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Reconstructibility analysis and state estimation of Drosophila melanogaster gene network

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Abstract For Drosophila melanogaster segmentation polarity gene network, relying solely on the analysis of gene logical nodes will pose significant challenges in issues such as reconstructibility and state estimation. Boolean network is an effective mathematical model which describes gene regulation networks. Based on the algebraic expression of Drosophila melanogaster gene network by semi-tensor product (STP) of matrices, the state space of gene network is split into the unreconstructible state set and the reconstructible state set by analyzing the measurable outputs of the system, and the concept of the local reconstructibility is further proposed. Second, for the unreconstructible state set, a pinning control scheme is proposed, where a pinning algorithm is given such that all states in the unreconstructible state set can simultaneously reach the reconstructible state set, and then a sufficient condition for the existence of a common control sequence and solution method are also given. Further, a pinning control-based state estimation method is proposed. Meanwhile, the conclusions are applied to the Drosophila melanogaster segmentation polarity gene network to verify the validity of the methods.

Keywords Boolean networks; semi-tensor product; reconstructibility; state estimation; pinning control; Drosophila melanogaster gene network

1. Introduction

Boolean network (BN) is an important model and can describe the interactions among entities of complex system biology. BN was firstly used to describe the dynamic behavior of genes by Kauffman [1]. Since then, BN has attracted much attention of scholars due to the advantages of simple structure and expanded to other aspects include BCN. Currently, Boolean networks have been successfully applied to gene regulatory networks include Drosophila segment polarity network, lac operon, and T-LGL signaling network. Recently, the matrix semi-tensor products [2-3] have been proposed to transform BN dynamics into discrete-time state space models. With the help of semi-tensor product (STP) of matrices, some excellent results of BNs and BCNs, such as controllability and reachability [4, 5, 6, 7], stability and stabilization [8, 9, 10], reconstructibility and observer design [11, 12, 13, 14, 15, 16, 17, 18, 19], pinning control [20, 21, 22, 23, 24, 25], have been developed.

State estimation is an important topic in control theory. In state-space modeling, the system output can be measured directly, but the state of most systems is unknown or difficult to measure. Therefore, how to uniquely determine the system state based on the obtained data is important to understand and control a system. Drosophila melanogaster is a fully metamorphic insect with the advantages of small size, short life cycle, as well as convenient genetic manipulation and transgenic modalities. Most importantly, the evolutionary conservatism of Drosophila melanogaster offers the possibility of translating Drosophila findings to human disease research. Thus, Drosophila has become one of the most desirable model insects for biological studies. This paper will analyze the state estimation problem of Drosophila melanogaster segment polarity gene network to provide new ideas for its study in biology. The reconstructibility of BNs reveals the ability to uniquely estimate the system state. At present, many excellent methods have been developed to tackle the problems of reconstructibility and state estimation. The reconstructibility was firstly developed in [11]. Then, paper [12] further discussed the reconstructibility of probabilistic BNs. Paper [13] analyzed the relationships between observability and reconstructibility of BCNs, and the reconstructibility of general BCNs was investigated. State estimation issue of stochastic time-varying BNs was investigated in [14]. Paper [15] investigated the optimal state estimation problem of BCNs with stochastic disturbances. The minimum reconstructibility of BCNs was solved in [16]. Paper [17] investigated the issues of state estimation and state estimation-based stabilization for Boolean control networks (BCNs). Recently, the reconstructibility was redefined in [18], where all states were used to determine the reconstructibility of BNs. Observability and reconstructibility criteria of BCNs were obtained via set controllability in [19]. After the outbreak of the COVID-19 pandemic, the development of vaccines became a top priority. And the primary prerequisite for the development of vaccines is the detection of the virus, thus it is particularly important to analyze the reconstructibility and state estimation of gene networks. However, the aforementioned papers do not fully consider the ability of output to estimate the system states. Therefore, this paper will propose a new method for reconstructibility analysis to uniquely determine the state of the Drosophila melanogaster segmentation polarity gene network.

Furthermore, the pinning control strategy aims to control some nodes of the system such that the system implements the desired goal. For example, the p53 network can be forced to enter the apoptotic attractor by controlling only some nodes in the presence of DNA damage [20]. Paper [21] investigated the stabilization problem of BCNs under pinning control. The paper [22] analyzed the single-input pinning control design problem for reachability of BNs, and the design method was extended to BNs under arbitrary disturbance inputs [23]. A pinning control design method for global stabilization of BNs was discussed in [24]. We know from the aforementioned papers that the pinning control technique plays a vital role in field of state feedback stabilization of BNs and BCNs. In this paper, we analyze the reconstructibility and state estimation of the Drosophila melanogaster segmentation polarity gene network by designing a pinning control strategy.

The main contributions of this paper are summarized as follows.

