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On Lebesgue-like Corresponding Function Spaces of Köthe-Toeplitz Duals of Generalized Cesàro Difference Sequence Spaces and Fixed Point Property for Asymptotically Nonexpansive Mappings

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Abstract In 1970, Cesàro Sequence Spaces was introduced by Shiue. In 1981, Kızmaz defined difference sequence spaces for ℓ^∞, c_0 and c. Then, in 1983, Orhan introduced Cesàro Difference Sequence Spaces. Later, Et and Tripathy et. al. generalized the space introduced by Orhan for any $m \in \mathbb{N}$. We will be interested in their generalizations such that some generalizations of the space introduced by Orhan were given by Et in 1996 and more by Tripathy and his friends in 2005 for each $m \in \mathbb{N}$ while they examined their duals and geometric properties. We investigate the corresponding function space of those Köthe-Toeplitz duals of some generalized Cesàro difference sequence spaces. In this study, first we recall that in 2004, Kaczor and Prus saw that there exists a large class of closed, bounded, convex subsets in ℓ^1 with fixed point property for affine asymptotically nonexpansive mappings. In the present study, we aim to discuss the analogous results for the corresponding function spaces of the Köthe-Toeplitz duals of some generalized Cesàro difference sequence spaces. Thus, we consider the generalized Köthe-Toeplitz duals of some generalized Cesàro difference sequence spaces written by $\Upsilon_m = \{a = (a_k)_k \subset \mathbb{R} : \ \|a\|_\Delta = \sum_{k=1}^{\infty} k^m |a_k| < \infty\}$ and its corresponding function space

$$\Sigma_m := \left\{ \begin{matrix} f: [0,1] \to \mathbb{R} \\ \text{measurable} \end{matrix} : \ \|f\| = \int_0^1 t^m |f(t)| dt < \infty \right\} \text{ for each } m \in \mathbb{N}.$$

Then, for any $m \in \mathbb{N}$, we show that there exists a very large class of closed, bounded, convex subsets in Σ_m with fixed point property for affine asymptotically nonexpansive mappings and so for affine nonexpansive mappings.

Keywords Fixed point property, nonexpansive mapping, Cesàro Difference Sequences, Köthe-Toeplitz dual
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1. Introduction

A Banach space $(X, \|.\|)$ is called to have the fixed point property for non-expansive mappings (fpp-ne) when any non-expansive self mappings defined on arbitrary non-empty closed, bounded and convex subset of the Banach space has a fixed point. Researchers have considered categorizing Banach spaces with this property. Firstly, in 1965, Browder [2] found that Hilbert spaces have the property and Kirk [14] generalized it to reflexive Banach spaces with normal structure. Then, researchers have especially investigated nonreflexive classical Banach spaces and wondered if they can be renormable and falls in the same category with their equivalent norm while they fail to be members of the category with their usual norm but they were able to detect some nonreflexive Banach spaces which have equivalent norms and they become to have the fixed point property with those renormings. The first example was given by Lin [15] for the Banach space of absolutely summable scalar sequences, ℓ^1. Because of sharing many common properties, it is natural to ask if the same is possible for the Banach space of scalar sequences converging to 0, c_0 as another well known classical non-reflexive Banach space. As a second interesting example, but under affinity condition, was given in [17] by Maria and Hernandes Lineares whose space experimented was the Banach space of Lebesgue integrable functions on [0,1], $L_1[0,1]$. It can be said that all these works are inspired by the work of Goebel and Kuczumow [11]. Goebel and Kuczumow showed that there exists very large class of non-weakly compact, closed, bounded and convex subsets of ℓ^1 respect to weak* topology of ℓ^1 with fixed point property for non-expansive mappings. Later, Kaczor and Prus [12] investigated if similar result could be done for asymptotically nonexpansive mappings and they saw that there exists a large class of closed, bounded, convex subsets in ℓ^1 with fixed point property for affine asymptotically non-expansive mappings. Moreover, Everest, in his Ph.D. thesis [9], written under supervision of Chris Lennard, considered large classes in ℓ^1 with fixed point property for affine asymptotically non-expansive mappings by generalizing Kaczor and Prus' work.

