*Available online www.jsaer.com*

*Journal of Scientific and Engineering Research*, **2024, 11(11):21-29**

**Research Article**

**ISSN: 2394-2630**
**CODEN(USA): JSERBR**

# On the Fixed Point Property for Nonexpansive Mappings on Large Classes in Some Banach Spaces in Close Relation to α-duals of Certain Difference Sequence Spaces

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**Abstract:** In 2000, Et and Esi introduced new type of generalized difference sequences by using the structure of Çolak's work from 1989 where he defined new types of sequence spaces while Çolak was also inspired by Kızmaz's idea about the difference operator he studied in 1981. Then, using Et and Esi's structure, Ansari and Chaudhry, in 2012, introduced a new type of generalized difference sequence spaces. Changing Ansari and Chaudhry's construction slightly, Et and Işık, in 2012, obtained new type of generalized difference sequence spaces which have equivalent norm to that of Ansari and Chaudhry's type Banach spaces. Then, Et and Işık found α-duals of the Banach spaces they got and investigated geometric properties for them. In this study, we consider α-duals of Et and Işık's generalized difference sequence spaces but we study some Banach spaces closely related with those such that they are degenerate Lorentz-Marcinkiewicz spaces. We take those Banach spaces related with the α-duals of Et and Işık's generalized difference sequence spaces in terms of fixed point theory and find large classes of closed, bounded and convex subsets in those with fixed point property for nonexpansive mappings.

**Keywords:** Nonexpansive Mapping, Fixed Point Property, Difference Sequences, α-duals.

## 1. Introduction

Researches have shown that the fixed point exists for some function classes defined on certain classes of sets in some spaces, while it cannot be found at all in others. Fixed point theory has examined how this happens or does not happen.