1) The concept of the locally reconstructible is proposed, where the state space of Drosophila melanogaster segmentation polarity gene network is split into the unreconstructible state set and the reconstructible state set.
2) A pinning scheme is designed such that all states in the unreconstructible state set can be simultaneously steered to the reconstructible state set.
3) Considering the pinning scheme for each unreconstructible state set, a sufficient condition of the existence of a common control sequence is provided, and an algorithm is proposed to summarize the pinging controller design.
4) Then a pinning control-based state estimation method is also discussed to solve the state estimation problem of the Drosophila melanogaster segmentation polarity gene network.

2. Preliminaries and Problem Formulation

2.1 Preliminaries

Notations: $\mathcal{D}:=\{0,1\}$, $\mathcal{D}^n=\underbrace{\mathcal{D}\times\mathcal{D}\times\cdots\times\mathcal{D}}_{n}$. $|S|$ is the cardinality of the set S. $\mathbb{Z}_+$ is the set of nonnegative integers, and $\mathcal{M}_{m\times n}$ is a set of $m\times n$ matrices. The elements in $\mathcal{D}$ are identified with a 2-D vector as $T=1\sim\delta_2^1$ and $F=0\sim\delta_2^2$. δ_n^i denotes the i-th column of identity matrix I_n. $\Delta_n:=\left\{\delta_n^i \middle| i=1,\cdots,n\right\}$. $\Delta:=\Delta_2$. $\boldsymbol{1}_{m\times n}:=\underbrace{[1_m,1_m,\cdots,1_m]}_{n}$. $Col_i(A)$ stands for the i-th column of the matrix A and the set of columns of the matrix A, respectively. For an $n\times mn$ matrix A, we split A into n-square blocks with dimension $n\times n$ and denote by $Blk_i(A)$ the i-th block of A, then $Blk(A)$ denotes the set of blocks of matrix A. We denote logical matrix $L=\left[\delta_m^{i_1},\delta_m^{i_2},\cdots,\delta_m^{i_n}\right]$ as $L=\delta_m[i_1,i_2,\cdots,i_n]$. Further, the set of $m\times n$ logical matrices is denoted by $\mathcal{L}_{m\times n}$. Then,

$C := A \ltimes_{\mathcal{B}} B$ denotes Boolean product of $A = (a_{ij}) \in \mathcal{L}_{m\times n}$ and $B = (b_{ij}) \in \mathcal{L}_{m\times n}$, where $C = (c_{ij}) \in \mathcal{L}_{m\times n}$ and $c_{ij} = \left(\sum_{k=1}^{n} a_{ik} \cdot b_{kj}\right)_{\mathcal{B}} \in \mathcal{D}$

Definition 1([2]): The STP of two matrices $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$ is defined as

$$A \ltimes B = \left(A \otimes I_{l/n}\right)\left(B \otimes I_{l/p}\right), \tag{1}$$

where $l = \operatorname{lcm}(n, p)$ is the least common multiple of n and p. "$\otimes$" is the Kronecker product.

Lemma 1([3]): For a Boolean function $f(X_1, \cdots, X_n): \mathcal{D}^n \to \mathcal{D}$, there exists a unique matrix $M_f \in \mathcal{L}_{2\times 2^n}$, called the structure matrix of $f(\cdot)$, such that

$$f(X_1, \cdots, X_n) = M_f \ltimes_{i=1}^{n} x_i \tag{2}$$

A BN is generally described as

$$\begin{cases} x_i(t+1) = f_i\left(x_1(t), x_2(t), \cdots, x_n(t)\right), i = 1, 2, \cdots, n \\ y_j(t) = h_j\left(x_1(t), x_2(t), \cdots, x_n(t)\right), j = 1, 2, \cdots, p \end{cases} \tag{3}$$

where $x_i(t) \in \mathcal{D}$ and $y_j(t) \in \mathcal{D}$ are state and output logical variables, respectively. $f_i(\cdot): \mathcal{D}^n \to \mathcal{D}$ and $h_i(\cdot): \mathcal{D}^n \to \mathcal{D}$ are logical functions. Based on Lemma 1 and Khatri-Rao product [2], the equivalent algebraic form of BN (3) is given as

$$\begin{cases} x(t+1) = L \ltimes x(t), \\ y(t) = H \ltimes x(t), \end{cases} \tag{4}$$

where $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_{2^n}$, $y(t) = \ltimes_{j=1}^{p} y_j(t) \in \Delta_{2^p}$. $L \in L_{2^n \times 2^n}$ and $H \in L_{2^p \times 2^n}$ are network transition matrix and output logical matrix, respectively.

