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Research Article

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The Coefficient Problem for the Certain Subclass of Bi-Univalent Functions

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Abstract: In this paper, defined a new subclass of bi-univalent functions. We examined some geometric properties such that coefficient estimates and Fekete-Sezöge problem for this defined class.

Keywords: Starlike function, convex function, bi-univalent function, pseudo-starlike function, pseudo-convex function

1. Introduction

In this study by $H(U)$ denoted the class of analytic functions on the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ in the complex plane $\mathbb{C}$. By A denoted the class of the functions $f\in H(U)$ given by series expansion

$$f(z)=z+a_2z^2+a_3z^3+a_4z^4+\cdots+a_nz^n+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}\ . \tag{1.1}$$

The subclass of A, which are univalent functions in U is denoted by S in the literature. Bieberbach published a paper [2] in which the coefficient hypothesis was proposed. This hypothesis states that if $f\in S$, then $|a_n|\le n$ for each $n\ge 2.$ On this subject are many articles in the literature (see [3-14]).

It is known that the function f is called bi-univalent function, if itself and inverse is univalent in U and $f(U)$, respectively. The class of bi-univalent functions in U is denoted by Σ in the literature.

For the inverse $g(w)=f^{-1}(w)$ of the function $f\in\Sigma$, can written

$$g(w)=w+A_2w^2+A_3w^3+A_4w^4+\ldots\ =w+\sum_{n=2}^{\infty}A_nz^n,\ w\in f(U), \tag{1.2}$$

where

$$A_2=-a_2,\ A_3=2a_2^2-a_3,\ A_4=-a_2^3+5a_2a_3-a_4,\ldots\ .$$

The starlike and bi-starlike function classes in the open unit disk U are defined analytically as follows and denoted by S^* and S^*_Σ, respectively

$$S^*=\left\{f\in S:\ \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in U\right\},$$

$$S^*_\Sigma=\left\{f\in S:\ \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in U \text{ and } \operatorname{Re}\left(\frac{wg'(w)}{g(w)}\right)>0,\ w\in f(U)\right\}.$$

Similarly the convex and bi-convex function classes in the open unit disk U are defined analytically as follows

$$C=\left\{ f\in S:\ \operatorname{Re}\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)>0,\ z\in U\right\},$$

$$C_{\Sigma}\left\{ f\in S:\ \operatorname{Re}\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)>0,\ z\in U \text{ and } \operatorname{Re}\left(\frac{\left(wg'(w)\right)'}{g'(w)}\right)>0,\ w\in f(U)\right\}$$

and denoted by C and C_{Σ}, respectively.

Let's $f,g\in H(U)$, then it is said that f is subordinate to g and denoted by $f\prec g$, if there exists a Schwartz function ω, such that $f(z)=g(\omega(z))$.

In the past few years, numerous subclasses of the class S have been introduced as special choices of the class $S^{*}(\varphi)$, C and S_{Σ}^{*}, C_{Σ} (see for example [4, 8-14, 16-22]).

2. Materials and Methods

Now, we will define new classes of functions and give some necessary lemmas for the proof of our main results.

Definition 2.1. For $\beta\in[0,1]$ and $\lambda>\frac{1}{2}$ the function $f\in\Sigma$ is said to be in the class $\chi_{\Sigma,\sinh}(\beta,\lambda)$, if the following conditions are satisfied

$$(1-\beta)\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}+\beta\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}\prec 1+\sinh z,\ z\in U \text{ and}$$

$$(1-\beta)\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}+\beta\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}\prec 1+\sinh w,\ w\in f(U).$$

In the cases $\beta=0$, $\beta=1$ and $\lambda=1$ from the Definition 2.1, we have the following classes.