Researchers have made classifications and characterizations. In 1965, Browder [4] proved that every Hilbert space has a property satisfying that every nonexpansive mapping defined on any closed, bounded, and convex (cbc) nonempty subset domain with the same range has a fixed point. Since that time, spaces with this property have been considered to have the fixed point property for nonexpansive mappings (fppne). Then, researchers considered looking for the spaces with the property and if the property still exists when larger classes of mappings are taken. Then also they have seen spaces failing the properties. For example, in 1965, Browder [5] and Göhde [16] with independent studies, they saw that uniform convex Banach spaces have the fppne. Then, Kirk [19] generalized the result for the reflexive Banach spaces with normal structure. In fact, Goebel and Kirk [13] noticed that Kirk's result was able to extend for uniformly Lipschitz mappings and some researchers have studied estimating the Lipschitz coefficient satisfying the property for uniform Lipschitz mappings on different Banach spaces. For example, Goebel and Kirk [14] showed that for Hilbert spaces, the best Lipschitz coefficient would be a scalar less than a number in the interval $\left[\sqrt{2}\ ,\ \ \frac{\pi}{2}\right]$, and Goebel and Kirk [13] and Lim [20] showed independently that for a Lebesgue space $L^p$ when $2 < p < \infty$, the coefficient is smaller by a scalar larger than or equal to $\left(1+\frac{1}{2^p}\right)^{\frac{1}{p}}$ while Alspach [1] showed that when $p = 2$, there exists a fixed point free Lipschitz mapping with Lispchitz coefficient $\sqrt{2}$ defined on a cbc subset. In fact, $\sqrt{2}$ is the smallest Lipschitz coefficient for Alspach's mapping. We need to note that, similar to the definition of the Banach spaces satisfying the fppne, if a Banach space has a property that every uniformly Lipschitz mapping defined on any cbc nonempty subset domain with the same range has a fixed point, then that Banach space has the fixed point property for uniformly Lischitz mapping (fppul). In terms of fixed point property for uniformly Lipschitz mappings, Dowling, Lennard, and Turett [7] showed that if a Banach space contains an isomorphic copy of $\ell^1$, then it fails the fppul. It is a well-known fact by researchers that $c_0$ or $\ell^1$ is almost isometrically embedded in every non-reflexive Banach space with an unconditional basis (see [23]). For this reason, classical non-reflexive Banach spaces fail the fixed point property for non-expansive mappings, that is, in these spaces, there can be a closed, convex and bounded subset and a non-expansive invariant $T$ mapping defined on that set such that $T$ has no fixed point. This result is based on well-known theorems in literature (see for example Theorem 1.c.12 in [23] and Theorem 1.c.5 in [24]). These theorems state that for a Banach lattice or Banach space with an unconditional basis to be reflexive, it is necessary and sufficient if it does not contain any isomorphic copies of $c_0$ or $\ell^1$. Therefore, this close relation to the reflexivity or nonreflexivity of Banach space, researchers have worked for years and questioned whether $c_0$ or $\ell^1$ can be renormed to have a fixed point for nonexpansive mappings. Lin [21] showed in his study that what was thought was not true and that at least $\ell^1$ could be renormed to have the fixed point property for nonexpansive mappings. Then, the remaining question was if the same could have been done for $c_0$, but the answer still remains open. Since the researchers have considered trying to obtain the analogous results for well-known other classical nonreflexive Banach spaces, another experiment was done for Lebesgue integrable functions space $L_1[0,1]$ by Hernandes-Lineares and Maria [22] but they were able to obtain the positive answer when they restricted the nonexpansive mappings by assuming they were affine as well. One can say that there is no doubt most research has been inspired by the ideas of the study [15] where Goebel and Kuczumow proved that while $\ell^1$ fails the fixed point property since one can easily find a cbc nonweakly compact subset there and a fixed point free invariant nonexpansive map, it is possible to find a very large class subsets in target such that invariant nonexpansive mappings defined on the members of the class have fixed points. In fact, it is easy to notice the traces of those ideas in [21] work. Even Goebel and Kuczumow's work has inspired many other researchers to investigate if there exist more example of nonreflexive Banach spaces with large classes satisfying fixed point property. For example, in 2004, Kaczor and Prus [17] wanted to generalize Goebel and Kuczumow's findings and they proved that affine asymptotically nonexpansive invariant mappings defined on a large class of cbc subsets in $\ell^1$ can have fixed points. Moreover, in [12], Kaczor and Prus' results were extended by having been found larger classes satisfying the fixed point property for affine asymptotically nonexpansive mappings. Thus, affinity condition became a tool for their works. In fact, another well-known nonreflexive Banach space, Lebesgue space $L_1[0,1]$, was studied in [22] and in their study they obtained an analogous result to [21] as they showed that $L_1[0,1]$ can be renormed to have the fixed point property for affine nonexpansive mappings. In this study, we will investigate some Banach spaces analogous to $\ell^1$. We actually consider $\alpha$-duals of Et and Işık's generalized difference sequence spaces but we study some Banach spaces closely related with those such that they are degenerate Lorentz-Marcinkiewicz spaces. We take those Banach spaces related with the $\alpha$-duals of Et and Işık's generalized difference sequence spaces and study Goebel-Kuczumow analogy for them. We prove that very large classes of closed, bounded and convex subsets in some Banach spaces which are closely related with $\alpha$-duals of their generalized difference sequence spaces investigated by Et and Işık [11] and actually are degenerate Lorentz-Marcinkiewicz spaces have the fixed point property for nonexpansive mappings. Therefore, firstly we would like to give the definition of Cesàro sequence spaces which was defined by Shiue [28] in 1970, and next we present Kızmaz's difference sequence space definition in [18] by noting that we work on a space which is derived from his ideas' generalizations such that many researchers (see for example [6, 8, 9, 10, 25, 28]) have generalized his work as well.

In fact, we need to note that Et and Esi's work [10] and Et and Çolak's work [9] used a common difference sequence definition from Çolak's work [6].

Now, first we recall that Shiue [28], in 1970, introduced the Cesàro sequence spaces written as

$$\text{ces}_p = \left\{ (x_n)_n \subset \mathbb{R} \;\middle|\; \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

such that $\ell^p \subset \text{ces}_p$ and

$$\text{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

such that $\ell^\infty \subset ces_\infty$ where $1 \le p < \infty$. Then, from the definition of Cesàro sequence spaces, Kızmaz [18], defined difference sequence spaces for $\ell^\infty$, c, and $c_0$ and symbolized them by $\ell^\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$, respectively. In his introduction, he defined the difference operator $\Delta$ applied to the sequence $x = (x_n)_n$ using the formula $\Delta x = (x_k - x_{k+1})_k$. In fact, he investigated Köthe-Toeplitz duals and their topological properties.