In this study, we aim to discuss the analogous results for the corresponding function spaces of the Köthe-Toeplitz duals of some generalized Cesàro difference sequence spaces. Thus, we consider the generalized Köthe-Toeplitz duals of some generalized Cesàro difference sequence spaces written by $\Upsilon_m = \{a = (a_k)_k \subset \mathbb{R} : \|a\|_\Delta = \sum_{k=1}^{\infty} k^m |a_k| < \infty\}$ and its corresponding function space

$$\Sigma_m := \left\{ \begin{matrix} f:[0,1] \to \mathbb{R} \\ \text{measurable} \end{matrix} : \|f\| = \int_0^1 t^m |f(t)| dt < \infty \right\} \text{ for each } m \in \mathbb{N}.$$

Then, for any $m \in \mathbb{N}$, we show that there exists a very large class of closed, bounded, convex subsets in Σ_m with fixed point property for affine asymptotically nonexpansive mappings and so for affine nonexpansive mappings.

First we recall that the Cesàro sequence spaces

$$\text{ces}_p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

and

$$\text{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

were introduced by Shiue [21] in 1970, where $1 \leq p < \infty$. It has been shown that $\ell^p \subset \text{ces}_p$ for $1 < p \leq \infty$. Moreover, it has been shown that Cesàro sequence spaces ces_p for $1 < p < \infty$ are seperable reflexive Banach spaces. Furthermore, it was also proved by Cui [4], Cui-Hudzik-Li [5] and Cui-Meng-Pluciennik [6] that Cesàro sequence spaces ces_p for $1 < p < \infty$ have the fixed point property. They prove this result using different methods. One method is to calculate Garcia-Falset coefficient. It is known that if Garcia-Falset coefficient is less than 2 for a Banach space, then it has the fixed point property for nonexpansive mappings [10]. Using this fact, since they calculate this coefficient for ces_p as $2^{1/p}$ similary to what it is for ℓ^p, they point the result for the Cesàro sequence spaces. Another fact is that they see that the space has normal structure for $1 < p < \infty$. Then using the fact via Kirk [14] that reflexive Banach spaces with normal structure has the fixed point property, they easily deduce that the space has the fixed point property for $1 < p < \infty$. Their results on Cesàro sequence spaces as a survey can be seen in [3].

Later, in 1981, Kızmaz [13] introduced difference sequence spaces for ℓ^∞, c and c_0 where they are the Banach spaces of bounded, convergent and null sequences $x = (x_n)_n$, respectively. As it is seen below, his definitions for these spaces were given using difference operator applied to the sequence x, $\triangle x = (x_k - x_{k+1})_k$.

$$\ell^\infty(\triangle) = \{x = (x_n)_n \subset \mathbb{R} | \triangle x \in \ell^\infty\},$$
$$c(\triangle) = \{x = (x_n)_n \subset \mathbb{R} | \triangle x \in c\},$$
$$c_0(\triangle) = \{x = (x_n)_n \subset \mathbb{R} | \triangle x \in c_0\}.$$

Kızmaz investigated Köthe-Toeplitz Duals and some properties of these spaces.

Furthermore, Cesàro sequence spaces X^p of non-absolute type were defined by Ng and Lee [18] in 1977 as follows:

$$X^p = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| < \infty \right\},$$

where $1 \le p < \infty$. They prove that X^p is linearly isomorphic and isometric to ℓ^p for $1 \le p \le \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Later, in 1983, Orhan [19] introduced Cesàro Difference Sequence Spaces by the following definitions:

$$C_p = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right| < \infty \right\},$$

where $1 \le p < \infty$ and $\triangle x_k = x_k - x_{k+1}$ for each $k \in \mathbb{N}$. He noted that their norms are given as below for any $x = (x_n)_n$:

$$\|x\|_p^* = |x_1| + \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p \right)^{1/p} \text{ and } \|x\|_\infty^* = |x_1| + \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|,$$

respectively.

Orhan showed that there exists a linear bounded operator $S: C_p \to C_p$ for $1 \le p \le \infty$ such that Köthe-Toeplitz β −Duals of these spaces are given respectively as follows:

$$S(C_p)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^q\} \text{ where } 1 < p < \infty \text{ and } q = \frac{p}{p-1},$$

$$S(C_1)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^\infty\} \text{ and}$$

$$S(C_\infty)^\beta = \{a = (a_n)_n \subset \mathbb{R} | (na_n)_n \in \ell^1\}.$$

It might be better to use the notation $X^p(\triangle)$ instead of C_p for $1 \le p \le \infty$ since we also recalled the difference sequence spaces and used similar type of notation.