Definition 2.2. For $\lambda>\frac{1}{2}$ the function $f\in\Sigma$ is said to be in the class $S_{\Sigma,\sinh}^{*}(\lambda)$, if the following conditions are satisfied

$$\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\prec 1+\sinh z,\ z\in U \text{ and } \frac{w\left(g'(w)\right)^{\lambda}}{g(w)}\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.3. For $\lambda>\frac{1}{2}$ the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}(\lambda)$, if the following conditions are satisfied

$$\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}\prec 1+\sinh z,\ z\in U \text{ and } \frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.4. For $\beta\in[0,1]$ the function $f\in\Sigma$ is said to be in the class $\chi_{\Sigma,\sinh}(\beta)$, if the following conditions are satisfied

$$(1-\beta)\frac{zf'(z)}{f(z)}+\beta\frac{(zf'(z))'}{f'(z)}\prec 1+\sinh z,\ z\in U \text{ and}$$

$$(1-\beta)\frac{wg'(w)}{g(w)}+\beta\frac{(wg'(w))'}{g'(w)}\prec 1+\sinh w,\ w\in f(U).$$

Let P be the class of analytic functions in U satisfied the conditions $p(0)=1$ and $\mathrm{Re}(p(z))>0,\ z\in U.$ It is clear that functions that satisfy these conditions have the following series expansion

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U. \tag{2.1}$$

The class P defined above is known as the class Caratheodory functions [23] in the literature.

Lemma 2.1 ([24]). Let the function p belong to the class P. Then,

$$|p_n|\le 2 \text{ for each } n\in\mathbb{N},\ |p_n-\nu p_k p_{n-k}|\le 2 \text{ for } n,\ k\in\mathbb{N},\ n>k \text{ and } \nu\in[0,1].$$

The equalities hold for the function

$$p(z)=\frac{1+z}{1-z}.$$

Lemma 2.2 ([24]) Let the an analytic function p be of the form (2.1), then

$$2p_2=p_1^2+(4-p_1^2)x,$$

$$4p_3=p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2+2(4-p_1^2)(1-|x|^2)y$$

for some $x,y\in\mathbb{C}$ with $|x|\le 1$ and $|y|\le 1$.

In this paper, we give some coefficient estimates and solve Fekete-Szegö problem for the class $\chi_{\Sigma,\sinh}(\beta,\lambda)$. Additionally, the results obtained for specific values of the parameters in our study are compared with the results available in the literature.

3. Results & Discussion

In this section of our study, we give some coefficient bound estimates for the functions belonging to the class $\chi_{\Sigma,\sinh}(\beta,\lambda)$ and solve Fekete-zegö problem for this class.

Theorem 3.1. Let the function f given by series expansions (1.1) belong to the class $\chi_{\Sigma,\sinh}(\beta,\lambda)$. Then, we have the following estimates

$$|a_2|\le\frac{1}{(2\lambda-1)(1+\beta)} \text{ and } |a_3|\le\begin{cases}\dfrac{1}{(3\lambda-1)(1+2\beta)} & \text{if } (3\lambda-1)(1+2\beta)\le(2\lambda-1)^2(1+\beta)^2,\\[2ex] \dfrac{1}{(2\lambda-1)^2(1+\beta)^2} & \text{if } (3\lambda-1)(1+2\beta)\ge(2\lambda-1)^2(1+\beta)^2.\end{cases} \tag{3.1}$$

Obtained here results are sharp.

Proof. Let $f\in\chi_{\Sigma,\sinh}(\beta,\lambda)$, then exists Schwartz functions $\omega:U\to U,\varpi:U_{r_0}\to U_{r_0}$, such that

$$(1-\beta)\frac{z(f'(z))^{\lambda}}{f(z)}+\beta\frac{\left[(zf'(z))'\right]^{\lambda}}{f'(z)}=1+\sinh\omega(z),\ z\in U \text{ and}$$

$$(1-\beta)\frac{w(g'(w))^{\lambda}}{g(w)}+\beta\frac{\left[(wg'(w))'\right]^{\lambda}}{g'(w)}=1+\sinh\varpi(w),w\in f(U). \quad (3.2)$$

Let's the functions $p,q\in P$ defined as follows:

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_2z^2+p_3z^3+...=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U,$$

$$q(w)=\frac{1+\varpi(w)}{1-\varpi(w)}=1+q_1w+q_2w^2+q_3w^3+...=1+\sum_{n=1}^{\infty}q_nw^n,\ w\in f(U). \quad (3.3)$$

It follows from that

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{p_1}{2}z+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)z^2+\frac{1}{2}\left(p_3-p_1p_2-\frac{p_1^3}{4}\right)z^3...,\ z\in U,$$

$$\varpi(w)=\frac{q(w)-1}{q(w)+1}=\frac{q_1}{2}w+\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)w^2+\frac{1}{2}\left(q_3-q_1q_2-\frac{q_1^3}{4}\right)w^3...,w\in f(U). \quad (3.4)$$