As one of the researchers generalizing his ideas, Çolak [6] in 1989, introduced firstly a generalized difference sequence space by taking an arbitrary sequence of nonzero complex values $v = (v_n)_n$ and then denoting a new difference operator by $\Delta_v$ such that for any sequence $x = (x_n)_n$, he defined the difference sequence of that $\Delta_v x = (v_k x_k - v_{k+1} x_{k+1})_k$. Then, Et and Esi [10] in 2000, generalized Çolak's difference sequence space by defining

$$\Delta_v(\ell^\infty) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v x \in \ell^\infty\},$$
$$\Delta_v(\text{c}) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v x \in \text{c}\}$$
$$\Delta_v(\text{c}_0) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v x \in \text{c}_0\}.$$

Furthermore, their $m^{th}$ order generalized difference sequence space is given for any $m \in \mathbb{N}$ by $\Delta_v^0 x = (v_k x_k)_k$, $\Delta_v^m x = (\Delta_v^m x_k)_k = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})_k$ with $\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$ for each $k \in \mathbb{N}$.

Next, in 2004, Bektaş, Et and Çolak [3] obtained the Köthe-Toeplitz duals for the generalized difference sequence space of Et and Esi's. We may recall here that their $m^{th}$ order difference sequence space has the following norm for any $m \in \mathbb{N}$:

$$\|x\|_v^{(m)} = \sum_{k=1}^{m} |v_k x_k| + \|\Delta_v^m x\|_\infty$$

Then, the corresponding Köthe-Toeplitz dual was obtained as in [3] and [10] such that it is written as below:

$$D_1^m = \{a = (a_n)_n \subset \mathbb{R} | (n^m v_n{}^{-1} a_n)_n \in \ell^1\} = \left\{ a = (a_n)_n \subset \mathbb{R}: \ \|a\|^{(m)} = \sum_{k=1}^{\infty} \frac{k^m |a_k|}{|v_k|} < \infty \right\}.$$

Note that $D_1^m \subset \ell^1$ if $k^m |v_k{}^{-1}| > 1$ for each $k, m \in \mathbb{N}$ and $\ell^1 \subset D_1^m$ if $k^m |v_k{}^{-1}| < 1$ for each $k, m \in \mathbb{N}$.

Ansari and Chaudhry [2], in 2012, introduced a new type of generalized difference sequence spaces by picking an arbitrary sequence of nonzero complex values $v = (v_n)_n$ as Çolak [6] did and next by symbolizing the new difference sequence space as $\Delta_{v,r}^m(E)$ for arbitrary $r \in \mathbb{R}, m \in \mathbb{N}$ and writing that space as below where $X$ is any of the sequence spaces $\ell^\infty, c$ or $c_0$.

$$\Delta_{v,r}^m(X) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v^m x \in X\}$$

where Ansari and Chaudhry [2] defined the norm by

$$\|x\|_{\Delta,v}^m = \sum_{k=1}^{m} |v_k x_k| + \sup_{k \in \mathbb{N}} |k^r \Delta_v^m x_k|$$

Then, by obtaining an equivalent norm to Ansari and Chaudhry's Banach space, Et and Işık [11] defined $m^{th}$ order generalized type difference sequence for any $m \in \mathbb{N}$ given by

$$\Delta_{v,r}^{(m)}(X) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v^m x \in X\}$$

where the norm is as follows:

$$\|x\|_{\Delta,v}^{(m)} = \sup_{k \in \mathbb{N}} |k^r \Delta_v^m x_k|$$

Then, Et and Işık found $\alpha$-duals of the Banach spaces they got and investigated geometric properties for them such that $m^{\text{th}}$ order $\alpha$-duals for their Banach spaces are written as

$$U_1^m = \{a = (a_n)_n \subset \mathbb{R} | (n^{m-r} v_n{}^{-1} a_n)_n \in \ell^1\} = \left\{ a = (a_n)_n \subset \mathbb{R}: \ \|a\|_\sim^{(m)} = \sum_{k=1}^{\infty} \frac{k^{m-r} |a_k|}{|v_k|} < \infty \right\}$$

Note that $U_1^m \subset \ell^1$ if $k^{m-r} |v_k{}^{-1}| > 1$ for each $k, m \in \mathbb{N}$ and $\ell^1 \subset U_1^m$ if $k^{m-r} |v_k{}^{-1}| < 1$ for each $k, m \in \mathbb{N}$.