We note that Orhan also proved that $X^p \subset X^p(\triangle)$ for $1 \le p \le \infty$ strictly. Also, one can clearly see that $X^p(\triangle)$ is linearly isomorphic and isometric to ℓ^p for $1 \le p \le \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Note also that Köthe-Toeplitz Dual for $p = \infty$ case in Orhan's study and ℓ^∞ case in Kızmaz study coincides.

Furthermore, Et and Çolak [8] generalized the spaces introduced in Kızmaz's work [13] in the following way for $m \in \mathbb{N}$.

$$\ell^\infty(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \ell^\infty\},$$

$$\mathrm{c}(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \mathrm{c}\},$$

$$\mathrm{c}_0(\triangle^m) = \{x = (x_n)_n \subset \mathbb{R} | \triangle^m x \in \mathrm{c}_0\}$$

where $\triangle x = (\triangle x_k) = (x_k - x_{k+1})_k$, $\triangle^0 x = (x_k)_k$, $\triangle^m x = (\triangle^m x_k) = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1})_k$ and $\triangle^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$ for each $k \in \mathbb{N}$.

Also, Et [7] and Tripathy et. al. [22] generalized the space introduced by Orhan in the following way for $m \in \mathbb{N}$.

$$X^p(\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle^m x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^\infty(\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \middle| \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle^m x_k \right| < \infty \right\},$$

Then, it is seen that that Köthe-Toeplitz Dual for $p = \infty$ case in Et's study [7] and ℓ^∞ case in Et and Çolak study [8] coincides such that Köthe-Toeplitz Dual was given as below for any $m \in \mathbb{N}$.

$$\Upsilon_m := \{a = (a_n)_n \subset \mathbb{R} | (n^m a_n)_n \in \ell^1\} = \left\{a = (a_k)_k \subset \mathbb{R} : \ \|a\| = \sum_{k=1}^{\infty} k^m |a_k| < \infty\right\}.$$

Note that $\Upsilon_m \subset \ell^1$ for any $m \in \mathbb{N}$.

One can see that corresponding function space for these duals can be given as below:

$$\Sigma_m := \left\{ \begin{matrix} f: [0,1] \to \mathbb{R} \\ \text{measurable} \end{matrix} : \|f\| = \int_0^1 t^m |f(t)| dt < \infty \right\}.$$

Note that $L_1[0,1] \subset \Sigma_m$ and Υ_m is the space when counting measure is used for Σ_m.

As we have already stated, in this study, we study the function spaces Σ_m for any $m \in \mathbb{N}$.

Now we provide some preliminaries before giving our main results.

Definition 1.1. *Let $(X, \|\cdot\|)$ be a Banach space and C is a non-empty closed, bounded, convex subset.*

1. If $T: C \to C$ is a mapping such that for all $\lambda \in [0,1]$ and for all $x, y \in C$, $T\big((1-\lambda)x + \lambda y\big) = (1-\lambda)T(x) + \lambda T(y)$ then T is said to be an affine mapping.

2. If $T: C \to C$ is a mapping such that $\| T(x) - T(y) \| \leq \| x - y \|$, $for\ all\ x, y \in C$ then T is said to be a nonexpansive mapping.

Also, if for every nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for nonexpansive mappings [fpp(ne)].

3. If $T: C \to C$ is a mapping such that there exists a sequence of scalars $(k_n)_{n \in N}$ decreasingly approach to 1 and $\| T^n(x) - T^n(y) \| \leq k_n \| x - y \|$, $for\ all\ x, y \in C$ and for all $n \in \mathbb{N}$ then T is said to be an asymptotically nonexpansive mapping.

Also, if for every asymptotically nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for asymptotically nonexpansive mappings [fpp(ane)].

