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Research Article

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Application of Elementary Transformation of Matrix to Solving Linear Equations

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Abstract: The elementary transformation of matrix is an important and basic operation law in the course of algebra, and it is an irreplaceable tool to study the solution of linear equations and matrices. In this paper, the elementary transformation is used to calculate the inverse of matrix and the rank of matrix, and the general problems in the solution of linear equations are sorted out and classified. Analyze the solution methods, skills and knowledge points of various questions through comments; Summarize similar problems, form a paper system, and discuss the application of matrix elementary transformation in solving linear equations.

Keywords: matrix theory; Elementary transformation of matrix; System of linear equations; Inverse matrix

1. Related Concepts and Methods

Definition 1 The combination of the elementary row transformation and the elementary column transformation of a matrix is called the elementary transformation of a matrix, and the matrix obtained by an elementary transformation of the identity matrix is called the elementary matrix, which can be divided into the following three types:

(1)Method transformation:Swap two rows (columns); The elementary matrix obtained by exchanging row i (column) with row j (column) of the identity matrix E, denoted $P(i,j)$;

(2)Multiplier transformation: Multiply $k \neq 0$ times all the elements of a row (column); The elementary matrix obtained by multiplying the i-th row (column) of the identity matrix E by the number $k(k \neq 0)$ is denoted $P(i(k))$.

(3)Double transform: Add k times of all the elements of a row (column) to the corresponding elements of another row (column); The elementary matrix obtained by adding the k times of the j-th row (column i) of the identity matrix E to the i-th row (column j) is denoted $P(i,j(k))$

The elementary matrix corresponding to the column transformation can also be obtained according to the above case, and is also included in the three matrices listed above. For example, if you add k times of column j of E to column i, you still get $P(i,j(k))$. In summary, these three matrices are all elementary matrices[0].

An elementary row transformation of an $s \times n$-matrix A is equivalent to multiplying matrix A by a corresponding $s \times s$-elementary matrix, an elementary column transformation of an $s \times n$-matrix A is equivalent to multiplying matrix A by a corresponding $n \times n$-elementary matrix, It can be found that there is an exact correspondence between matrix elementary transformation and matrix multiplication [2].

Definition 2 Row echelon matrix: ① the zero row (the row with all elements) is at the bottom; ② The column label of the non-zero leading element (that is, the first non-zero element of the non-zero row) increases strictly with the increase of the label, then this matrix A is called the row echelon matrix.
Like

$$\begin{pmatrix} 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

matrix for line stepped matrix. All rows are zero from the first element to below the first non-zero element of the row; If one row is all zeros, then the next row is all zeros.
Column echelon matrix: ① The zero column (the row with all elements) is on the far right; (2) the non-zero song yuan (i.e., the first nonzero column not to element) of the line label strictly increasing with the increase of column label, is called the matrix For the column step matrix[1].
Since the elementary transformation does not change the rank of the matrix, and any matrix can be transformed into a $m \times n$-echelon matrix through a series of elementary row transformations. Therefore, to determine the rank of the matrix, we should first transform into a echelon matrix by elementary row transformation, and then judge the rank of the original matrix according to the obtained matrix.

Example 1 Let $A = \begin{pmatrix} 3 & -1 & -4 & 2 & -2 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & 2 & 1 & 3 & 4 \\ -1 & 4 & 3 & -3 & 0 \end{pmatrix}$, Find the rank of matrix A.

$$A = \begin{pmatrix} 3 & -1 & -4 & 2 & -2 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & 2 & 1 & 3 & 4 \\ -1 & 4 & 3 & -3 & 0 \end{pmatrix} \xrightarrow[r_4+r_2]{\substack{r_1-3r_2 \\ r_3-r_2}} \begin{pmatrix} 0 & -1 & -1 & -1 & -2 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 4 \\ 0 & 4 & 2 & -2 & 0 \end{pmatrix}$$

$$\xrightarrow[r_1 \leftrightarrow r_2 \quad r_3 \leftrightarrow r_4]{\substack{r_3+2r_1 \\ r_4+4r_1}} \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & -2 \\ 0 & 0 & -2 & -6 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So the rank of matrix A is 3.
If the rank of a group of vectors can be found, each vector can be regarded as a row in a matrix, the vector is organized into a matrix, and the rank of a group of vectors is reduced to the rank of a matrix, which is obviously a much simpler problem.
When A is an invertible matrix of order n, A and E are placed side by side to form a matrix $(A|E)$ of $n \times 2n$ .Since $A^{-1}(A|E) = (E|A^{-1})$, A series of elementary row transformations are performed on the matrix $(A|E)$, transforming its left half into the identity matrix, while the right half is A^{-1}.

