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On the Pseudo Convex Bi-univalent Function Class of Complex Order

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Abstract In this study, we defined a new subclass of convex bi-univalent functions and examine some geometric properties this function class. For this class, we gave some coefficient estimates and solve Fekete-Szegö problem.

Keywords Convex function, bi-univalent function, pseudo convex function.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ of the complex plane $\mathbb{C}$. Let A be the class of the functions $f\in H(U)$ given by the following series expansions

$$f(z)=z+a_2z^2+a_3z^3+a_4z^4+\cdots+a_nz^n+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}\ . \tag{1.1}$$

It is clear that the function $f\in A$ satisfies the conditions $f(0)=0$ and $f'(0)=1$. The subclass of A, which is univalent functions in the open unit disk U, denoted by S. This class was introduced by Köebe [1] first time. Within a short period, in 1916 Bieberbach [2] published a paper in which the famous coefficient hypothesis was proposed. This hypothesis states that if $f\in S$ and has the series form (1.1), then $|a_n|\le n$ for each $n\ge 2$. In 1985, it was De-Branges [3], who settled this long-lasting hypothesis. There were a lot of papers devoted to this hypothesis and its related coefficient problems (see [4-18, 26]).

As is known that the function f is called a bi-univalent function, if itself and inverse is univalent in U and $f(U)$, respectively. The class of bi-univalent functions is denoted by Σ [18].

For the inverse $g(w)=f^{-1}(w),\ w\in f(U)$ of the function $f\in\Sigma$, we can write

$$g(w)=w+A_2w^2+A_3w^3+A_4w^4+\ldots\ =w+\sum_{n=2}^{\infty}A_nz^n,\ w\in f(U), \tag{1.2}$$

Where,

$$A_2=-a_2,\ A_3=2a_2^2-a_3,\ A_4=-a_2^3+5a_2a_3-a_4,\ldots\ .$$

The bi-convex function class defined in the open unit disk U is defined analytically as follows

$$C_{\Sigma}=\left\{ f\in S:\ \operatorname{Re}\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)>0,\ z\in U \text{ and } \operatorname{Re}\left(\frac{\left(zg'(w)\right)'}{g'(w)}\right)>0,\ w\in f(U) \right\}.$$

Let's $f,g\in H(U)$, then it is said that f is subordinate to g and denoted by $f\prec g$, if there exists a Schwartz function ω, such that $f(z)=g(\omega(z))$.

In the past few years, numerous subclasses of the collection S have been introduced as special choices of the class $C(\varphi)$ and C_{Σ} (see for example [5, 8, 10-26]).

2. Materials and Methods

Now, let's define some new subclass of bi-univalent functions in the open unit disk U.

Definition 2.1 For $\lambda>\frac{1}{2}$ and $\tau\in\mathbb{C}-\{0\}$ the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}(\lambda,\tau)$, if the following conditions are satisfied

$$\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}-1\right]\right\}\prec 1+\sinh z,\ z\in U \text{ and}$$

$$\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}-1\right]\right\}\prec 1+\sinh w,\ w\in f(U).$$

From the Definition 2.1, in the special values of the parameters $\lambda=1$, $\tau=1$ and $\lambda=1=\tau$, we obtain the following function classes, respectively.

Definition 2.2 For $\tau\in\mathbb{C}-\{0\}$ the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}(\tau)$, if the following conditions are satisfied

$$\left\{1+\frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)}-1\right]\right\}\prec 1+\sinh z,\ z\in U \text{ and } \left\{1+\frac{1}{\tau}\left[\frac{\left(wg'(w)\right)'}{g'(w)}-1\right]\right\}\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.3 For $\lambda>\frac{1}{2}$ the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}(\lambda)$, if the following conditions are satisfied

$$\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}\prec 1+\sinh z,\ z\in U \text{ and } \frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.4 For the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}$, if the following conditions are satisfied

$$\frac{\left(zf'(z)\right)'}{f'(z)}\prec 1+\sinh z,\ z\in U,\ \frac{\left(wg'(w)\right)'}{g'(w)}\prec 1+\sinh w,\ w\in f(U).$$

Let P be the class of analytic functions in U satisfied the conditions $p(0)=1$ and $\mathrm{Re}(p(z))>0$, $z\in U$. From the subordination principle easily can written

$$\mathrm{P}=\left\{p\in A:\ p(z)\prec\frac{1+z}{1-z}, z\in U\right\},$$

and in that case the function p has a series expansion of the following form

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U. \tag{2.1}$$

The class P defined above is known as the class Caratheodory functions [27].