Remark 1.1. *In 1979, Goebel and Kuczumow [11] showed there exists a large class of closed, bounded and convex subsets of ℓ^1 using a key lemma they obtained. Their lemma says that if $\{x_n\}$ is a sequence in ℓ^1 converging to x in weak-star topology, then for any $y \in \ell^1$,*

$$r(y) = r(x) + \|y - x\|_1 \ where\ r(y) = \limsup_n \|x_n - y\|_1 \ .$$

We will call this fact ∴.

The analogue of this lemma for $L_1[0,1]$ is observed via the result in Brezis and Lieb [1]. Note that Hernández-Linares pointed this fact in his Ph.D. thesis [16], written under supervision of Maria Japon Pineda. Now we provide the lemma which is deduced by their results and will be key for our results in this section.

Lemma 1.1. *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of real valued measurable functions which are uniformly bounded in $L_1[0,1]$. Assume that f_n converges to an $f \in L_1[0,1]$ pointwise almost everywhere (a.e.). Then for any $g \in L_1[0,1]$,*

$$S(g) = S(f) + \|f - g\|_1 \ where\ S(g) = \limsup_n \|f_n - g\|_1 \ .$$

Since the corresponding function space of a Köthe-Toeplitz Dual of a Cesàro Difference Sequence Space which contains Lebesgue space $L_1[0,1]$ and in fact it is isometrically isomorphic to $L_1[0,1]$, for any $m \in \mathbb{N}$, for the corresponding function spaces Σ_m the following lemma can be given as straight and quick result.

Lemma 1.2. *Fix $m \in \mathbb{N}$. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of real valued measurable functions which are uniformly bounded in Σ_m. Assume that f_n converges to an $f \in \Sigma_m$ pointwise almost everywhere (a.e.). Then for any $g \in \Sigma_m$,*

$$S(g) = S(f) + \|f - g\| \ where\ S(g) = \limsup_n \|f_n - g\|.$$

2. Main Result

In this section, we work on Kaczor and Prus analogy for a Banach space containing Lebesgue space $L_1[0,1]$. The space we consider is the corresponding function spaces Σ_m for the Köthe-Toeplitz Dual of a Cesàro difference sequence space $X^\infty(\triangle^m)$ for any $m \in \mathbb{N}$. We show that there exists a very large class of closed, bounded and convex subsets of the space with the fixed point property for affine asymptoticallys non-expansive mappings and so for affine nonexpansive mappings. Note that in his Master's thesis [20], in 2022, Oymak studied the case $m=1$ and also cases $m=2$ with $m=3$ have been recently submitted by the same authors of this article.

Now, for any $m \in \mathbb{N}$, let us first consider the following classes of closed, bounded and convex subsets for Banach spaces Σ_m by the following examples. We should note here that we will be using similar ideas to those in in the section 2 of Ph.D. thesis of Everest [9], written under supervision of Chris Lennard, where Everest firstly provides Goebel and Kuczumow's proofs in detailed.

Here, we first consider some sample sets that represent the broad set classes we mentioned, and then we give a relevant theorem for each of these sets.

Example 2.1. *Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1$ and $f_n := e_n$ for all integers $n \geq 2$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n := (n+1)t^{n-m}$, $\forall n \in \mathbb{N}$. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of Σ_m by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \beta_n f_n : \ \forall n \in \mathbb{N}, \quad t_n \geq 0 \ \ and \ \sum_{n=1}^{\infty} \beta_n = 1 \right\}.$$

Example 2.2. *Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1$ and $f_n := e_n$ for all integers $n \geq 2$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n := \frac{ne^{nt}}{t^m(e^n-1)}$, $\forall n \in \mathbb{N}$. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of Σ_m by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \beta_n f_n : \ \forall n \in \mathbb{N}, \quad \beta_n \geq 0 \ \ and \ \sum_{n=1}^{\infty} \beta_n = 1 \right\}.$$

Example 2.3. *Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1$ and $f_n := e_n$ for all integers $n \geq 2$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n := \frac{ne^{nt}}{t^m(e^n-1)} \chi_{[0,\frac{1}{n}]}$, $\forall n \in \mathbb{N}$, where χ is the characteristics funtion. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of Σ_m by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \beta_n f_n : \ \forall n \in \mathbb{N}, \quad \beta_n \geq 0 \ \ and \ \sum_{n=1}^{\infty} \beta_n = 1 \right\}.$$