The space we are studying is given by below for any $m \in \mathbb{N}$.

$$W_1^m = \left\{ a = (a_n)_n \subset \mathbb{R} \,\middle|\, \left(\frac{a_n}{n^{m-r} v_n}\right)_n \in \ell^1 \right\} = \left\{ a = (a_k)_k \subset \mathbb{R} \ : \quad \|a\|_{\widetilde{(m)}} = \sum_{k=1}^{\infty} \frac{|a_k|}{k^{m-r}|v_k|} < \infty \right\}.$$

Note that $\ell^1 \subset W_1^m$ if $k^{m-r}|v_k| > 1$ for each $k, m \in \mathbb{N}$.

Now, we will need the following well-known preliminaries before giving our main results. [14] may be suggested as a good reference for these fundamentals.

**Definition 1.1** *Consider that* $(X, \|\cdot\|)$ *is a Banach space and let* $C$ *be a non-empty cbc subset. Let* $: C \to C$ *be a mapping. We say that*

1. $T$ *is an affine mapping if for every* $t \in [0,1]$ *and* $a, b \in C$, $T\big((1-t)a + tb\big) = (1-t)T(a) + t\ T(b)$.
2. $T$ *is a nonexpansive mapping if for every* $a, b \in C$, $\| T(a) - T(b) \| \leq \| a - b \|$.

Then, we will easily obtain an analogous key lemma from the below lemma in the work [15].

**Lemma 1.2** *Let* $\{u_n\}$ *be a sequence in* $\ell^1$ *converging to* $u$ *in weak-star topology. Then, for every* $w \in \ell^1$,

$$Q(w) = Q(u) + \|w - u\|_1$$

*where*

$$Q(w) = \limsup_n \|u_n - w\|_1.$$

Note that our scalar field in this study will be real numbers although Çolak [6] considers complex values of $v = (v_n)_n$ while introducing his structer of the difference sequence which is taken as the fundamental concept in this study.

## 2. Main Results

In this section, we will present our results. As mentioned in the first section, we investigate Goebel and Kuzmunow analogy for the space $W_1^m$ for each $m \in \mathbb{N}$. We aim to show that there is a large class of cbc subsets in $W_1^m$ such that every nonexpansive invariant mapping defined on the subsets in the class taken has a fixed point. Recall that the invariant mappings have the same domain and the range. Note that we will assume that $r \in \mathbb{R}$ is arbitrary due to the definition of the space.

First, due to isometric isomorphism, using Lemma 1.2, we will provide the straight analogous result as a lemma below which will be a key step as in the works such as [15], and [12] and in fact the methods in the study [12] will be our lead in this work.

**Lemma 2.1** *Let* $m \in \mathbb{N}$ *and* $\{u_n\}$ *be a sequence in the Banach space* $W_1^m$ *and assume* $\{u_n\}$ *converges to* $u$ *in weak-star topology. Then, for every* $w \in W_1^m$,

$$Q(w) = Q(u) + \|w - u\|_{\sim}^{(m)}$$

where

$$Q(w) = \limsup_n \|u_n - w\|_{\sim}^{(m)}.$$

Then, we obtain our results by the following theorems.

**Theorem 2.2** *Fix* $m \in \mathbb{N}$, $r \in \mathbb{R}$ *and* $t \in (0,1)$. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence defined by* $f_1 := t\ v_1\ e_1$, $f_2 := t 2^{m-r}\ v_2\ e_2$, *and* $f_n := n^{m-r} v_n e_n$ *for all integers* $n \geq 3$ *where the sequence* $(e_n)_{n\in\mathbb{N}}$ *is the canonical basis of both* $c_0$ *and* $\ell^1$. *Then, consider the cbc subset* $E^{(m)} = E_t{}^{(m)}$ *of* $W_1^m$ *by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \alpha_n f_n : \forall n \in \mathbb{N}, \qquad \alpha_n \geq 0 \ \ and \ \sum_{n=1}^{\infty} \alpha_n = 1 \right\}.$$

Then, $E^{(m)}$ has the fixed point property for $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mappings.