Let us give the following lemmas known in the literature.

Lemma 2.1 ([28]). Let the function p belong in the class P. Then,

$$|p_n|\le 2 \text{ for each } n\in\mathbb{N} \text{ and } |p_n-\nu p_kp_{n-k}|\le 2 \text{ for } n,k\in\mathbb{N},\ n>k \text{ and } \nu\in[0,1].$$

The equalities holds for the function

$$p(z)=\frac{1+z}{1-z}.$$

Lemma 2.2 ([28]) Let the an analytic function p be of the form (2.1), then

$$2p_2=p_1^2+(4-p_1^2)x,$$

$$4p_3=p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2+2(4-p_1^2)(1-|x|^2)y$$

for $x,y\in\mathbb{C}$ with $|x|\le 1$ and $|y|\le 1$.

In this paper, we give some coefficient estimates and examine Fekete-Szegö problem for the class $C_{\Sigma,\sinh}(\lambda,\tau)$. Additionally, the results obtained here are compared with the results obtained in the literatüre for specific values of the parameters.

3. Results & Discussion

In this section of the our study, we examine the coefficient estimates problem for the class $C_{\Sigma,\sinh}(\lambda,\tau)$.

Theorem 3.1 Let the function f given by series expansions (1.1) belong to the class $C_{\Sigma,\sinh}(\lambda,\tau)$. Then are provided the following inequalities

$$|a_2|\le\frac{|\tau|}{2(2\lambda-1)} \text{ and } |a_3|\le|\tau|\cdot\begin{cases}\dfrac{1}{3(3\lambda-1)}, & |\tau|\le\dfrac{4(2\lambda-1)^2}{3(3\lambda-1)},\\[2ex] \dfrac{|\tau|}{4(2\lambda-1)^2}, & |\tau|\ge\dfrac{4(2\lambda-1)^2}{3(3\lambda-1)}.\end{cases} \tag{3.1}$$

Obtained here results are sharp.

Proof. Let $f\in C_{\Sigma,\sinh}(\lambda,\tau)$, $\lambda>\frac{1}{2}$ and $\tau\in\mathbb{C}-\{0\}$. Then, exists Schwartz functions $\omega:U\to U$, $\varpi:U_{r_0}\to U_{r_0}$, such that

$$\frac{1}{\tau}\left[\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}-1\right]=\sinh\omega(z), z\in U \text{ and } \frac{1}{\tau}\left[\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}-1\right]=\sinh\varpi(w), w\in f(U). \tag{3.2}$$

Let's the functions $p,q\in P$ defined as follows:

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_2z^2+p_3z^3+...=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U,$$

$$q(w)=\frac{1+\varpi(w)}{1-\varpi(w)}=1+q_1w+q_2w^2+q_3w^3+...=1+\sum_{n=1}^{\infty}q_nw^n,\ w\in f(U). \tag{3.3}$$

From these equalities, easily can written

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{p_1}{2}z+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)z^2+\frac{1}{2}\left(p_3-p_1p_2-\frac{p_1^3}{4}\right)z^3...,\ z\in U,$$

$$\varpi(w)=\frac{q(w)-1}{q(w)+1}=\frac{q_1}{2}w+\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)w^2+\frac{1}{2}\left(q_3-q_1q_2-\frac{q_1^3}{4}\right)w^3...,w\in f(U). \tag{3.4}$$