2.2 Problem Formulation

The segment polarity genes play a significant role in the embryonic development of Drosophila. Consider the following Drosophila melanogaster segmentation polarity gene network [25],

$$\begin{cases} x_1(t+1) = x_1(t) \wedge \neg x_2(t) \wedge \neg x_4(t), \\ x_2(t+1) = \neg x_1(t) \wedge x_2(t) \wedge \neg x_3(t), \\ x_3(t+1) = x_1(t) \vee x_3(t), \\ x_4(t+1) = x_2(t) \vee x_4(t), \\ x_5(t+1) = \neg x_2(t) \wedge \neg x_4(t) \vee \left(x_5(t) \wedge \neg x_1(t) \wedge \neg x_3(t)\right), \\ x_6(t+1) = \neg x_1(t) \wedge \neg x_3(t) \vee \left(x_6(t) \wedge \neg x_2(t) \wedge \neg x_4(t)\right), \end{cases} \tag{5}$$

where x_1, x_2, x_3, x_4, x_5, x_6 represent the genes wg_1, wg_2, wg_3, wg_4, PTC_1 and PTC_2, respectively. Then, the output $y(t)$ is given below: $y_1(t) = x_1(t)$, $y_2(t) = x_2(t)$ and $y_3(t) = x_3(t) \vee x_4(t) \vee x_5(t) \vee x_6(t)$. By Lemma 1, the equivalent algebraic form of BN (5) is

$$\begin{cases} x(t+1) = L \ltimes x(t), \\ y(t) = H \ltimes x(t), \end{cases} \tag{6}$$

where the network transition matrix L and output logical matrix H are given as follows.

$$\begin{aligned} L = \delta_{64}[&52,52,52,52,52,52,52,52,52,52,52,52,52,52,52,52, \\ &52,52,52,52,21,22,21,22,52,52,52,52,21,22,21,22, \\ &52,52,52,52,52,52,52,52,41,41,43,43,41,41,43,43, \\ &52,52,52,52,53,54,53,54,57,57,59,59,61,61,61,61], \end{aligned}$$

$$\begin{aligned} H = \delta_8[&1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,4,5,5, \\ &5,5,5,5,5,5,5,5,5,5,5,5,5,5,6,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,8]. \end{aligned}$$

First, this paper will investigate the reconstructibility and state estimation problem of BN (5) based on the system output, and the observer does not need to be designed. Then, for the unreconstructible states in system (5), it is considered whether the unreconstructible states can be steered to a reconstructible state by applying an external signal to the system, namely, pinning control, to realize that the state of the BN (5) is uniquely estimated by the measurable outputs and pinning control technique.

3. Main results

3.1 System analysis

The following two cases are discussed for BN (6).

Given the output matrix H . On the one hand, one can obviously see that the $Col_i\left(H\right)$, $i \in \{16,32,48,64\}$ is different from the other ones. That indicates that the output of system forms one-to-one mapping with the corresponding state in the system. Therefore, if the current output of the system is any one of these four outputs, the system state can be uniquely determined immediately. The state $\delta_{2^n}^i \in \Delta_{2^n}$ are called reconstructible, furthermore, the set of all such states is defined as the reconstructible state set and denoted by S_r

On the other hand, one can see that the output corresponding to the state $\delta_{2^n}^j$, $j \in \{1,2,\cdots,15\}$ is identical, namely δ_8^1 . That means that the state $\delta_{2^n}^j$ cannot be uniquely determined based on the output. Then consider all such outputs $Col_i\left(H\right) = \delta_{2^p}^\tau$, the state $\delta_{2^n}^j \in \Delta_{2^n}$ is called unreconstructible, furthermore, the set of all such states is defined as an unreconstructible state set and denoted by S_u^τ , where $\left|\mathrm{S}_u^\tau\right| \geq 2$. Then we also observe three other similar cases from the output matrix H , namely, $\tau = 3$, $\tau = 5$ 和 $\tau = 7$. Thus, the unreconstructible state set of BN may be not unique.

Next, the sets S_r and S_u^τ of the BN (6) were calculated. By above analysis, one can see that $Col_{16}\left(H\right) = \delta_8^2$, $Col_{32}\left(H\right) = \delta_8^4$, $Col_{48}\left(H\right) = \delta_8^6$ and $Col_{64}\left(H\right) = \delta_8^8$ are different from other columns of H , thus the reconstructible state set of BN (6) is $\mathrm{S}_r = \left\{\delta_{64}^{16}, \delta_{64}^{32}, \delta_{64}^{48}, \delta_{64}^{64}\right\}$. Next, one can obtain that the unreconstructible state set are as follows.

1) If $y\left(t\right) = \delta_8^1$, $\mathrm{S}_u^1 = \left\{\delta_{64}^1, \delta_{64}^2, \delta_{64}^3, \delta_{64}^4, \delta_{64}^5, \delta_{64}^6, \delta_{64}^7, \delta_{64}^8, \delta_{64}^9, \delta_{64}^{10}, \delta_{64}^{11}, \delta_{64}^{12}, \delta_{64}^{13}, \delta_{64}^{14}, \delta_{64}^{15}\right\}$,

2) If $y\left(t\right) = \delta_8^3$, $\mathrm{S}_u^3 = \left\{\delta_{64}^{17}, \delta_{64}^{18}, \delta_{64}^{19}, \delta_{64}^{20}, \delta_{64}^{21}, \delta_{64}^{22}, \delta_{64}^{23}, \delta_{64}^{24}, \delta_{64}^{25}, \delta_{64}^{26}, \delta_{64}^{27}, \delta_{64}^{28}, \delta_{64}^{29}, \delta_{64}^{30}, \delta_{64}^{31}\right\}$,