*Proof.* Fix $m \in \mathbb{N}$, $r \in \mathbb{R}$ and $t \in (0,1)$. Let $T: E^{(m)} \to E^{(m)}$ be a $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\big(u^{(n)}\big)_{n\in\mathbb{N}} \in E^{(m)}$ such that $\big\|Tu^{(n)} - u^{(n)}\big\|_{\sim}^{(m)} \underset{n}{\to} 0$. Due to isometric isomorphism, $W_1^m$ shares common geometric properties with $\ell^1$ and so both $W_1^m$ and its predual have similar fixed point theory properties to $\ell^1$ and $c_0$, respectively. Thus, considering that on bounded subsets the weak star topology on $\ell^1$ is equivalent to the coardinate-wise convergence topology, and $c_0$ is separable, in $W_1^m$, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then, we can say that we may denote the weak* closure of the set $E^{(m)}$ by

$$C^{(m)} := \overline{E^{(m)}}^{w^*} = \left\{\sum_{n=1}^{\infty} \alpha_n f_n: \ each \ \ \alpha_n \geq 0 \ \ and \ \ \sum_{n=1}^{\infty} \alpha_n \leq 1\right\}$$

and without loss of generality, we may pass to a subsequence if necessary and get a weak* limit $u \in C^{(\mathrm{m})}$ of $u^{(n)}$. Then, by Lemma 2.1, we have a function $r: W_1^m \to [0,\infty)$ defined by

$$Q(w) = \limsup_n \left\|u^{(n)} - w\right\|_{\sim}^{(m)} \ , \quad \forall w \in W_1^m$$

such that for every $w \in W_1^m$,

$$Q(w) = Q(u) + \|u - w\|_{\sim}^{(m)}.$$

**Case 1.** $u \in E^{(m)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|_{\sim}^{(m)}$ and

$$Q(Tu) = \limsup_n \left\|Tu - u^{(n)}\right\|_{\sim}^{(m)} \leq \limsup_n \left\|Tu - T\left(u^{(n)}\right)\right\|_{\sim}^{(m)} + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|_{\sim}^{(m)}$$

$$\leq \limsup_n \left\|u - u^{(n)}\right\|_{\sim}^{(m)} + 0 \ = Q(u). \qquad (1)$$

Thus, $Q(Tu) = Q(u) + \|Tu - u\|_{\sim}^{(m)} \leq r(u)$ and so $\|Tu - u\|_{\sim}^{(m)} = 0$. Therefore, $Tu = u$.

**Case 2.** $u \in C^{(m)} \setminus E^{(m)}$.

Then, we may find scalars satisfying $u = \sum_{n=1}^{\infty} \delta_n f_n$ such that $\sum_{n=1}^{\infty} \delta_n < 1$ $and$ $\delta_n \geq 0, \forall n \in \mathbb{N}$. Define $\xi := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\beta \in \left[\frac{-\delta_1}{\xi}, \frac{\delta_2}{\xi} + 1\right]$ define

$$h_\beta := (\delta_1 + \beta\xi) f_1 + (\delta_2 + (1-\beta)\xi) f_2 + \sum_{n=3}^{\infty} \delta_n f_n.$$

Then,

$$\left\|h_\beta - u\right\|_{\sim}^{(m)} = \|\beta t \xi v_1 e_1 + (1-\beta)\xi t \ \ v_2 2^{\mathrm{m-r}} e_2\|_{\sim}^{(m)} = t|\beta|\xi + t|1-\beta|\xi.$$

$\left\|h_\beta - u\right\|_{\sim}^{(m)}$ is minimized for $\beta \in [0,1]$ and its minimum value would be $t\xi$.

Now fix $w \in E^{(m)}$. Then, we may find scalars satisfying $w = \sum_{n=1}^{\infty} \alpha_n f_n$ such that $\sum_{n=1}^{\infty} \alpha_n \ = 1$ with $\alpha_n \geq 0, \ \forall n \in \mathbb{N}$. We may also write each $f_k$ with coefficients $\gamma_k$ for each $k \in \mathbb{N}$ where $\gamma_1 := t \ \ v_1$, $\gamma_2 := t\ 2^{m-r}\ v_2$, and $\gamma_n := n^{m-r} v_n$ for all integers $n \geq 3$ such that for each $n \in \mathbb{N}$, $f_n = \gamma_n e_n$.

