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# On the Pseudo Starlike Bi-univalent Function Class of Complex Order

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**Abstract** In this paper, we defined a new subclass of starlike bi-univalent functions and examine some geometric properties this function class. For this definition class, we gave some coefficient estimates and solve Fekete-Sezöge problem.

**Keywords** Starlike function, bi-univalent function, pseudo starlike function

## 1. Introduction

In this section, we give some basic information which we will use in our study.

Let $H(U)$ be the class of analytic functions in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ of the complex plane $\mathbb{C}$. By $A$, we will denote the class of the functions $f\in H(U)$ given by series expansions

$$f(z)=z+a_2z^2+a_3z^3+a_4z^4+\cdots+a_nz^n+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}\ . \qquad (1.1)$$

It is clear that $f(0)=0$ and $f'(0)=1$ for $f\in A$. The subclass of $A$, which are univalent functions in $U$ is denoted by $S$ in the literature. This class was introduced by Köebe [1] first time and has become the core ingredient of advanced research in this field. After a short time, in 1916 Bieberbach [2] published a paper in which the coefficient hypothesis was proposed. This hypothesis states that if $f\in S$ and has the series form (1.1), then $|a_n|\le n$ for each $n\ge 2$. There are many articles in the literature regarding to this hypothesis (see [3-16]).

It is well known that the function $f(z)$ is called a bi-univalent function, if itself and inverse is univalent in $U$ and $f(U)$, respectively. The class of bi-univalent functions in $U$ is denoted by $\Sigma$ [17].

For the inverse $g(w)=f^{-1}(w)$ of the function $f\in\Sigma$, we can write

$$g(w)=w+A_2w^2+A_3w^3+A_4w^4+\ldots=w+\sum_{n=2}^{\infty}A_nz^n,\ w\in f(U), \qquad (1.2)$$

where

$$A_2=-a_2,\ A_3=2a_2^2-a_3,\ A_4=-a_2^3+5a_2a_3-a_4,\ldots .$$

It is well known that the starlike and bi-starlike function classes in the open unit disk $U$ are defined analytically as follows

$$S^* = \left\{ f \in S: \ \mathrm{Re}\left( \frac{zf'(z)}{f(z)} \right) > 0, \ z \in U \right\},$$

$$S_\Sigma^* = \left\{ f \in S: \ \mathrm{Re}\left( \frac{zf'(z)}{f(z)} \right) > 0, \ z \in U \text{ and } \mathrm{Re}\left( \frac{wg'(w)}{g(w)} \right) > 0, \ w \in f(U) \right\}$$

and denoted by $S^*$ and $S_\Sigma^*$, respectively.

Let's $f, g \in H(U)$, then it is said that $f$ is subordinate to $g$ and denoted by $f \prec g$, if there exists a Schwartz function $\omega$, such that $f(z) = g(\omega(z))$.

In the past few years, numerous subclasses of the class $S$ have been introduced as special choices of the class $S^*(\varphi)$ and $S_\Sigma^*$ (see for example [4, 7, 9-16, 18-24]).

## 2. Materials and Methods

Now, let's we define new subclass of bi-univalent functions in the open unit disk $U$.

**Definition 2.1.** For $\lambda > \frac{1}{2}$ and $\tau \in \mathbb{C} - \{0\}$ the function $f \in \Sigma$ is said to be in the class $S_{\Sigma,\sinh}^*(\lambda, \tau)$, if the following conditions are satisfied

$$\left\{ 1 + \frac{1}{\tau}\left[ \frac{z\left(f'(z)\right)^\lambda}{f(z)} - 1 \right] \right\} \prec 1 + \sinh z, \ z \in U \text{ and}$$

$$\left\{ 1 + \frac{1}{\tau}\left[ \frac{w\left(g'(w)\right)^\lambda}{g(w)} - 1 \right] \right\} \prec 1 + \sinh w, \ w \in f(U).$$

In the special values of the parameters $\lambda$ and $\tau$ from the Definition 1.1, we have the following classes of bi-univalent functions.