2. Application of elementary transformation of matrix to solving linear equations

Solving homogeneous linear equations

Theorem 1 considers a system of homogeneous linear equations over a number field

$$A_{m \times n} X_{n \times 1} = 0_{m \times 1} \tag{1}$$

Let $C=\begin{bmatrix} A_{m\times n} \\ E_{n\times n} \end{bmatrix}$, A series of elementary column transformations on C can be reduced to $\begin{bmatrix} B_{m\times n} \\ Q_{n\times n} \end{bmatrix}$, Where $B_{m\times n}$ is a column echelon matrix and $Q_{n\times n}$ is an invertible matrix of order n (That is, the product of a series of elementary matrices corresponding to a series of elementary column transformations by C), if $r(B_{m\times n})=r$, then $Q_{n\times n}$ after $(n-r)$ column is a fundamental solution system of (1).

Theorem 2 considers a system of homogeneous linear equations over a number field

$$X_{1\times n}A_{n\times m}=0_{1\times m} \qquad (2)$$

Let $C=\begin{bmatrix} A_{n\times m} & E_{n\times n} \end{bmatrix}$, A series of elementary column transformations on C can be reduced to $\begin{bmatrix} B_{n\times m} & P_{n\times n} \end{bmatrix}$, Where $B_{n\times m}$ is a column echelon matrix and $P_{n\times n}$ is an n-rank invertible matrix of order (that is, the product of a series of elementary matrices corresponding to a series of elementary column transformations performed by C), if $r(B_{m\times n})=r$, then $P_{n\times n}$ after $(n-r)$ is a fundamental solution system of (2).

Example 2 solves the basic solution system and general solution of the following homogeneous linear equations

$$\begin{cases} x_1-x_2-x_3+x_4=0 \\ x_1-2x_2+x_3-3x_4=0 \\ 2x_1-x_2-4x_3+6x_4=0 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -2 & 1 & -3 \\ 2 & -1 & -4 & 6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{c_1+c_3 \\ -c_1+c_4}]{c_1+c_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & -4 \\ 2 & 1 & -2 & 4 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-4c_2+c_4]{2c_2+c_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & -5 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So the fundamental solution system: $\alpha_1=(3,2,1,0)^T, \alpha_2=(-5,-4,0,1)^T$

The general solution is $X=k_1\alpha_1+k_2\alpha_2=k_1\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}+k_2\begin{bmatrix} -5 \\ -4 \\ 0 \\ 1 \end{bmatrix}$, k_1, k_2 is an arbitrary constant.

Solving non-homogeneous linear equations

Theorem 3 considers a system of nonhomogeneous linear equations over a number field

$$A_{m\times n}X_{n\times 1}=b_{m\times 1} \qquad (3)$$

Let $C=\begin{bmatrix} A_{m\times n} & -b_{m\times 1} \\ E_{n\times n} & 0_{n\times 1} \end{bmatrix}$, A series of elementary column transformations on C can be reduced to $\begin{bmatrix} B_{m\times n} & E_{m\times 1} \\ Q_{n\times n} & U_{n\times 1} \end{bmatrix}$, Where $B_{m\times n}$ is a column echelon matrix and $Q_{n\times n}$ is a invertible matrix of order, then

（1） If $E_{m\times 1}\neq 0$， be $r(A_{m\times n}\ -b)=r+1\neq r(A_{m\times n})=r$， The equations (2-3) have no solution.If $E_{m\times 1}=0$， be $r(A_{m\times n}\ -b)=r(A_{m\times n})=r$， The system (3) has a solution. (2) When there is a solution, $U_{n\times 1}$ is a nonhomogeneous particular solution, if $r(B_{m\times n})=r$， $Q_{n\times n}$ after $(n-r)$ column is a basic solution system of the (2-3) derived group, be written as $q_{r+1}\quad q_{r+2}\quad \cdots \quad q_n$， Then the general solution is obtained from the structure of the solution of the non-homogeneous linear system : $X=k_{r+1}q_{r+1}+k_{r+2}q_{r+2}+\cdots+k_nq_n+U_{n\times 1}$.

Theorem 4 considers a system of nonhomogeneous linear equations over a number field

$$X_{1\times n}A_{n\times m}=b_{1\times m} \qquad (4)$$

Let $C=\begin{bmatrix} A_{n\times m} & E_{n\times n} \\ -b & 0 \end{bmatrix}$, A series of elementary column transformations on C can be reduced to $\begin{bmatrix} B_{n\times m} & P_{n\times n} \\ E_{1\times m} & U_{1\times n} \end{bmatrix}$, Where $B_{n\times m}$ is a column echelon matrix and $P_{n\times n}$ is an invertible matrix of order n, then

（1） If $E_{1\times m}\neq 0$, The equations (4) have no solution;If $E_{1\times m}=0$, be $r(A_{m\times n}\ \ -b)=r(A_{m\times n})=r$, The system (4) has a solution. (2) When there is a solution, $U_{1\times n}$ Is a nonhomogeneous particular solution, if $r(B_{n\times m})=r$, $P_{n\times n}$ post- $(n-r)$ behavior (4) A fundamental solution system of the derived group, denoted $p_{r+1}\quad p_{r+2}\quad \cdots \quad p_n$, Then the general solution is obtained from the structure of the solution of the non-homogeneous linear system $X=k_{r+1}p_{r+1}+k_{r+2}p_{r+2}+\cdots+k_np_n+U_{1\times n}$.