3) If $y\left(t\right) = \delta_8^5$, $\mathrm{S}_u^5 = \left\{\delta_{64}^{33}, \delta_{64}^{34}, \delta_{64}^{35}, \delta_{64}^{36}, \delta_{64}^{37}, \delta_{64}^{38}, \delta_{64}^{39}, \delta_{64}^{40}, \delta_{64}^{41}, \delta_{64}^{42}, \delta_{64}^{43}, \delta_{64}^{44}, \delta_{64}^{45}, \delta_{64}^{46}, \delta_{64}^{47}\right\}$,

4) If $y\left(t\right) = \delta_8^7$, $\mathrm{S}_u^7 = \left\{\delta_{64}^{49}, \delta_{64}^{50}, \delta_{64}^{51}, \delta_{64}^{52}, \delta_{64}^{53}, \delta_{64}^{54}, \delta_{64}^{55}, \delta_{64}^{56}, \delta_{64}^{57}, \delta_{64}^{58}, \delta_{64}^{59}, \delta_{64}^{60}, \delta_{64}^{61}, \delta_{64}^{62}, \delta_{64}^{63}\right\}$.

Then, we define the union of the unreconstructible state set of BN (6) is $\mathrm{S}_u = \mathrm{S}_u^1 \bigcup \mathrm{S}_u^3 \bigcup \mathrm{S}_u^5 \bigcup \mathrm{S}_u^7$. According to the above analysis, an reconstructible state set S_r and the unreconstructible state sets S_u^τ satisfy $\mathrm{S}_r \bigcup \mathrm{S}_u = \Delta_{2^n}$ and $\mathrm{S}_r \bigcup \mathrm{S}_u = \varnothing$. The concept of the locally reconstructible BN is given below.

Definition 2: Given Boolean networks (3). If the sets $\mathrm{S}_r = \varnothing$ and $\mathrm{S}_u \neq \varnothing$, the BN (3) is locally reconstructible.

According to Definition 2, it is known that the BN (6) is the locally reconstructible. Obviously, the practical meaning of dividing the state space into two categories is equivalent to Definition 4 in [18]. However, different from the reconstructibility in [18] depend on the network transition matrix L and the output matrix H , this paper defines the reconstructibility of BNs directly from the output matrix H . There is no need to design observers and the conclusions are more direct and concise. For the unreconstructible state sets S_u^τ of system (6), the state of the BN still cannot be uniquely determined. Next, this restriction will be broken by using pinning control.

3.2 Pinning control design

In order to make any state in S_u^τ of system (6) be uniquely determined by the outputs, pinning control is imposed on BN (6) such that all states in S_u^τ can simultaneously reach the set S_r. How to select pinning nodes and design the network transition matrix will be given. Suppose that BN (5) with pinning control is described as

$$\begin{cases} x_\xi(t+1) = \varphi_\xi(u(t), x_1(t), \cdots, x_6(t)), \xi \in \{\xi_1, \cdots, \xi_\tau\}, \\ x_\varsigma(t+1) = f_\varsigma(x_1(t), \cdots, x_6(t)), \varsigma \in \{1,2,\cdots,6\} \setminus \{\xi_1, \cdots, \xi_\tau\}, \end{cases} \tag{7}$$

where $\varphi_{\xi_i}(\cdot)$ and $f_\varsigma(\cdot)$ are logical functions. For $\xi_i \in \{1,2,\cdots,6\}$ and $i = 1,2,\cdots,\rho$, x_{ξ_i} are the pinned nodes, where ρ is the number of the controlled nodes. $u(t) \in \Delta$ is the free single control sequence. Based on Lemma 1, the equivalent algebraic form of BN (7) is given as

$$x(t+1) = \bar{L}u(t)x(t) \tag{8}$$

where $\bar{L}$ is network transition matrix. Assume the structure matrix $M_i \in L_{2\times 2^n}$ of the logical function $f_i(\cdot)$, $i = 1,2,\cdots,n$ in BN (5). $\bar{M}_{\xi_i}$ and $\bar{M}_\varsigma$ stand for the structure matrix of the logical function $\varphi_{\xi_i}(\cdot)$ and $f_\varsigma(\cdot)$ of system (7), respectively. Note that paper [5] given the reachability condition from one state to another one. Here, we extend the results of [5] to implement the simultaneous reachability from an unreconstructible state set to a reconstructible state set. Afterward, the termination condition for judging whether a set to a collection is reachable at the same time is given. This is the important step to analyze the reconstructibility of BN (6) and finish state estimation. The reachability matrix is defined as $M = \sum_{i=1}^{2^m} Blk_i(\bar{L})$. Based on the result of [5], the reachability from S_u^τ to S_r is identified by the following Lemma.