**Definition 2.2.** For $\tau \in \mathbb{C} - \{0\}$ the function $f \in \Sigma$ is said to be in the class $S_{\Sigma,\sinh}^*(\tau)$, if the following conditions are satisfied

$$\left\{ 1 + \frac{1}{\tau}\left[ \frac{zf'(z)}{f(z)} - 1 \right] \right\} \prec 1 + \sinh z, \ z \in U \text{ and } \left\{ 1 + \frac{1}{\tau}\left[ \frac{wg'(w)}{g(w)} - 1 \right] \right\} \prec 1 + \sinh w, \ w \in f(U).$$

**Definition 2.3.** For $\lambda > \frac{1}{2}$ the function $f \in \Sigma$ is said to be in the class $S_{\Sigma,\sinh}^*(\lambda)$, if the following conditions are satisfied

$$\frac{z\left(f'(z)\right)^\lambda}{f(z)} \prec 1 + \sinh z, \ z \in U \text{ and } \frac{w\left(g'(w)\right)^\lambda}{g(w)} \prec 1 + \sinh w, \ w \in f(U).$$

**Definition 2.4.** The function $f \in \Sigma$ is said to be in the class $S_{\Sigma,\sinh}^*$, if the following conditions are satisfied

$$\frac{zf'(z)}{f(z)} \prec 1 + \sinh z, \ z \in U \text{ and } \frac{wg'(w)}{g(w)} \prec 1 + \sinh w, \ w \in f(U).$$

Let $\mathrm{P}$ be the class of analytic functions in $U$ satisfied the conditions $p(0)=1$ and $\mathrm{Re}(p(z))>0$, $z\in U$.

From the definition of subordination easily can written

$$\mathrm{P}=\left\{p\in H(U):\ p(z)\prec\frac{1+z}{1-z},z\in U\right\}.$$

Also, $p(z)$ has a series expansion as follows

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U. \tag{2.1}$$

The class $\mathrm{P}$ defined above is known as the class Caratheodory functions [25] in the literature.

Now, let us give some necessary lemmas for the proof of our main results.

**Lemma 2.1** ([26]). Let the function $p$ belong to the class $\mathrm{P}$. Then,

$$|p_n|\le 2 \text{ for each } n\in\mathbb{N},\ |p_n-\nu p_k p_{n-k}|\le 2 \text{ for } n,k\in\mathbb{N},\ n>k \text{ and } \nu\in[0,1].$$

The equalities hold for the function

$$p(z)=\frac{1+z}{1-z}.$$

**Lemma 2.2** ([26]) Let the an analytic function $p$ be of the form (2.1), then

$$2p_2=p_1^2+(4-p_1^2)x,$$

$$4p_3=p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2+2(4-p_1^2)\left(1-|x|^2\right)y$$

for some $x,y\in\mathbb{C}$ with $|x|\le 1$ and $|y|\le 1$.

In this paper, we give some coefficient estimates and solve Fekete-Szegö problem for the class $S^*_{\Sigma,\sinh}(\lambda,\tau)$. Additionally, the results obtained for specific values of the parameters in our study are compared with the results obtained in the literature.

## 3. Results & Discussion

In this section, we give some coefficient estimates for the functions belonging to the class $S^*_{\Sigma,\sinh}(\lambda,\tau)$.

**Theorem 3.1.** Let the function $f$ given by series expansions (1.1) belong to the class $S^*_{\Sigma,\sinh}(\lambda,\tau)$. Then, are provided the following inequalities

$$|a_2|\le\frac{|\tau|}{2\lambda-1} \text{ and } |a_3|\le|\tau|\begin{cases}\dfrac{1}{3\lambda-1} & \text{if } |\tau|\le\dfrac{(2\lambda-1)^2}{3\lambda-1},\\ \dfrac{|\tau|}{(2\lambda-1)^2} & \text{if } |\tau|\ge\dfrac{(2\lambda-1)^2}{3\lambda-1}.\end{cases} \tag{3.1}$$

Obtained here results are sharp.