Example 3 solves the general solution of the following non-homogeneous system of linear equations

$$\begin{cases} x_1+2x_2+4x_3-3x_4=0 \\ 3x_1+5x_2+6x_3-4x_4=1 \\ 4x_1+5x_2-2x_3+3x_4=3 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 4 & -3 & 0 \\ 3 & 5 & 6 & -4 & -1 \\ 4 & 5 & -2 & 3 & -3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[3c_1+c_4]{\substack{-2c_1+c_2 \\ -4c_1+c_3}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -6 & 5 & -1 \\ 2 & -3 & -18 & 15 & -3 \\ 1 & -2 & -4 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[5c_2+c_4]{\substack{-c_2+c_5 \\ -6c_2+c_3}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 \\ 1 & -2 & 8 & -7 & 2 \\ 0 & 1 & -6 & 5 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The basic solution system of the derived group is $\alpha_1=(8,-6,1,0)^T,\alpha_2=(-7,5,0,1)^T$ .Specially solve to $\beta_1=(2,-1,0,0)^T$.

The general solution is $X=k_1\alpha_1+k_2\alpha_2+\beta=k_1\begin{bmatrix} 8 \\ -6 \\ 1 \\ 0 \end{bmatrix}+k_2\begin{bmatrix} -7 \\ 5 \\ 0 \\ 1 \end{bmatrix}+\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. k_1,k_2 is an arbitrary constant.

By observing the solution process of the above examples, it is found that using elementary transformation to solve the matrix is indeed simple and practical, and achieves twice the result with half the effort.

Solving matrix equation

We know that the solution to $AX = B$ is $X = A^{-1}B$.It's actually the product of matrices that look like $A^{-1}B$, because $A^{-1}(A,B)=(E,A^{-1}B)$, So after an elementary row transformation, (A,B) becomes $(E,A^{-1}B)$, that's an elementary row transformation of matrix $n\times 2n$ and matrix (A,B) , When A becomes the identity matrix, The matrix at B is $A^{-1}B$.

Example 4 Solving matrix equations $AX = B$, among

$$A=\begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}, B=\begin{pmatrix} 4 & 2 & 3 \\ 1 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$$

$$(A,B)=\begin{pmatrix} 2 & 2 & 3 & 4 & 2 & 3 \\ 1 & -1 & 0 & 1 & 1 & 0 \\ -1 & 2 & 1 & -1 & 2 & 3 \end{pmatrix} \xrightarrow[r_3+r_1]{\substack{r_1\leftrightarrow r_2 \\ r_2-2r_1}} \begin{pmatrix} 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 4 & 3 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 3 & 3 \end{pmatrix}$$

$$\xrightarrow{\substack{r_3\leftrightarrow r_2 \\ r_3-4r_1 \\ -r_3}} \begin{pmatrix} 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & -2 & 12 & 9 \end{pmatrix} \xrightarrow{\substack{r_2-r_3 \\ r_1+r_2}} \begin{pmatrix} 1 & 0 & 0 & 3 & -8 & -6 \\ 0 & 1 & 0 & 2 & -9 & -6 \\ 0 & 0 & 1 & -2 & 12 & 9 \end{pmatrix}$$

therefore $X = A^{-1}B = \begin{pmatrix} 3 & -8 & -6 \\ 2 & -9 & -6 \\ -2 & 12 & 9 \end{pmatrix}$

When A, B is irreversible, a different elementary transformation method is needed to solve the problem.

3. Conclusion

In this paper, by analyzing some related problems of linear equations and observing coefficient matrix and augmented matrix of the properties of equations, it is found that the solution of linear equations can be completed by a series of matrix transformations, that is to say, the process of elementary transformation is the whole process of solving. Article mainly discussed how to use the elementary transformation to solve linear equations, the results show that this method compared to previous learning with elimination method to solve the linear system of equations is much easier, and save a lot of cumbersome steps, make the solution process more clear and reduces the chance of error. And matrix rank and related problems, such as maximal linearly independent group, can only solve a matrix into a step. No matter what knowledge is involved in this area, as long as you want to make the problem simple or efficient to solve, there is no doubt that elementary transformation is a good way.

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