Lemma 2 [5]: For BN (8), the set S_r is reachable from S_u^τ at time step s, if for $\forall \delta_{2^n}^j \in S_u^\tau$ and some $\delta_{2^n}^i \in S_r$

$$\left((M^s)_B\right)_{i,j} = 1 \tag{9}$$

holds. Moreover, the set S_r is not reachable from S_u^τ, if there exists a positive integer $k \in \mathbb{Z}_+$ such that

$$\begin{cases} (M^{k+1})_B \in \{(M^1)_B, (M^2)_B, \cdots, (M^k)_B\} \\ ((M^\lambda)_B)_{i,j} = 0, \lambda = 1,2,\cdots,k \end{cases} \tag{10}$$

hold. $A = (M_i), i \in [1,n]$ is a matrix array, where $M_i \in L_{2\times 2^n}$, and C_A^j denotes all combinations of the selected j structural matrices from the array A. We first assume $S_u^\tau = \{\delta_{2^n}^{j_1}, \delta_{2^n}^{j_2}, \cdots, \delta_{2^n}^{j_\eta}\}$ and $S_r = \{\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \cdots, \delta_{2^n}^{i_\mu}\}$, where $\eta + \mu \le 2^n$. Next, Algorithm 1 is provided to choose the pinning nodes $x_{\xi_1}, \cdots, x_{\xi_\rho}$ and compute the network transition matrix $\bar{L}$ of system (8).

Algorithm 1: The construction of Lc and the selection of pinning nodes

Input: network node number n, sets S_r and S_u^τ, M_i and $i = 1,2,\cdots,n$

Output: pinning node and network transition matrix

Procedure:

Step 1：Initialize $j = 1$. $A = \{M_1, M_2, \cdots, M_n\}$. $cel = A$

Step2：Let $t = 1$ and $\Lambda = C_A^j$.

Step3： $B = \Lambda(t,:)$, and let $i = 1$.

Step4：let $k = 1$.

Step5：If $B\{i\} = A\{k\}$, then $\bar{M}_{\xi_k} = \begin{bmatrix} M_k & \mathbf{1}_{2\times 2^n} - M_k \end{bmatrix}$, and $cel\{k\} = \bar{M}_{\xi_k}$, x_{ξ_k} is pinning nodes. Otherwise, go to step 6.

Step6：If $k = n$,go to step 7. Otherwise, $k = k+1$ and go to step 5.

Step7：If $i = size(B)$, go to step 8. Otherwise, $i = i+1$, and go to step 4. Meanwhile, let $q = 1$.

Step8：let $p = 1$.

Step9：If $cel\{q\} = \mathrm{A}\{p\}$, then construct matrix $\bar{M}_{\varsigma} = \begin{bmatrix} M_p & M_p \end{bmatrix}$. $cel\{p\} = \bar{M}_{\varsigma}$ and go to step10.

Step10：If $p = n$, go to step 11. Otherwise, $p = p+1$ and go to step 9.

Step11：If $q = n$, go to step 12. Otherwise, $q = q+1$ and go to step 8.

Step12：Calculate $\bar{L}$ by $Col(\bar{L}) = \hat{a}_{r=1}^{n} Col_i(cel\{r\})$, $i \in [1, 2^n]$, and split it into $\bar{L} = \begin{bmatrix} Blk_1(\bar{L}) & Blk_2(\bar{L}) \end{bmatrix}$. Meanwhile, calculate $M = \sum_{\alpha=1}^{2} Blk_{\alpha}(\bar{L})$, and let $s = 1$.

Step13：Given the set S_r and S_u^{τ}. If the condition (8) holds, output $\bar{L}$ and pinning node, end this algorithm. Otherwise, $s = s+1$ and go to step 14.

Step14：If the condition (8) holds, go to step15. Otherwise, $s = s+1$ and go to step13.

Step15：If $t = size(\Lambda)$, go to step16. Otherwise, $t = t+1$ and go to step 3.

Step16：If $j = n$, end this algorithm. Otherwise, $j = j+1$ and go to step 2.

By Algorithm 1, the designed pinning scheme can simultaneously steer all states in the set S_u^{τ} to the target state set S_r. As few nodes as possible will be chosen such that all states of set S_u^{τ} can simultaneously reach set S_r by Algorithm 1. while minimizing the number of pinning nodes and reducing computational complexity. Next, the feasibility of Algorithm 1 will be verified by introducing pinning control to BN (5). The results of subsection 3.1 show that the reconstructible state set and the unreconstructible state sets of BN (5) are S_r, S_u^1, S_u^3, S_u^5 and S_u^7. First, consider the initial state set S_u^1, the BN (5) is transformed into its equivalent algebraic form by Lemma 1.

$$x_i(t+1) = M_i x(t) \ i = 1, 2, \cdots, 6 \qquad (11)$$

Where

$$\begin{aligned} M_1 = \delta_2[&2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,1,1,1,1,2,2,2,2,1,1,1,1,\\ &2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2], \\ M_2 = \delta_2[&2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,\\ &2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2], \\ M_3 = \delta_2[&1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,\\ &1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2], \\ M_4 = \delta_2[&1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,1,1,1,1,2,2,2,2,\\ &1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,1,1,1,1,2,2,2,2], \\ M_5 = \delta_2[&2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,1,1,1,1,2,2,2,2,1,1,1,1,\\ &2,2,2,2,2,2,2,2,1,1,2,2,1,1,2,2,2,2,2,2,1,1,1,1,1,1,2,2,1,1,1,1], \\ M_6 = \delta_2[&2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,1,2,1,2,2,2,2,2,1,2,1,2,\\ &2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1,2,2,2,2,1,2,1,2,1,1,1,1,1,1,1,1]。 \end{aligned}$$