**Proof.** Let $f\in S^*_{\Sigma,\sinh}(\lambda,\tau)$, $\lambda>\frac{1}{2}$ and $\tau\in\mathbb{C}-\{0\}$. Then, exists Schwartz functions $\omega:U\to U,\varpi:U_{r_0}\to U_{r_0}$, such that

$$\frac{1}{\tau}\left[\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right]=\sinh\omega(z), z\in U \text{ and } \frac{1}{\tau}\left[\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}-1\right]=\sinh\varpi(w), w\in f(U). \quad (3.2)$$

Let's the functions $p,q\in P$ defined as follows:

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_2z^2+p_3z^3+...=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U,$$

$$q(w)=\frac{1+\varpi(w)}{1-\varpi(w)}=1+q_1w+q_2w^2+q_3w^3+...=1+\sum_{n=1}^{\infty}q_nw^n,\ w\in f(U). \quad (3.3)$$

We get from there

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{p_1}{2}z+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)z^2+\frac{1}{2}\left(p_3-p_1p_2-\frac{p_1^3}{4}\right)z^3...,\ z\in U,$$

$$\varpi(w)=\frac{q(w)-1}{q(w)+1}=\frac{q_1}{2}w+\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)w^2+\frac{1}{2}\left(q_3-q_1q_2-\frac{q_1^3}{4}\right)w^3...,w\in f(U). \quad (3.4)$$

If we perform the necessary operations in the left side of the equalities (3.2), take into account the expressions (3.4) and use the series expansion of the $\sinh$ function, we obtain the following equalities

$$\frac{1}{\tau}\left\{a_2(2\lambda-1)z+\left[(3\lambda-1)a_3+\left(2\lambda^2-4\lambda+1\right)a_2^2\right]z^2+...\right\}=\frac{p_1}{2}z+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+...,\ z\in U,$$

$$\frac{1}{\tau}\left\{A_2(2\lambda-1)w+\left[(3\lambda-1)A_3+\left(2\lambda^2-4\lambda+1\right)A_2^2\right]w^2+...\right\}=\frac{q_1}{2}w+\left(\frac{q_2}{2}-\frac{q_1^2}{4}\right)w^2+...,\ w\in f(U). \quad (3.5)$$

Comparing the coefficients of the same degree terms on the right and left sides of the equalities (3.5), we obtain the following equalities for the coefficients $a_2$ and $a_3$ of the function $f$

$$\frac{1}{\tau}a_2(2\lambda-1)=\frac{p_1}{2}, \quad (3.6)$$

$$\frac{1}{\tau}\left[(3\lambda-1)a_3+\left(2\lambda^2-4\lambda+1\right)a_2^2\right]=\frac{p_2}{2}-\frac{p_1^2}{4}, \quad (3.7)$$

$$-\frac{1}{\tau}a_2(2\lambda-1)=\frac{q_1}{2}, \quad (3.8)$$

$$\frac{1}{\tau}\left[(3\lambda-1)\left(2a_2^2-a_3\right)+\left(2\lambda^2-4\lambda+1\right)a_2^2\right]=\frac{q_2}{2}-\frac{q_1^2}{4}. \quad (3.9)$$

It follows from (3.6) and (3.8) that

$$\frac{\tau p_1}{2(2\lambda-1)}=a_2=-\frac{\tau q_1}{2(2\lambda-1)} \quad (3.10)$$

and it can be seen that

$$p_1=-q_1. \quad (3.11)$$

Applying Lemma 2.1 to the equality (3.10), we obtain the first inequality of (3.1)

$$|a_2|\le\frac{|\tau|}{2\lambda-1}.$$

Considering the equation (3.11), from the equalities (3.7) and (3.9) we obtain the following equality for the coefficient $a_3$

$$a_3 = \frac{\tau^2}{4(2\lambda-1)^2} p_1^2 + \frac{\tau(p_2-q_2)}{4(3\lambda-1)}. \tag{3.12}$$

Also, from the Lemma 2.2, we can write

$$p_2 - q_2 = \frac{4-p_1^2}{2}(x-y)$$

for some $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$.