From the (3.2) and (3.4) we can written

$$\begin{aligned}&(1-\beta)\left\{1+a_2(2\lambda-1)z+\left((3\lambda-1)a_3+\left(2\lambda^2-4\lambda+1\right)a_2^2\right)z^2+\cdots\right\}\\&+\beta\left\{1+2a_2(2\lambda-1)z+\left(3(3\lambda-1)a_3+\left(8\lambda^2-16\lambda+4\right)a_2^2\right)z^2+\cdots\right\}\\&=1+\frac{p_1}{2}z+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\cdots,\ z\in U,\end{aligned}$$

$$\begin{aligned}&(1-\beta)\left\{1+A_2(2\lambda-1)w+\left((3\lambda-1)A_3+\left(2\lambda^2-4\lambda+1\right)A_2^2\right)w^2+\cdots\right\}\\&+\beta\left\{1+2A_2(2\lambda-1)w+\left(3(3\lambda-1)A_3+\left(8\lambda^2-16\lambda+4\right)A_2^2\right)w^2+\cdots\right\} \quad (3.5)\\&=1+\frac{q_1}{2}w+\left(\frac{q_2}{2}-\frac{q_1^2}{4}\right)w^2+\cdots,\ w\in f(U).\end{aligned}$$

Comparing the coefficients of the same degree terms on the right and left sides of the equalities (3.5), we obtain the following equalities

$$a_2=\frac{p_1}{1+\beta},\ \frac{5}{4}(1+2\beta)a_3-\frac{7}{8}(1+3\beta)a_2^2=\frac{p_2}{2}-\frac{p_1^2}{4}, \quad (3.6)$$

$$a_2=-\frac{q_1}{1+\beta},\ \frac{5}{4}(1+2\beta)\left(2a_2^2-a_3\right)-\frac{7}{8}(1+3\beta)a_2^2=\frac{q_2}{2}-\frac{q_1^2}{4}. \quad (3.7)$$

Then,

$$\frac{p_1}{1+\beta}=a_2=-\frac{q_1}{1+\beta};\ \text{that is, }\ p_1=-q_1. \quad (3.8)$$

Applying Lemma 2.1 to the equality (3.8), we have first inequality of theorem.

Considering the second equality of (3.8), from the equalities (3.6) and (3.7) we obtain

$$a_3=\frac{1}{4(2\lambda-1)^2(1+\beta)^2}p_1^2+\frac{p_2-q_2}{4(3\lambda-1)(1+2\beta)}. \quad (3.9)$$

For some $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$ from the Lemma 2.2, we can write

$$p_2 - q_2 = \frac{4 - p_1^2}{2}(x - y).$$

Substitute this expression for the difference $p_2 - q_2$ in the equality (3.9), we get

$$a_3 = \frac{1}{4(2\lambda-1)^2(1+\beta)^2} p_1^2 + \frac{4 - p_1^2}{8(3\lambda-1)(1+2\beta)}(x - y). \quad (3.10)$$

Applying triangle inequality to the last equality, we obtain

$$|a_3| \le \frac{t^2}{4(2\lambda-1)^2(1+\beta)^2} + \frac{4 - t^2}{8(3\lambda-1)(1+2\beta)}(\xi + \eta), \ (\xi, \eta) \in [0,1]^2, \quad (3.11)$$

where $\xi = |x|$, $\eta = |y|$ and $t = |p_1|$. From here can written

$$|a_3| \le \frac{1}{4}\left[a(\lambda, \beta) t^2 + \frac{4}{(3\lambda-1)(1+2\beta)} \right], \ t \in [0,2], \quad (3.12)$$

where $a(\lambda, \beta) = \frac{(3\lambda-1)(1+2\beta) - (2\lambda-1)^2(1+\beta)^2}{(2\lambda-1)^2(1+\beta)^2(3\lambda-1)(1+2\beta)}$.