Then the equation (11) can be further transformed into (8). By Algorithm 1, we know that the appropriate pinning node is x_3 and x_4, namely, $\varsigma_1 = 1$, $\varsigma_2 = 2$, $\xi_3 = 3$, $\xi_4 = 4$, $\varsigma_5 = 5$, $\varsigma_6 = 6$. The structure matrices of node logical functions are obtained as

$$\begin{cases} \bar{M}_{\xi_k} = \begin{bmatrix} M_k & \mathbf{1}_{2\times 2^n} - M_k \end{bmatrix}, \ k = 3, 4 \\ \bar{M}_{\varsigma} = \begin{bmatrix} M_l & M_l \end{bmatrix}, \ l = 1, 2, 5, 6 \end{cases}$$

Hence, one has

$$\begin{aligned}\bar{L} = *_{i=1}^{4}\bar{M}_i = \delta_{64}[&52,52,52,52,52,52,52,52,52,52,52,52,52,52,52,52,\\ &52,52,52,52,21,22,21,22,52,52,52,52,21,22,21,22,\\ &52,52,52,52,52,52,52,52,41,41,43,43,41,41,43,43,\\ &52,52,52,52,53,54,53,54,57,57,59,59,61,61,61,61,\\ &64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,\\ &64,64,64,64,25,26,25,26,64,64,64,64,25,26,25,26,\\ &64,64,64,64,64,64,64,64,37,37,39,39,37,37,39,39,\\ &64,64,64,64,57,58,57,58,53,53,55,55,49,49,49,49]。\end{aligned}$$

The reachability matrix M of (8) is as follow.

$$\begin{aligned}M &= \sum_{i=1}^{2^m} Blk_i\left(\bar{L}\right) = Blk_1\left(\bar{L}\right) + Blk_2\left(\bar{L}\right)\\ &= \delta_{64}[52,52,52,52,52,52,52,52,52,52,52,52,52,52,52,52,\\ &\qquad 52,52,52,52,21,22,21,22,52,52,52,52,21,22,21,22,\\ &\qquad 52,52,52,52,52,52,52,52,41,41,43,43,41,41,43,43,\\ &\qquad 52,52,52,52,53,54,53,54,57,57,59,59,61,61,61,61],\\ &+ \delta_{64}[64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,\\ &\qquad 64,64,64,64,25,26,25,26,64,64,64,64,25,26,25,26\\ &\qquad 64,64,64,64,64,64,64,64,37,37,39,39,37,37,39,39,\\ &\qquad 64,64,64,64,57,58,57,58,53,53,55,55,49,49,49,49]。\end{aligned}$$

So, there have $\left(\left(M^1\right)_{\mathrm{B}}\right)_{64,j} = 1$, $j \in \{1,2,\cdots,15\}$, which implies that all states in S_u^1 can simultaneously reach S_r at $s=2$ based on Lemma 2. Similarly, the reachability from S_u^3, S_u^5 and S_u^7 to S_r can be verified by designing pinning control.

3.3 Control sequence design

In subsection 3.2, although the pinning control can be designed such that all states in S_u^τ can simultaneously reach S_r, the control sequences may be different for different state $x \in \mathrm{S}_u^\tau$. However, we can only determine from the output which the unreconstructible state set the current state is in, not the system state. Therefore, a common control sequence needs to be designed such that all states in S_u^τ can reach S_r. To analyze the existence conditions of the common control sequence and design it, the sets $\mathrm{S}_u^\tau = \left\{\delta_{2^n}^{j_1}, \delta_{2^n}^{j_2}, \cdots, \delta_{2^n}^{j_\eta}\right\}$ and $\mathrm{S}_r = \left\{\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \cdots, \delta_{2^n}^{i_\mu}\right\}$ are still assumed to be given.

Theorem 1: Consider BN (7) with pinning control, a common control sequence $u_s = \left(u(t_0), u(t_1), \cdots, u(t_{s-1})\right)$ can guide all states in S_u^τ reaching S_r at $t = t_s$, if there exist a positive integer number β and $1 \le \beta \le 2^{ms}$ such that for $\forall \delta_{2^n}^{j_\sigma} \in \mathrm{S}_u^\tau$, $\sigma \in [1,\eta]$,

$$Blk_\beta\left(\bar{L}^s\right)\delta_{2^n}^{j_\sigma} \in \mathrm{S}_r \tag{12}$$

holds, where $\bar{L}^s = \hat{a}_{i=0}^{s-1}\left(I_{2^{im}} \otimes \bar{L}\right) \in \mathcal{L}_{2^n \times 2^{n+ms}}$.

