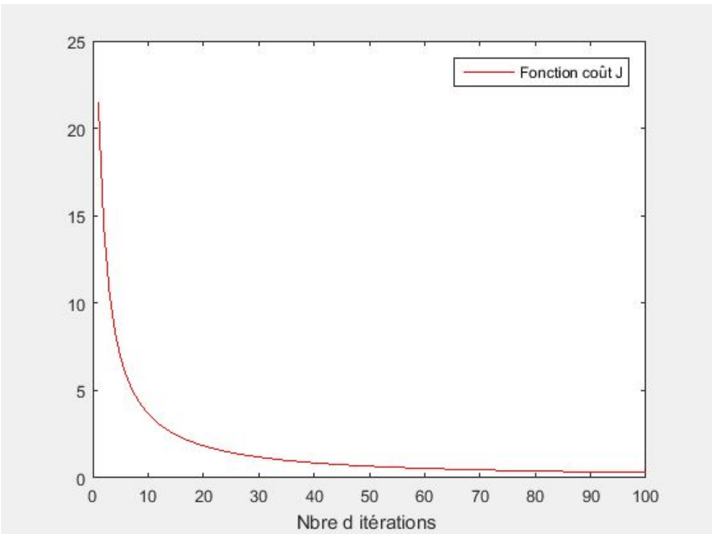

Figure 25: Evolution of the cost function as a function of the number of iterations for $N = 250$ and 936 nodes

Synthesis:

After implementation in Matlab, we obtained the results shown in the figures above. They show:

1. By observing and analysing these figures: for figures $refFigure01$ and $refFigure02$, we can see that the optimal solution tends to approach the target function. This tendency increases remarkably when we increase the number of nodes in the mesh. Indeed, the various figures 6, 11, 17, 22 representing the target function are practically identical to the respective figures 7, 12, 18, 23, representing the optimal solution obtained by progressively varying the number of nodes and the time step.
2. Figures 5, 10, 15, 19 represent the evolution of the functional J as the iterations increase in the cases with 64 nodes and $N = 50$, 189 nodes and $N = 100$, 376 nodes and $N = 150$, 625 nodes and $N = 200$, 936 nodes and $N = 250$. The first thing to notice when looking at these figures is that the curves they represent all have the same behaviour and are decreasing. This suggests that the cost function is decreasing and tending towards low values. We can therefore say that the optimal solution is very close to the target function when the control is optimal. The representations of optimal controls are given in figures 3, 8, 13, 18.

7. Conclusion

We assume that to control a pollution problem, it is easier to act on the pollution source than on the domain boundary. In this chapter, based on our model problem, we have proposed a distributed pollution control model. This optimal control problem is a constrained optimisation problem in which we act on the source. In the theoretical part, we studied the existence and uniqueness of the solution to this problem, deduced its adjoint problem and an explicit expression for the optimality condition.

From the initial problem, the adjoint problem and the optimality condition, which gives an expression for the gradient of the function J, we have constructed an algorithm in which, for each iteration, we simultaneously calculate the state solution, the adjoint solution and the gradient of J. Since two systems of partial differential equations have to be solved, the solutions calculated are approximate solutions found using the Lagrange finite element method $P1$. The optimisation method chosen is the Wolfe step gradient descent method.

The numerical results have enabled us to give representations of the optimal controls by varying the number of mesh nodes and the time step. As a conclusion, we can therefore find a control for which the concentration of the pollutant can be brought as close as possible to a given target function.

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