Then, maximizing the function

$$\chi(t) = a(\lambda, \beta) t^2 + \frac{4}{(3\lambda-1)(1+2\beta)},$$

it can easily be seen that $\chi(t) \le \frac{4}{(3\lambda-1)(1+2\beta)}$ if $a(\lambda, \beta) \le 0$ and $\chi(t) \le \frac{4}{(2\lambda-1)^2(1+\beta)^2}$ if $a(\lambda, \beta) \ge 0.$

Thus, the proof of second inequality of (3.1) is provided.

The result of theorem is sharp for the function

$$f_1(z) = z + \frac{z^2}{(2\lambda-1)(1+\beta)} + \frac{z^3}{(3\lambda-1)(1+2\beta)}, \ z \in U$$

in the case $(3\lambda-1)(1+2\beta) \le (2\lambda-1)^2(1+\beta)^2$ and for the function

$$f_2(z) = z + \frac{z^2}{(2\lambda-1)(1+\beta)} + \frac{z^3}{(2\lambda-1)^2(1+\beta)^2}, \ z \in U$$

in the case $(3\lambda-1)(1+2\beta) \ge (2\lambda-1)^2(1+\beta)^2$.

With this, the proof of theorem is completed.

Taking $\beta = 0$, $\beta = 1$ and $\lambda = 1$ in the Theorem 3.1, we obtain the following results, respectively.

Corollary 3.1. If $f \in S^*_{\Sigma,\sinh}(\lambda)$, then

$$|a_2| \le \frac{1}{2\lambda-1} \text{ and } |a_3| \le \begin{cases} \frac{1}{(2\lambda-1)^2} & if \ \lambda \in \left(\frac{1}{2}, \frac{7+\sqrt{17}}{8}\right], \\ \frac{1}{3\lambda-1} & if \qquad \lambda \ge \frac{7+\sqrt{17}}{8}. \end{cases}$$

Corollary 3.2. If $f \in C_{\Sigma,\sinh}(\lambda)$, then

$$|a_2| \le \frac{1}{2(2\lambda-1)} \text{ and } |a_3| \le \begin{cases} \frac{1}{4(2\lambda-1)^2} & if \ \lambda \in \left(\frac{1}{2}, \frac{25+\sqrt{117}}{32}\right], \\ \frac{1}{3(3\lambda-1)} & if \qquad \lambda \ge \frac{25+\sqrt{117}}{32}. \end{cases}$$

Corollary 3.3. If $f \in \chi_{\Sigma,\sinh}(\beta)$, then

$$|a_2| \le \frac{1}{1+\beta} \text{ and } |a_3| \le \frac{1}{(1+\beta)^2}.$$

Now, we give the following theorem on the Fekete-Szegö problem for the class $\chi_{\Sigma,\sinh}(\beta,\lambda)$.

Theorem 3.2. Let $f \in \chi_{\Sigma,\sinh}(\beta,\lambda)$ and $\mu \in \mathbb{C}$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{1}{(3\lambda-1)(1+2\beta)} & if \ |1-\mu| \le l(\lambda,\beta), \\ \frac{|1-\mu|}{(2\lambda-1)^2(1+\beta)^2} & if \ |1-\mu| \ge l(\lambda,\beta), \end{cases} \tag{3.13}$$

Proof. Let $f \in \chi_{\Sigma,\sinh}(\beta,\lambda)$, then from the equalities (3.8) and (3.10), we can write

$$a_3 - \mu a_2^2 = \frac{(1-\mu)p_1^2}{4(2\lambda-1)^2(1+\beta)^2} + \frac{(4-p_1^2)(x-y)}{8(3\lambda-1)(1+2\beta)} \tag{3.14}$$

for some $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$.

Applying triangle inequality to the equality (3.14), we obtain

$$|a_3 - \mu a_2^2| \le \frac{|1-\mu| t^2}{4(2\lambda-1)^2(1+\beta)^2} + \frac{(4-t^2)(\xi+\eta)}{8(3\lambda-1)(1+2\beta)}, \ \xi,\eta \in [0,1],$$

where $\xi = |x|$, $\eta = |y|$ and $t = |p_1|$.

From the last inequality, easily can written

$$|a_3 - \mu a_2^2| \le \frac{\left[|1-\mu| - l(\lambda,\beta)\right]}{4(2\lambda-1)^2(1+\beta)^2} t^2 + \frac{1}{(3\lambda-1)(1+2\beta)}, \ t \in [0,2], \tag{3.15}$$

where $l(\lambda,\beta) = \frac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}$.