Proof. Consider BN (8).

$$x(t+1) = \bar{L}u(t)x(t) \tag{13}$$

For a given common control sequence $u_s = \left(u(t_0), u(t_1), \cdots, u(t_{s-1})\right)$ and $\forall \delta_{2^n}^{j_\sigma} \in \mathrm{S}_u^\tau, \sigma \in [1,\eta]$, one can iteratively obtain

$$\begin{aligned} x(t_s) &= \bar{L}u(t_{s-1})\bar{L}u(t_{s-2})\cdots\bar{L}u(t_0)\delta_{2^n}^{j_\sigma} \\ &= \bar{L}\left(I_{2^m}\otimes\bar{L}\right)\left(I_{2^{2m}}\otimes\bar{L}\right)\cdots\left(I_{2^{(s-1)m}}\otimes\bar{L}\right)\ltimes_{k=s-1}^{0}u(t_k)\delta_{2^n}^{j_\sigma} \\ &= \ltimes_{i=0}^{t_{s-1}}\left(I_{2^{im}}\otimes\bar{L}\right)\ltimes_{k=s-1}^{0}u(t_k)\delta_{2^n}^{j_\sigma}. \end{aligned}$$

where $\bar{L}^s = \ltimes_{i=0}^{t_{s-1}}\left(I_{2^{im}}\otimes\bar{L}\right)$. Then the matrix $\bar{L}^s$ is split into 2^{ms} blocks.

$$\bar{L}^s = \left[Blk_1\left(\bar{L}^s\right) \quad Blk_2\left(\bar{L}^s\right) \quad \cdots \quad Blk_{2^{ms}}\left(\bar{L}^s\right)\right]$$

Then, the condition (12) means that the common control sequence u_s makes the state $\delta_{2^n}^{j_\sigma}$ reach S_r, and the STP of u_s can be calculated by $\ltimes_{k=s-1}^{0}u(t_k)=\delta_{2^{ms}}^{\beta}$. So, the condition (12) guarantees that all states in S_u^τ reach S_r at $t=s$ since the state $\delta_{2^n}^{j_\sigma}\in S_u^\tau$ is arbitrarily given.

For any initial state $x\in S_u^\tau$, the final state of the system (8) will eventually keep switching among many cycles or evolving in some cycle [5]. Thus, there exists a positive integer s such that

$$x = \ltimes_{i=0}^{s-1}\left(I_{2^{im}}\otimes\bar{L}\right)\ltimes_{k=s-1}^{0}u(k)x$$

After all initial states in S_u^τ stabilize in cycles in finite time steps, there exists a positive integer s such that the equation

$$Blk\left(\bar{L}^{s+1}\right)\subseteq\bigcup_{i=1}^{s}Blk\left(\bar{L}^{i}\right) \tag{14}$$

holds. If the common control sequence cannot be obtained before the equation (11) holds, that indicates there is not a common control sequence for the pinning scheme. Next, an algorithm will be provided to compute the common control sequence for $S_u^\tau=\left\{\delta_{2^n}^{j_1},\delta_{2^n}^{j_2},\cdots,\delta_{2^n}^{j_\eta}\right\}$ and $S_r=\left\{\delta_{2^n}^{i_1},\delta_{2^n}^{i_2},\cdots,\delta_{2^n}^{i_\mu}\right\}$. First, let us assume that the pinning nodes and $\bar{L}$ have been obtain from Algorithm 1. u_τ and s are the STP and length of control sequence, respectively.

Algorithm 2:

Input: $\bar{L}$, S_u^τ, S_r

Output: u_τ, s

Procedure:

Step 1: Initialize $s=1$.

Step 2: Let $\beta=1$ and calculate $\bar{L}^s=\ltimes_{i=0}^{s-1}\left(I_{2^{im}}\otimes\bar{L}\right)$, then split $\bar{L}^s$ into a block matrix

$$\bar{L}^s = \left[Blk_1\left(\bar{L}^s\right) \quad Blk_2\left(\bar{L}^s\right) \quad \cdots \quad Blk_{2^{ms}}\left(\bar{L}^s\right)\right]$$

Step 3: If we can find a positive integer β, $1\le\beta\le 2^{ms}$, such that $\forall\delta_{2^n}^{j_\sigma}\in S_u^\tau,\sigma\in[1,\eta]$, $Blk_\beta\left(\bar{L}^s\right)\delta_{2^n}^{j_\sigma}\in S_r$ holds, then go to step 5. Otherwise, go to step 4.

Step 4: If $Blk\left(\bar{L}^{s+1}\right)\subseteq\bigcup_{i=1}^{s}Blk\left(\bar{L}^{i}\right)$, end this Algorithm. Otherwise, $s=s+1$ and go to step 2.

Step 5: Let $u_\tau=\ltimes_{k=s-1}^{0}u(t_k)=\delta_{2^{ms}}^{\beta}$, output u_τ and s, and end this Algorithm.