Substitute this expression for the difference $p_2 - q_2$ in the equality (3.12), we get

$$a_3 = \frac{\tau^2}{4(2\lambda-1)^2} p_1^2 + \frac{\tau(4-p_1^2)}{8(3\lambda-1)}(x-y). \tag{3.13}$$

Applying triangle inequality to the last equality, we obtain

$$|a_3| \le \frac{|\tau|^2}{4(2\lambda-1)^2} t^2 + \frac{|\tau|(4-p_1^2)}{8(3\lambda-1)}(\xi+\eta), \ \xi,\eta \in [0,1], \tag{3.14}$$

where $\xi = |x|$, $\eta = |y|$ and $t = |p_1|$.

From the inequality (3.14), can written

$$|a_3| \le \frac{|\tau|}{4}\left[a(\tau,\lambda)t^2 + \frac{4}{3\lambda-1}\right], \ t \in [0,2], \tag{3.15}$$

where

$$a(\tau,\lambda) = \frac{|\tau|}{(2\lambda-1)^2} - \frac{1}{3\lambda-1}.$$

Then, maximizing the function

$$\chi(t) = a(\tau,\lambda)t^2 + \frac{4}{3\lambda-1},$$

it can easily be seen that

$$\chi(t) \le \frac{4}{3\lambda-1}$$

if $a(\tau,\lambda) \le 0$ and

$$\chi(t) \le \frac{4|\tau|}{(2\lambda-1)^2}$$

if $a(\tau,\lambda) \ge 0$.

Thus, the proof of second inequality of (3.1) is provided.

The result of theorem is sharp for the function

$$f_1(z) = z + \frac{\tau}{2\lambda-1} z^2 + \frac{\tau}{3\lambda-1} z^3, \ z \in U$$

in the case $(3\lambda-1)|\tau| \le (2\lambda-1)^2$ and for the function

$$f_2(z) = z + \frac{\tau}{2\lambda - 1} z^2 + \frac{\tau^2}{(2\lambda - 1)^2} z^3,\ z \in U$$

in the case $(3\lambda - 1)|\tau| \geq (2\lambda - 1)^2$.

With this, the proof of theorem is completed.

Taking $\lambda = 1$, $\tau = 1$ and $\lambda = \tau = 1$ in the Theorem 3.1, we obtain the following results, respectively.

**Corollary 3.1.** If $f \in S^*_{\Sigma,\sinh}(\tau)$, then

$$|a_2| \leq |\tau| \text{ and } |a_3| \leq |\tau| \begin{cases} \frac{1}{2} \text{ if } & 0 < |\tau| \leq \frac{1}{2}, \\ |\tau| \text{ if } & |\tau| \geq \frac{1}{2}. \end{cases}$$

**Corollary 3.2.** If $f \in S^*_{\Sigma,\sinh}(\lambda)$, then

$$|a_2| \leq \frac{1}{2\lambda - 1} \text{ and } |a_3| \leq \begin{cases} \frac{1}{(2\lambda - 1)^2} & \text{if } \lambda \in \left(\frac{1}{2}, \frac{7 + \sqrt{17}}{8}\right], \\ \frac{1}{3\lambda - 1} & \text{if } \frac{7 + \sqrt{17}}{8} \leq \lambda. \end{cases}$$

**Corollary 3.3.** If $f \in S^*_{\Sigma,\sinh}$, then

$$|a_2| \leq 1 \text{ and } |a_3| \leq 1.$$

Now, we focused on the solution of the Fekete-Szegö problem for the class $S^*_{\Sigma,\sinh}(\lambda, \tau)$.