Example 2.4. *Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1$ and $f_n := e_n$ for all integers $n \geq 2$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is given by the formula $e_n := \frac{4n}{\pi t^m(1+n^2t^2)} \chi_{[0,\frac{1}{n}]}$, $\forall n \in \mathbb{N}$, where χ is the characteristics funtion. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of Σ_m by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \beta_n f_n : \ \forall n \in \mathbb{N}, \quad \beta_n \geq 0 \ \ and \ \sum_{n=1}^{\infty} \beta_n = 1 \right\}.$$

Theorem 2.1. *For $m \in \mathbb{N}$ and $b \in (0,1)$, then each of the sets $E^{(m)}$ defined as in the examples above has the fixed point property for affine asymptotically nonexpansive mappings.*

Proof. Fix $m \in \mathbb{N}$ *and* $b \in (0,1)$. Let $T : E^{(m)} \to E^{(m)}$ be an affine asymptotically nonexpansive mapping. Then, since T is affine, by Lemma 1.1.2 in the Ph.D. thesis of Everest [9] written under supervision of Lennard, there exists a sequence $\left(f^{(n)}\right)_{n\in\mathbb{N}} \in E^{(m)}$ such that $\left\|Tf^{(n)} - f^{(n)}\right\| \underset{n}{\to} 0$. Without loss of generality, passing to a subsequence if necessary, there exists $f \in E^{(m)}$ such that $f^{(n)}$ converges to f in weak* topology. Then, by Goebel Kuczumow analog fact, Lemma 1.2 given in the last part of the previous section, we can define a function $s : \Sigma_m \to [0,\infty)$ by

$$s(f) = \limsup_n \left\|f^{(n)} - g\right\| \quad , \ \forall g \in \Sigma_m$$

and so

$$s(g) = s(g) + \|f - g\| \ , \ \forall g \in \Sigma_m .$$

Now define the weak* closure of the set $E^{(m)}$ as it is seen below.

$$W := \overline{E^{(m)}}^{w^*} = \left\{ \sum_{n=1}^{\infty} \beta_n f_n : \ each \ \beta_n \geq 0 \ \ and \ \sum_{n=1}^{\infty} \beta_n \leq 1 \right\}$$

Since T is asymptotically nonexpansive mapping, there exists a decreasing sequence $(k_n)_{n\in\mathbb{N}} \in [1,\infty)$ converging to 1 such that $\forall f, g \in \Sigma_m$ and $\forall n \in \mathbb{N}$,

$$\|T^n f - T^n g\| \leq k_n\|f-g\|.$$

Case 1: $f \in E^{(m)}$

Fix $q \in \mathbb{N}$ and take $k_0 = 1$. Then, we have $s(T^q f) = S(f) + \|T^q f - f\|$ and

$$s(T^q f) = \limsup_n \left\|T^q f - f^{(n)}\right\| \leq \limsup_n \left\|T^q f - T^q\left(f^{(n)}\right)\right\| + \limsup_n \left\|T^q\left(f^{(n)}\right) - f^{(n)}\right\| \qquad (2.1)$$

$$\leq \limsup_n k_q \left\|f - f^{(n)}\right\| + \limsup_n \sum_{j=1}^{q} \left\|T^j\left(f^{(n)}\right) - T^{j-1}\left(f^{(n)}\right)\right\|$$

$$\leq k_q \limsup_n \left\|f - f^{(n)}\right\| + \limsup_n \sum_{j=1}^{q} k_{j-1}\left\|T\left(f^{(n)}\right) - f^{(n)}\right\|$$

$$= k_q S(f).$$

Therefore, $\|T^q f - f\| \leq S(f)(k_q - 1)$ and so by taking limit as $q \to \infty$, we have $\lim_q \|T^q f - f\| = 0$ but then since $\lim_q \|T^{q+1} f - Tf\| \leq \lim_q \|T^q f - f\|$, $\lim_q \|T^{q+1} f - Tf\| = 0$ and so $T^q f$ converges both Tf and f; thus, $Tf = f$ by the uniqueness of the limits.

Case 2: $f \in W \backslash E^{(m)}$.

Then, f is of the form $\sum_{n=1}^{\infty} \gamma_n f_n$ such that $\sum_{n=1}^{\infty} \gamma_n < 1$ *and* $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$.