Then,

$$\begin{aligned}
\|\mathrm{w} - u\|^{(m)} &= \left\|\sum_{k=1}^{\infty} \alpha_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|^{(m)} \\
&= \left\|\sum_{k=1}^{\infty} \alpha_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|^{(m)} \\
&= \left\|\sum_{k=1}^{\infty} (\alpha_k - \delta_k) f_k\right\|^{(m)} \\
&= \sum_{k=1}^{\infty} \left|(\alpha_k - \delta_k)\frac{\gamma_k}{k^{m-r} v_k}\right| .
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathrm{w} - u\|^{(m)} &\geq \sum_{k=1}^{\infty} t \ \ |\alpha_k - \delta_k| \\
&\geq t\left|\sum_{k=1}^{\infty} (\alpha_k - \delta_k)\right| \\
&= t\left|1 - \sum_{k=1}^{\infty} \delta_k\right| \\
&= t\xi.
\end{aligned}$$

Hence,

$$\|w-u\|^{(m)} \geq t\xi = \|h_\beta - u\|^{(m)}$$

and the equality is obtained if and only if $(1-t)\sum_{k=3}^{\infty}|\alpha_k-\delta_k| = 0$; that is, we have $\|w-u\|_{\sim}^{(m)} = t\xi$ if and only if $\alpha_k = \delta_k$ for every $k \geq 3$; or say, $\|w-u\|_{\sim}^{(m)} = t\xi$ if and only if $w = h_\beta$ for some $\beta \in [0,1]$.

Then, there exists a continuous function $\rho$: $[0,1] \to E^{(m)}$ defined by $\rho(\beta) = h_\beta$ and $\Lambda\rho([0,1])$ is a compact convex subset and so $\|w-u\|_{\sim}^{(m)}$ achieves its minimum value at $w = h_\beta$ and for any $h_\beta \in \Lambda$, we get

$$\begin{aligned} Q(h_\beta) &= Q(u) + \|h_\beta - u\|_{\sim}^{(m)} \\ &\leq Q(u) + \|Th_\beta - u\|_{\sim}^{(m)} \\ &= Q(Th_\beta) = \limsup_n \|Th_\beta - u^{(n)}\|_{\sim}^{(m)} \end{aligned}$$

then, like the inequality (1), we get

$$\begin{aligned} Q(h_\beta) &\leq \limsup_n \|Th_\beta - T(u^{(n)})\|_{\sim}^{(m)} + \limsup_n \|u^{(n)} - T(u^{(n)})\|_{\sim}^{(m)} \\ &\leq \limsup_n \|h_\beta - u^{(n)}\|_{\sim}^{(m)} + \limsup_n \|u^{(n)} - T(u^{(n)})\|_{\sim}^{(m)} \\ &\leq \limsup_n \|h_\beta - u^{(n)}\|_{\sim}^{(m)} + 0 = Q(h_\beta). \end{aligned}$$

Hence, $r(h_\beta) \leq Q(Th_\beta) \leq r(h_\beta)$ and so $Q(Th_\beta) = Q(h_\beta)$.

Therefore,

$$Q(u) + \|Th_\beta - u\|_{\sim}^{(m)} = Q(u) + \|h_\beta - u\|_{\sim}^{(m)}.$$

Thus, $\|Th_\beta - u\|_{\sim}^{(m)} = \|h_\beta - u\|_{\sim}^{(m)}$ and so $Th_\beta \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's fixed point theorem [27] easily we get the result $T$ has a fixed point since $T$ is continuous; thus, $h_\beta$ is the unique minimizer of $\|w-u\|_{\sim}^{(m)}$ $: w \in E^{(m)}$ and $Th_\beta = h_\beta$.

Therefore, $E^{(m)}$ has the fixed point property for nonexpansive mappings.

**Theorem 2.3** *Fix $m \in \mathbb{N}$, $r \in \mathbb{R}$ and $t \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := t\ v_1\ e_1$, $f_2 := t\ 2^{m-r}\ v_2\ e_2$, $f_3 := t\ 3^{m-r}\ v_3\ e_3$, and $f_n := n^{m-r} v_n e_n$ for all integers $n \geq 4$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both $c_0$ and $\ell^1$. Then, consider the cbc subset $E^{(m)} = E_t{}^{(m)}$ of $W_1^m$ by*

$$E^{(m)} := \left\{\sum_{n=1}^{\infty} \alpha_n f_n :\ \forall n \in \mathbb{N}, \qquad \alpha_n \geq 0\ \ and\ \ \sum_{n=1}^{\infty} \alpha_n = 1\right\}.$$

Then, $E^{(m)}$ has the fixed point property for $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mappings.