From the (3.2), (3.4) we obtain

$$2a_2(2\lambda-1)z+\left[3(3\lambda-1)a_3+\left(8\lambda^2-16\lambda+4\right)a_2^2\right]z^2+...=\tau\left\{\frac{p_1}{2}z+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+...\right\},\ z\in U,$$

$$2A_2(2\lambda-1)w+\left[3(3\lambda-1)A_3+\left(8\lambda^2-16\lambda+4\right)A_2^2\right]w^2+...=\tau\left\{1+\frac{q_1}{2}w+\left(\frac{q_2}{2}-\frac{q_1^2}{4}\right)w^2+...\right\},$$

$$w\in f(U). \tag{3.5}$$

Which, follows that

$$a_2=\frac{p_1\tau}{4(2\lambda-1)}, \tag{3.6}$$

$$\frac{1}{\tau}\left[3(3\lambda-1)a_3+4\left(2\lambda^2-4\lambda+1\right)a_2^2\right]=\frac{p_2}{2}-\frac{p_1^2}{4}, \tag{3.7}$$

$$a_2=-\frac{q_1\tau}{4(2\lambda-1)}, \tag{3.8}$$

$$\frac{1}{\tau}\left[3(3\lambda-1)\left(2a_2^2-a_3\right)+4\left(2\lambda^2-4\lambda+1\right)a_2^2\right]=\frac{q_2}{2}-\frac{q_1^2}{4}. \tag{3.9}$$

From the equalities (3.6) and (3.8), we can write

$$-\frac{q_1\tau}{4(2\lambda-1)}=a_2=\frac{p_1\tau}{4(2\lambda-1)} \text{ or } p_1=-q_1. \tag{3.10}$$

Thus, according to the Lemma 2.1 from the sekond equality (3.10), we obtain the first result of the theorem.

Using the equality (3.10), from the equalities (3.7) and (3.9) we obtain the following equality for a_3

$$a_3 = \frac{\tau^2}{16(2\lambda-1)^2} p_1^2 + \frac{\tau}{12(3\lambda-1)}(p_2 - q_2). \quad (3.11)$$

From the Lemma 2.2, we can easily write

$$p_2 - q_2 = \frac{4-p_1^2}{2}(x-y)$$

for $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$.

Substitute this expression for the difference $p_2 - q_2$ in (3.11), we get

$$a_3 = \frac{\tau^2}{16(2\lambda-1)^2} p_1^2 + \frac{\tau}{12(3\lambda-1)} \cdot \frac{4-p_1^2}{2}(x-y). \quad (3.12)$$

Applying triangle inequality to the last equality, we obtain

$$|a_3| \le \frac{|\tau|^2}{16(2\lambda-1)^2} t^2 + \frac{|\tau|}{12(3\lambda-1)} \cdot \frac{4-t^2}{2}(\xi+\eta), \ (\xi,\eta) \in [0,1], \quad (3.13)$$

where $\xi = |x|$, $\eta = |y|$ and $t = |p_1|$.

From the inequality (3.13), we easily obtain

$$|a_3| \le \frac{|\tau|}{4}\left[a(|\tau|,\lambda)t^2 + \frac{4}{3(3\lambda-1)}\right], \ t \in [0,2], \quad (3.14)$$

where

$$a(|\tau|,\lambda) = \frac{|\tau|}{4(2\lambda-1)^2} - \frac{1}{3(3\lambda-1)}.$$

Then, maximiziring the function

$$\psi(t) = a(\lambda,\beta)t^2 + \frac{4}{3(3\lambda-1)},$$

we obtain the second inequality of (3.1).

The result of theorem is sharp for the function

$$f_1(z) = z + \frac{\tau}{2(2\lambda-1)} z^2 + \frac{\tau}{3(3\lambda-1)} z^3, \ z \in U$$

in the case $3(3\lambda-1)|\tau| \le 4(2\lambda-1)^2$ and for the function

$$f_2(z) = z + \frac{\tau}{2(2\lambda-1)} z^2 + \frac{\tau^2}{4(2\lambda-1)^2} z^3, \ z \in U$$

in the case $3(3\lambda-1)|\tau| \ge 4(2\lambda-1)^2$.

With this, the proof of theorem is completed.

Taking $\lambda = 1$, $\tau = 1$ and $\lambda = 1 = \tau$ in the Theorem 3.1, we obtain the following results, respectively.