Next, the control sequence u_s for the set S_u^τ in BN (6) are designed. Considering the pinning control scheme in Subsection 3.2, the common control sequence is calculated by Algorithm 2.

Let $s=1$, calculate $\bar{L}^1$ and split it into $2^{ms}=2$ blocks.

$$\bar{L}^1 = \left[Blk_1\left(\bar{L}^s\right) \quad Blk_2\left(\bar{L}^s\right)\right]$$

Where

$$Blk_1\left(\overline{\boldsymbol{L}}^1\right)=\delta_{64}[52,52,52,52,52,52,52,52,52,52,52,52,52,52,52,52,\\ 52,52,52,52,21,22,21,22,52,52,52,52,21,22,21,22,\\ 52,52,52,52,52,52,52,52,41,41,43,43,41,41,43,43,\\ 52,52,52,52,53,54,53,54,57,57,59,59,61,61,61,61],$$

$$Blk_2\left(\overline{\boldsymbol{L}}^1\right)=\delta_{64}[64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,\\ 64,64,64,64,25,26,25,26,64,64,64,64,25,26,25,26,\\ 64,64,64,64,64,64,64,64,37,37,39,39,37,37,39,39,\\ 64,64,64,64,57,58,57,58,53,53,55,55,49,49,49,49].$$

Obviously, if $\beta=2$, the condition $Blk_2\left(\overline{\boldsymbol{L}}^1\right)\delta_{64}^j\in S_r$ holds for $j\in\{1,2,\cdots,15\}$. Thus, $u_\tau=\hat{a}_{k=s-1}^{0}u(t_k)=\delta_{2^{ms}}^{\beta}$. Then, the common control sequence is $u_s=\left(\delta_2^2\right)$. All states in set S_u^1 can simultaneously reach the set S_r under u_s. Similarly, for other S_u^τ of BN (6), their common control sequences can be calculated according to the above proposed method.

4. Pinning control-based state estimation

For BN (5), a new method about state estimation is proposed. Consider the initial time t_0, first, the state $x(t_0)\in S_r$ and its corresponding output is one to one. Thus, the state $x(t_0)\in S_r$ can be uniquely determined from output $y(t_0)$. Hereafter, the state can be derived from $x(t+1)=Lx(t)$ for $t\ge t_0$. Second, if the current state $x(t_0)\in S_u^\tau$ is determined based on the output $y(t_0)$, then the pinning control is introduced into BN (5). The pinning scheme and common control sequence u_s are designed by Algorithm 1 and Algorithm 2. Then, all states in set S_u^τ of BN (5) will reach S_r at $t=t_s$. The state estimator of BN (5) for $\forall x(t_0)\in S_u^\tau$ is as follow.

$$\hat{x}(t+1)=\begin{cases}\overline{L}u_s(t)x(t), & t_0\le t\le t_{s-1},\\ L\hat{x}(t), & t\ge t_s,\end{cases}\tag{15}$$

The states in set S_u^τ of BN (5) reach S_r at $t=t_{s-1}$ while the controller is removed at t_{s-1}. Then the state is successfully estimated by output $y(t_{s-1})$. Hereafter, $x(t+1)=Lx(t)$ for $t\ge t_s$ .The network state will be uniquely estimated by pinning control sequence u_s at time $t=s$.

Next, the state estimation process for the BN (6) is discussed. Assume that output trajectory of a state is as follow.

$\left(\delta_8^8,\delta_8^7,\delta_8^7,\delta_8^7,\delta_8^7,\delta_8^7,\cdots\right)$

One can see the output $y(t_0)=\delta_8^8$ at the initial time step t_0. Thus, the initial state $x(t_0)=\delta_{64}^{64}$ can be uniquely determined by the system output. Similarly, if the output $y(t_0)=\delta_8^2$, δ_8^4 and δ_8^6, the corresponding states and the outputs are one-to-one, thus the states of the system (6) can also be uniquely determined. The above network state can be estimated based on the output.

If the output $y(t_0)=\delta_8^7$, then the state $x(t_0)\in S_u^7$ cannot be uniquely determined. Then pinning control is introduced into the system. The common control sequence $u_s=\left(\delta_2^2\right)$ is obtained by Algorithm 2, this means that the control sequence u_s is applied to the BN (6) at time t_0, then all states in S_u^1 can simultaneously reach S_r after one time steps, and thus the state of the BN can be uniquely estimated using the estimator (12).

5. Conclusion

This paper discusses the reconstructibility and state estimation problems of Drosophila melanogaster segmentation polarity gene network by system output. The concept of the locally reconstructible BN is proposed by system analysis. Additionally, for the unreconstructible state set, an algorithm is proposed to select pinning nodes and construct the network transition matrix such that all states in the unreconstructible state sets can simultaneously reach the reconstructible state set. Furthermore, a sufficient condition for the existence of a common control sequence and design algorithms are provided. Then, a state estimation method based on pinning control is provided to analyze the state estimation problem of Drosophila melanogaster segmentation polarity gene network. How to reduce the computational complexity will become our next consideration.

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