*Proof.* Fix $m \in \mathbb{N}$, $r \in \mathbb{R}$ *and* $t \in (0,1)$. Let $T: E^{(m)} \to E^{(m)}$ be a $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $(u^{(n)})_{n\in\mathbb{N}} \in E^{(m)}$ such that $\|Tu^{(n)} - u^{(n)}\|_{\sim}^{(m)} \underset{n}{\to} 0$. Due to isometric isomorphism, $W_1^m$ shares common geometric properties with $\ell^1$ and so both $W_1^m$ and its predual have similar fixed point theory properties to $\ell^1$ and $c_0$, respectively. Thus, considering that on bounded subsets the weak star topology on $\ell^1$ is equivalent to the coardinate-wise convergence topology and $c_0$ is separable, in $W_1^m$, the unit closed ball is weak*-sequentially compact due to Banach-Alaoğlu theorem. Then, we can say that we may denote the weak* closure of the set $E^{(\mathrm{m})}$ by

$$C^{(m)} := \overline{E^{(m)}}^{w^*} = \left\{\sum_{n=1}^{\infty} \alpha_n f_n :\ each\ \ \alpha_n \geq 0\ \ and\ \ \sum_{n=1}^{\infty} \alpha_n \leq 1\right\}$$

and without loss of generality, we may pass to a subsequence if necessary and get a weak* limit $u \in C^{(\mathrm{m})}$ of $u^{(n)}$. Then, by Lemma 2.1, we have a function $r: W_1^m \to [0,\infty)$ defined by

$$Q(w) = \limsup_n \|u^{(n)} - w\|_{\sim}^{(m)}\ ,\quad \forall w \in W_1^m$$

such that for every $w \in W_1^m$,

$$Q(w) = Q(u) + \|u - w\|_{\sim}^{(m)}.$$

**Case 1.** $u \in E^{(m)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|_{\sim}^{(m)}$ and

$$Q(Tu) = \limsup_n \|Tu - u^{(n)}\|_{\sim}^{(m)} \leq \limsup_n \|Tu - T(u^{(n)})\|_{\sim}^{(m)} + \limsup_n \|u^{(n)} - T(u^{(n)})\|_{\sim}^{(m)}$$
$$\leq \limsup_n \|u - u^{(n)}\|_{\sim}^{(m)} + 0 = Q(u). \qquad (2)$$

Thus, $Q(Tu) = Q(u) + \|Tu - u\|_{\sim}^{(m)} \leq r(u)$ and so $\|Tu - u\|_{\sim}^{(m)} = 0$. Therefore, $Tu = u$.

**Case 2.** $u \in C^{(m)} \backslash E^{(m)}$.

Then, we may find scalars satisfying $u = \sum_{n=1}^{\infty} \delta_n f_n$ such that $\sum_{n=1}^{\infty} \delta_n < 1$ *and* $\delta_n \geq 0, \forall n \in \mathbb{N}$. Define $\xi := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\beta \in \left[\frac{-\delta_1}{\xi}, \frac{\delta_2}{\xi} + 1\right]$, define

$$\mathrm{h}_\beta := \left(\delta_1 + \frac{\beta}{2}\xi\right) f_1 + \left(\delta_2 + \frac{\beta}{2}\xi\right) f_2 + (\delta_3 + (1-\beta)\xi) f_3 + \sum_{n=4}^{\infty} \delta_n f_n.$$

Then,

$$\|h_\beta - u\|_{\sim}^{(m)} = \left\|\frac{\beta}{2} t\xi v_1 e_1 + \frac{\beta}{2} t\xi 2^{m-r} v_2 e_2 + (1-\beta)\xi t\, 3^{m-r} v_3 e_3\right\|_{\sim}^{(m)}$$
$$= t\left|\frac{\beta}{2}\right|\xi + t\left|\frac{\beta}{2}\right|\xi + t|1-\beta|\xi.$$

$\|h_\beta - u\|_{\sim}^{(m)}$ is minimized for $\beta \in [0,1]$ and its minimum value would be $t\xi$.

Now fix $w \in E^{(m)}$. Then, we may find scalars satisfying $w = \sum_{n=1}^{\infty} \alpha_n f_n$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$ with $\alpha_n \geq 0$, $\forall n \in \mathbb{N}$. We may also write each $f_k$ with coefficients $\gamma_k$ for each $k \in \mathbb{N}$ where $\gamma_1 := t\ v_1$, $\gamma_2 := t2^{m-r}\ v_2$, $\gamma_3 := t3^{m-r}\ v_3$, and $\gamma_n := n^{m-r} v_n$ for all integers $n \geq 4$ such that for each $n \in \mathbb{N}$, $f_n = \gamma_n e_n$.