Corollary 3.1 If $f \in C_{\Sigma,\sinh}(\tau)$, then

$$|a_2| \le \frac{|\tau|}{2} \text{ and } |a_3| \le |\tau| \cdot \begin{cases} \frac{1}{6}, & |\tau| \le \frac{2}{3}, \\ \frac{|\tau|}{4}, & |\tau| \ge \frac{2}{3}. \end{cases}$$

Corollary 3.2 If $f \in C_{\Sigma,\sinh}(\lambda)$, then

$$|a_2| \le \frac{1}{2(2\lambda-1)} \text{ and } |a_3| \le \begin{cases} \frac{1}{4(2\lambda-1)^2}, \text{ if } \lambda \in \left(\frac{1}{2}, \frac{25+\sqrt{177}}{32}\right], \\ \frac{1}{3(3\lambda-1)}, \quad \text{if} \quad \lambda \ge \frac{25+\sqrt{177}}{32}. \end{cases}$$

Corollary 3.3 If $f \in C_{\Sigma,\sinh}$, then

$$|a_2| \le \frac{1}{2} \text{ and } |a_3| \le \frac{1}{4}.$$

Now, we focused on the solution of the Fekete-Szegö problem for the class $C_{\Sigma,\sinh}(\lambda,\tau)$.

Theorem 3.2 Let $f \in C_{\Sigma,\sinh}(\lambda,\tau)$, $\tau \in \mathbb{C}-\{0\}$ and $\mu \in \mathbb{C}$. Then,

$$|a_3 - \mu a_2^2| \le |\tau| \begin{cases} \frac{1}{3(3\lambda-1)} \text{ if } |1-\mu||\tau| \le \frac{4(2\lambda-1)^2}{3(3\lambda-1)}, \\ \frac{|\tau||1-\mu|}{4(2\lambda-1)^2} \text{ if } |1-\mu||\tau| \ge \frac{4(2\lambda-1)^2}{3(3\lambda-1)}. \end{cases} \tag{3.15}$$

Obtained here result is sharp.

Proof. Let $f \in C_{\Sigma,\sinh}(\lambda,\tau)$, $\lambda > \frac{1}{2}$ and $\tau \in \mathbb{C}-\{0\}$. From the equalities (3.6) and (3.12), we can write

$$a_3 - \mu a_2^2 = (1-\mu)\frac{\tau^2 p_1^2}{16(2\lambda-1)^2} + \frac{\tau(4-p_1^2)}{24(3\lambda-1)}(x-y) \tag{3.16}$$

for $x, y \in \mathbb{D}$ with $|x| \le 1$ and $|y| \le 1$.

Applying triangle inequality to the equality (3.16), we obtain

$$|a_3 - \mu a_2^2| \le |1-\mu| \frac{|\tau|^2 t^2}{16(2\lambda-1)^2} + \frac{|\tau|(4-t^2)}{24(3\lambda-1)}(\xi+\eta),\ (\xi,\eta) \in [0,1]^2,$$

where $\xi = |x|$, $\eta = |y|$ and $t = |p_1|$.

It follows that

$$|a_3 - \mu a_2^2| \le \frac{|\tau|}{16(2\lambda-1)^2}\left[|1-\mu||\tau| - \frac{4(2\lambda-1)^2}{3(3\lambda-1)}\right]t^2 + \frac{|\tau|}{3(3\lambda-1)},\ t \in [0,2]. \tag{3.17}$$

Maximiziring the function on the right hand side of the inequality (3.17) according to the parameter t, we get

$$\left|a_3-\mu a_2^2\right|\le|\tau|\begin{cases}\dfrac{1}{3(3\lambda-1)} & \text{if } |1-\mu||\tau|\le\dfrac{4(2\lambda-1)^2}{3(3\lambda-1)},\\ \dfrac{|\tau||1-\mu|}{4(2\lambda-1)^2} & \text{if } |1-\mu||\tau|\ge\dfrac{4(2\lambda-1)^2}{3(3\lambda-1)}.\end{cases}$$

The result of the theorem is sharp for the function

$$f_1(z)=z+\frac{\sqrt{|\tau|}}{\sqrt{3|1-\mu|(3\lambda-1)}}z^2+\frac{|\tau|}{3|1-\mu|(3\lambda-1)}z^3,\ z\in U$$

in the case $3(3\lambda-1)|1-\mu||\tau|\le4(2\lambda-1)^2$ and for the function

$$f_2(z)=z+\frac{|\tau|}{2(2\lambda-1)}z^2+\frac{|\tau|^2}{4(2\lambda-1)^2}z^3,\ z\in U$$

in the case $3(3\lambda-1)|1-\mu||\tau|\ge4(2\lambda-1)^2$.