Maximizing the function $\varphi : [0,2] \to \mathbb{R}$ defined as follows

$$\varphi(t)=\frac{\left[|1-\mu|-l(\lambda,\beta)\right]}{4(2\lambda-1)^2(1+\beta)^2}t^2+\frac{1}{(3\lambda-1)(1+2\beta)},\ t\in[0,2],$$

we can easily see that $\varphi(t)\le\frac{1}{(3\lambda-1)(1+2\beta)}$ if $|1-\mu|\le l(\lambda,\beta)$ and $\varphi(t)\le\frac{|1-\mu|}{(2\lambda-1)^2(1+\beta)^2}$

if $|1-\mu|\ge l(\lambda,\beta)$.

Thus, the proof of theorem is completed.

Taking $\beta=0$, $\beta=1$ and $\lambda=1$ in the Theorem 3.2, we obtain the following results, respectively.

Corollary 3.4. If $f\in S^*_{\Sigma,\sinh}(\lambda)$ and $\mu\in\mathbb{C}$, then

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{1}{3\lambda-1} & if\ |1-\mu|\le\frac{(2\lambda-1)^2}{(3\lambda-1)},\\ \frac{|1-\mu|}{(2\lambda-1)^2} & if\ |1-\mu|\ge\frac{(2\lambda-1)^2}{(3\lambda-1)}.\end{cases}$$

Corollary 3.5. If $f\in C_{\Sigma,\sinh}(\lambda)$ and $\mu\in\mathbb{C}$, then

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{1}{3(3\lambda-1)} & if\ |1-\mu|\le\frac{4(2\lambda-1)^2}{3(3\lambda-1)},\\ \frac{|1-\mu|}{4(2\lambda-1)^2} & if\ |1-\mu|\ge\frac{4(2\lambda-1)^2}{3(3\lambda-1)}.\end{cases}$$

Corollary 3.6. If $f\in\chi_{\Sigma,\sinh}(\beta)$ and $\mu\in\mathbb{C}$, then

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{1}{2(1+2\beta)} & if\ |1-\mu|\le\frac{(1+\beta)^2}{2(1+2\beta)},\\ \frac{|1-\mu|}{(1+\beta)^2} & if\ |1-\mu|\ge\frac{(1+\beta)^2}{2(1+2\beta)}.\end{cases}$$

Also, taking $\mu=0$ and $\mu=1$ in the Theorem 3.2, we obtain the following results, respectively.

Corollary 3.7. If $f\in\chi_{\Sigma,\sinh}(\beta,\lambda)$, then

$$|a_3|\le\begin{cases}\frac{1}{(3\lambda-1)(1+2\beta)} & if\ (3\lambda-1)(1+2\beta)\le(2\lambda-1)^2(1+\beta)^2,\\ \frac{1}{(2\lambda-1)^2(1+\beta)^2} & if\ (3\lambda-1)(1+2\beta)\ge(2\lambda-1)^2(1+\beta)^2.\end{cases}$$

Corollary 3.8. If $f\in\chi_{\Sigma,\sinh}(\beta,\lambda)$, then

$$|a_3-a_2^2|\le\frac{1}{(3\lambda-1)(1+2\beta)}.$$

Remark 3.1. We note that Corollary 3.7 confirms the second result of Theorem 3.1.
In the case $\mu\in\mathbb{R}$, we can prove the following theorem similarly.

Theorem 3.3. Let $f \in \chi_{\Sigma,\sinh}(\beta,\lambda)$ and $\mu \in \mathbb{R}$. Then,

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{1}{(3\lambda-1)(1+2\beta)} & if \;\; \mu \in \left[1-l(\lambda,\beta), 1+l(\lambda,\beta)\right], \\ \dfrac{|1-\mu|}{(2\lambda-1)^2(1+\beta)^2} & if \;\; \begin{cases} \mu \le 1-l(\lambda,\beta) \;\; or \\ \mu \le 1+l(\lambda,\beta), \end{cases} \end{cases}$$

where $l(\lambda,\beta)=\dfrac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}$.

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