Define $\delta := 1 - \sum_{n=1}^{\infty} \gamma_n$ and next define

$$h := (\gamma_1 + \delta) f_1 + \sum_{n=2}^{\infty} \gamma_n f_n.$$

Then, $\|h - f\| = \|b\delta e_1\| = b\delta$.

Now fix $g \in E^{(m)}$ of the form $\sum_{n=1}^{\infty} \beta_n f_n$ such that $\sum_{n=1}^{\infty} \beta_n = 1$ with $\beta_n \geq 0$, $\forall n \in \mathbb{N}$. We may also write each f_k with coefficients γ_k for each $k \in \mathbb{N}$ where $\xi_1 := b\ v_1$, and $\xi_n := n^{-1} v_n$ for all integers $n \geq 2$ such that for each $n \in \mathbb{N}$, $f_n = \xi_n e_n$.

Then,

$$\|g - f\| = \left\|\sum_{k=1}^{\infty} \beta_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k\right\| = \left\|\sum_{k=1}^{\infty} \beta_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k\right\|$$

$$= \left\|\sum_{k=1}^{\infty} (\beta_k - \gamma_k) f_k\right\|$$

$$= \int_0^1 t^m \left|\sum_{k=1}^{\infty} (\beta_k - \gamma_k) f_k\right| dm = \int_0^1 \left|\sum_{k=1}^{\infty} (\beta_k - \gamma_k) t^m f_k\right| dm$$

$$\geq \left|\int_0^1 \sum_{k=1}^{\infty} (\beta_k - \gamma_k) t^m f_k dm\right|$$

$$\geq b\left|\sum_{k=1}^{\infty} (\beta_k - \gamma_k)\right|$$

$$= b\left|1 - \sum_{k=1}^{\infty} \gamma_k\right|$$

$$= b\delta$$

Hence,

$$\|g - f\| \geq b\delta = \|h - f\|.$$

Next, we have the following.

$s(h) = s(f) + \|h - f\| \leq s(f) + \|T^q h - f\| = s(T^q h)$ but this follows

$$= \limsup_n \|T^q h - f^{(n)}\| \text{ then similarly to the inequality (2.1)}$$
$$\leq \limsup_n \|T^q h - T^q(f^{(n)})\| + \limsup_n \|f^{(n)} - T^q(f^{(n)})\|$$
$$\leq k_q \limsup_n \|h - f^{(n)}\| + \limsup_n \sum_{j=1}^{q} \|T^j(f^{(n)}) - T^{j-1}(f^{(n)})\|$$
$$\leq k_q \limsup_n \|h - f^{(n)}\| + \limsup_n \sum_{j=1}^{q} k_{j-1} \|T(f^{(n)}) - f^{(n)}\|$$
$$\leq k_q \limsup_n \|h - f^{(n)}\| + 0$$
$$= k_q s(h).$$

Hence, $s(h) \leq s(T^q h) \leq k_q s(h)$ and so taking limit as $q \to \infty$, we have $\lim_q s(T^q h) = s(h)$; that is, $\lim_q s(f) + \|T^q h - f\| = \lim_q s(f) + \|h - f\|$ which means $\lim_q \|T^q h - f\| = \|h - f\|$ (2.2)