Thus, the proof of teorem is completed.

Taking $\lambda=1$, $\tau=1$ and $\lambda=1=\tau$ in the Theorem 3.2, we obtain the following results, respectively.

Corollary 3.4 If $f\in C_{\Sigma,\sinh}(\tau)$, $\tau\in\mathbb{C}-\{0\}$ and $\mu\in\mathbb{C}$, then

$$\left|a_3-\mu a_2^2\right|\le\frac{|\tau|}{24}\begin{cases}4 & \text{if } 6|1-\mu||\tau|\le4,\\ 6|\tau||1-\mu| & \text{if } 6|1-\mu||\tau|\ge4.\end{cases}$$

Corollary 3.5 If $f\in C_{\Sigma,\sinh}(\lambda)$, $\tau\in\mathbb{C}-\{0\}$ and $\mu\in\mathbb{C}$, then

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{1}{3(3\lambda-1)} & \text{if } |1-\mu|\le\dfrac{4(2\lambda-1)^2}{3(3\lambda-1)},\\ \dfrac{|1-\mu|}{4(2\lambda-1)^2} & \text{if } |1-\mu||\tau|\ge\dfrac{4(2\lambda-1)^2}{3(3\lambda-1)}.\end{cases}$$

Corollary 3.6 If $f\in S^*_{\Sigma,\sinh}$ and $\mu\in\mathbb{C}$, then

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{2}\begin{cases}1 & \text{if } 2|1-\mu|\le1,\\ 2|1-\mu| & \text{if } 2|1-\mu|\ge1.\end{cases}$$

Also, taking $\mu=0$ and $\mu=1$ in the Theorem 3.2, we obtain the following results, respectively.

Corollary 3.7 If $f\in S^*_{\Sigma,\sinh}(\lambda,\tau)$, then,

$$|a_3|\le|\tau|\begin{cases}\dfrac{1}{3\lambda-1} & \text{if } |\tau|\le\dfrac{(2\lambda-1)^2}{3\lambda-1},\\ \dfrac{|\tau|}{(2\lambda-1)^2} & \text{if } |\tau|\ge\dfrac{(2\lambda-1)^2}{3\lambda-1}.\end{cases}$$

Corollary 3.8 If $f\in S^*_{\Sigma,\sinh}(\lambda,\tau)$, then,

$$\left|a_3 - a_2^2\right| \leq \frac{|\tau|}{3\lambda - 1}.$$

Rematk 3.1 We note that Corollary 3.7 confirms the second result of Theorem 3.1.

In the case $\mu, \tau \in \mathbb{C}$ and $\tau \neq 0$, we give the following theorem.

Theorem 3.3 Let $f \in S^{*}_{\Sigma,\sinh}(\lambda, \tau)$ and $\tau \neq 0,\ \tau, \mu \in \mathbb{C}$. Then,

$$\left|a_3 - \mu a_2^2\right| \leq |\tau| \begin{cases} \dfrac{1}{3\lambda - 1} & \text{if } 1 - \dfrac{(2\lambda-1)^2}{(3\lambda-1)|\tau|} \leq \mu \leq 1 + \dfrac{(2\lambda-1)^2}{(3\lambda-1)|\tau|}, \\ \dfrac{|\tau|(1-\mu)}{(2\lambda-1)^2} & \text{if } \mu \leq 1 - \dfrac{(2\lambda-1)^2}{(3\lambda-1)|\tau|}, \\ \dfrac{|\tau|(\mu-1)}{(2\lambda-1)^2} & \text{if } 1 + \dfrac{(2\lambda-1)^2}{(3\lambda-1)|\tau|} \leq \mu. \end{cases}$$

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