**Theorem 3.2.** Let $f \in S^*_{\Sigma,\sinh}(\lambda, \tau)$, $\tau \in \mathbb{C} - \{0\}$ and $\mu \in \mathbb{C}$. Then,

$$|a_3 - \mu a_2^2| \leq |\tau| \begin{cases} \frac{1}{3\lambda - 1} & \text{if } (3\lambda - 1)|1 - \mu||\tau| \leq (2\lambda - 1)^2, \\ \frac{|\tau||1 - \mu|}{(2\lambda - 1)^2} & \text{if } (3\lambda - 1)|1 - \mu||\tau| \geq (2\lambda - 1)^2. \end{cases} \tag{3.16}$$

Obtained here result is sharp.

**Proof.** Let $f \in S^*_{\Sigma,\sinh}(\lambda, \tau)$, $\lambda > \frac{1}{2}$ and $\tau \in \mathbb{C} - \{0\}$. From the equalities (3.10) and (3.13), we can write the following equality for the expression $a_3 - \mu a_2^2$

$$a_3 - \mu a_2^2 = (1 - \mu) \frac{\tau^2 p_1^2}{4(2\lambda - 1)^2} + \frac{\tau(4 - p_1^2)}{8(3\lambda - 1)}(x - y) \tag{3.17}$$

for some $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$.

Applying triangle inequality to the equality (3.17), we obtain

$$|a_3 - \mu a_2^2| \leq |1 - \mu| \frac{|\tau|^2 t^2}{4(2\lambda - 1)^2} + \frac{|\tau|(4 - t^2)}{8(3\lambda - 1)}(\xi + \eta),\ \xi, \eta \in [0,1],$$

where $\xi = |x|$, $\eta = |y|$ and $t = |p_1|$.

From the last inequality, easily can written

$$\left|a_3-\mu a_2^2\right| \le \frac{|\tau|}{4(2\lambda-1)^2}\left[|1-\mu||\tau|-\frac{(2\lambda-1)^2}{3\lambda-1}\right]t^2+\frac{|\tau|}{3\lambda-1},\ t\in[0,2]. \tag{3.18}$$

Maximizing the function $\varphi:[0,2]\to\mathbb{R}$ defined as follows

$$\varphi(t)=\frac{1}{4(2\lambda-1)^2}\left[|1-\mu||\tau|-\frac{(2\lambda-1)^2}{3\lambda-1}\right]t^2+\frac{1}{3\lambda-1},\ t\in[0,2],$$

we can easily see that

$$\varphi(t)\le\frac{1}{3\lambda-1}$$

if $(3\lambda-1)|1-\mu||\tau|\le(2\lambda-1)^2$ and

$$\varphi(t)\le\frac{|\tau|}{(2\lambda-1)^2}|1-\mu|$$

if $(3\lambda-1)|1-\mu||\tau|\ge(2\lambda-1)^2$.

With this, the inequality (3.16) is proven.

Moreover, it can be easily seen that the result of theorem is sharp for the function

$$f_1(z)=z+\frac{\sqrt{|\tau|}}{\sqrt{|1-\mu|(3\lambda-1)}}z^2+\frac{|\tau|}{|1-\mu|(3\lambda-1)}z^3,\ z\in U$$

in the case $(3\lambda-1)|1-\mu||\tau|\le(2\lambda-1)^2$ and for the function

$$f_2(z)=z+\frac{|\tau|}{2\lambda-1}z^2+\frac{|\tau|^2}{(2\lambda-1)^2}z^3,\ z\in U$$

in the case $(3\lambda-1)|1-\mu||\tau|\ge(2\lambda-1)^2$.

Thus, the proof of theorem is completed.