Then,

$$\|\mathrm{w} - u\|^{(m)} = \left\|\sum_{k=1}^{\infty} \alpha_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|^{(m)}$$
$$= \left\|\sum_{k=1}^{\infty} \alpha_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|^{(m)}$$
$$= \left\|\sum_{k=1}^{\infty} (\alpha_k - \delta_k) f_k\right\|^{(m)}$$
$$= \sum_{k=1}^{\infty} \left|(\alpha_k - \delta_k)\frac{\gamma_k}{k^{m-r} v_k}\right|$$
$$\geq \sum_{k=1}^{\infty} t\ |\alpha_k - \delta_k|$$
$$\geq t\left|\sum_{k=1}^{\infty} (\alpha_k - \delta_k)\right|$$
$$= t\left|1 - \sum_{k=1}^{\infty} \delta_k\right|$$
$$= t\xi.$$

Hence,

$$\|\mathrm{w} - u\|^{(m)} \geq t\xi = \|h_\beta - u\|^{(m)}$$

and the equality is obtained if and only if $(1-t)\sum_{k=4}^{\infty} |\alpha_k - \delta_k| = 0$; that is, we have $\|\mathrm{w} - u\|_{\sim}^{(m)} = t\xi$ if and only if $\alpha_k = \delta_k$ for every $k \geq 3$; or say, $\|\mathrm{w} - u\|_{\sim}^{(m)} = t\xi$ if and only if $\mathrm{w} = h_\beta$ for some $\beta \in [0,1]$.

Then, there exists a continuous function $\rho: [0,1] \to E^{(m)}$ defined by $\rho(\beta) = h_\beta$ and $\Lambda\rho([0,1])$ is a compact convex subset and so $\|\mathrm{w} - u\|_{\sim}^{(m)}$ achieves its minimum value at $\mathrm{w} = h_\beta$ and for any $\mathrm{h}_\beta \in \Lambda$, we get

$$Q(h_\beta) = Q(u) + \|h_\beta - u\|_{\sim}^{(m)}$$
$$\leq Q(u) + \|Th_\beta - u\|_{\sim}^{(m)}$$

$$= Q(Th_\beta) = \limsup_n \|Th_\beta - u^{(n)}\|_\sim^{(m)}$$

then same as the inequality (2), we get

$$Q(h_\beta) \leq \limsup_n \|Th_\beta - T(u^{(n)})\|_\sim^{(m)} + \limsup_n \|u^{(n)} - T(u^{(n)})\|_\sim^{(m)}$$
$$\leq \limsup_n \|h_\beta - u^{(n)}\|_\sim^{(m)} + \limsup_n \|u^{(n)} - T(u^{(n)})\|_\sim^{(m)}$$
$$\leq \limsup_n \|h_\beta - u^{(n)}\|_\sim^{(m)} + 0 = Q(h_\beta).$$

Hence, $r(h_\beta) \leq Q(Th_\beta) \leq r(h_\beta)$ and so $Q(Th_\beta) = Q(h_\beta)$.

Therefore,

$$Q(u) + \|Th_\beta - u\|_\sim^{(m)} = Q(u) + \|h_\beta - u\|_\sim^{(m)}.$$

Thus, $\|Th_\beta - u\|_\sim^{(m)} = \|h_\beta - u\|_\sim^{(m)}$ and so $Th_\beta \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's fixed point theorem [27] we can easily we get the result $T$ has a fixed point since $T$ is continuous. Thus, $h_\beta$ is the unique minimizer of $\|w - u\|_\sim^{(m)} \ : w \in E^{(m)}$ and $Th_\beta = h_\beta$.

Therefore, $E^{(m)}$ has the fixed point property for nonexpansive mappings.

## 3. Conclusion

As it has been mentioned in earlier sections of the study, investigating and looking for large classes of closed, bounded and convex subsets in Banach spaces alike the Banach spaces of absolutely summable scalars are center of interests for many fixed point theorists. One can investigate to get larger classes for more general spaces than those in the present study and due to isometry, that would not be hard by following the ideas of Goebel and Kuczumows. However, trying to generalize their ideas and looking for different examples of the sets and spaces would be valuable studies.

**Acknowledgement**

We would like to note that the first author is currently supported by The Scientific and Technological Research Council of Türkiye with the grant number 1059B192300789. The work had been conducted way before his grant.

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