Moreover, for any $g \in E^{(m)}$,

$$\|g - h\| = \left\|\sum_{k=1}^{\infty} \beta_k f_k - (\gamma_1 + \delta) f_1 - \sum_{n=2}^{\infty} \gamma_n f_n\right\| = \left\|\sum_{k=2}^{\infty} (\beta_k - \gamma_k) f_k + (\beta_1 - \gamma_1 - \delta) f_1\right\|$$
$$\leq \left\|\sum_{k=2}^{\infty} (\beta_k - \gamma_k) f_k\right\| + \|(\beta_1 - \gamma_1 - \delta) f_1\| = \int_0^1 t^m \left|\sum_{k=2}^{\infty} (\beta_k - \gamma_k) f_k\right| dm + \int_0^1 t^m |(\beta_1 - \gamma_1 - \delta) f_1| dm$$
$$\leq \sum_{k=2}^{\infty} \int_0^1 t^m |(\beta_k - \gamma_k) f_k| dm + \int_0^1 t^m |(\beta_1 - \gamma_1 - \delta) f_1| dm = \sum_{k=2}^{\infty} |\beta_k - \gamma_k| + b|\beta_1 - \gamma_1 - \delta|$$
$$= \sum_{k=2}^{\infty} |\beta_k - \gamma_k| + b\left|\beta_1 + \sum_{k=2}^{\infty} \beta_k - \sum_{k=2}^{\infty} \beta_k - \gamma_1 - 1 + \sum_{k=1}^{\infty} \gamma_k\right|$$
$$= \sum_{k=2}^{\infty} |\beta_k - \gamma_k| + b\left|\sum_{k=2}^{\infty} \gamma_k - \sum_{k=2}^{\infty} \beta_k\right|$$
$$\leq \sum_{k=2}^{\infty} |\beta_k - \gamma_k| + b \sum_{k=2}^{\infty} |\beta_k - \gamma_k| = (1 + b) \sum_{k=2}^{\infty} |\beta_k - \gamma_k| = \frac{1+b}{1-b}(1 - b) \sum_{k=2}^{\infty} |\beta_k - \gamma_k|$$
$$= \frac{1+b}{1-b}\left[b\delta - b\delta + (1 - b) \sum_{k=2}^{\infty} |\beta_k - \gamma_k|\right] = \frac{1+b}{1-b}\left[b(1 - (1 - \delta)) - b\delta + (1 - b) \sum_{k=2}^{\infty} |\beta_k - \gamma_k|\right]$$
$$= \frac{1+b}{1-b}\left[b(1 - (1 - \delta)) + (1 - b) \sum_{k=2}^{\infty} |\beta_k - \gamma_k| - b\delta\right]$$
$$= \frac{1+b}{1-b}\left[b\left(\sum_{k=1}^{\infty} \beta_k - \sum_{k=1}^{\infty} \gamma_k\right) + (1 - b) \sum_{k=2}^{\infty} |\beta_k - \gamma_k| - b\delta\right]$$
$$\leq \frac{1+b}{1-b}\left[b \sum_{k=1}^{\infty} |\beta_k - \gamma_k| + (1 - b) \sum_{k=2}^{\infty} |\beta_k - \gamma_k| - b\delta\right].$$

Hence,

$$\|g - h\| \leq \frac{1+b}{1-b}\left[b|\beta_1 - \gamma_1| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k| - b\delta\right] = \frac{1+b}{1-b}[\|g - f\| - \|h - f\|].$$

Now, fix $\varepsilon > 0$ and recall that $b \in (0,1)$. Then, we can choose $\mu(\varepsilon) := \frac{1-b}{1+b}\varepsilon \in (0, \infty)$ such that for any $g = \sum_{k=1}^{\infty} \beta_k f_k \in E^{(m)}$,

$$|\|g - f\| - \|h - f\|| \leq \|g - f\| - \|h - f\| < \mu.$$

Then, $\|g - h\| < \frac{1+b}{1-b}\mu = \varepsilon$.

So for every $\varepsilon > 0$, there exists $\mu = \mu(\varepsilon)$ such that if $|\|g - f\| - \|h - f\|| < \mu$ then $\|g - h\| < \varepsilon$ so this implies for any sequence $(s_n)_n$ in $E^{(m)}$ with $\lim_n \|s_n - f\| = \|h - f\|$ implies $\lim_n \|s_n - h\| = 0$. But then since in (2.2) we obtained $\lim_q \|T^q h - f\| = \|h - f\|$, we have $\lim_q \|T^q h - h\| = 0$.

Furthermore,

$$\|h - Th\| \leq \lim_q \|T^q h - h\| + \lim_q \|T^q h - Th\| \leq k_1 \lim_q \|T^{q-1} h - h\| = 0$$

Hence, $Th = h$ and so $E^{(m)}$ has fpp(ane) as desired.

From Theorem 2.1, the following Corollary is straightforward since every nonexpansive mappings is also an asymptotically nonexpansive mapping.

Corollary 2.2. *For $m \in \mathbb{N}$ and $b \in (0,1)$, each of the sets $E^{(m)}$ defined as in the examples above has the fixed point property for affine nonexpansive mappings.*

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