Taking $\lambda=1$, $\tau=1$ and $\lambda=\tau=1$ in the Theorem 3.2, we obtain the following results, respectively.

**Corollary 3.4.** If $f\in S^*_{\Sigma,\sinh}(\tau)$, $\tau\in\mathbb{C}-\{0\}$ and $\mu\in\mathbb{C}$, then

$$\left|a_3-\mu a_2^2\right|\le|\tau|\begin{cases}\frac{1}{2} & \text{if } 2|1-\mu||\tau|\le1,\\ |\tau||1-\mu| & \text{if } 2|1-\mu||\tau|\ge1.\end{cases}$$

**Corollary 3.5.** If $f\in S^*_{\Sigma,\sinh}(\lambda)$, $\tau\in\mathbb{C}-\{0\}$ and $\mu\in\mathbb{C}$, then

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{1}{3\lambda-1} & \text{if } |1-\mu|\le\frac{(2\lambda-1)^2}{3\lambda-1},\\ \frac{|1-\mu|}{(2\lambda-1)^2} & \text{if } |1-\mu|\ge\frac{(2\lambda-1)^2}{3\lambda-1}.\end{cases}$$

**Corollary 3.6.** If $f\in S^*_{\Sigma,\sinh}$, $\tau\in\mathbb{C}-\{0\}$ and $\mu\in\mathbb{C}$, then

$$\left|a_3-\mu a_2^2\right| \leq \frac{1}{2}\begin{cases}1 & \text{if } 2\left|1-\mu\right| \leq 1, \\ 2\left|1-\mu\right| & \text{if } 2\left|1-\mu\right| \geq 1.\end{cases}$$

Also, taking $\mu=0$ and $\mu=1$ in the Theorem 3.2, we obtain the following results, respectively.

**Corollary 3.7.** If $f \in S_{\Sigma,\sinh}^{*}(\lambda,\tau)$ and $\tau \in \mathbb{C}-\{0\}$, then,

$$\left|a_3\right| \leq \left|\tau\right| \begin{cases}\dfrac{1}{3\lambda-1} & \text{if } \left|\tau\right| \leq \dfrac{(2\lambda-1)^2}{3\lambda-1}, \\ \dfrac{\left|\tau\right|}{(2\lambda-1)^2} & \text{if } \left|\tau\right| \geq \dfrac{(2\lambda-1)^2}{3\lambda-1}.\end{cases}$$

**Corollary 3.8.** If $f \in S_{\Sigma,\sinh}^{*}(\lambda,\tau)$ and $\tau \in \mathbb{C}-\{0\}$, then,

$$\left|a_3-a_2^2\right| \leq \frac{\left|\tau\right|}{3\lambda-1}.$$

**Remark 3.1.** We note that Corollary 3.4 confirms the second result of Theorem 3.2.

In the case $\mu,\tau \in \mathbb{C}$ and $\tau \neq 0$, we can prove the following theorem similarly.

**Theorem 3.3.** Let $f \in S_{\Sigma,\sinh}^{*}(\lambda,\tau)$ and $\tau \neq 0,\ \tau,\mu \in \mathbb{C}$. Then,

$$\left|a_3-\mu a_2^2\right| \leq \left|\tau\right| \begin{cases}\dfrac{1}{3\lambda-1} & \text{if } 1-\dfrac{(2\lambda-1)^2}{(3\lambda-1)\left|\tau\right|} \leq \mu \leq 1+\dfrac{(2\lambda-1)^2}{(3\lambda-1)\left|\tau\right|}, \\ \dfrac{\left|\tau\right|\left|1-\mu\right|}{(2\lambda-1)^2} & \text{if } \begin{cases}\mu \leq 1-\dfrac{(2\lambda-1)^2}{(3\lambda-1)\left|\tau\right|}, \\ 1+\dfrac{(2\lambda-1)^2}{(3\lambda-1)\left|\tau\right|} \leq \mu.\end{cases}\end{cases}$$

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