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Research Article

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Boundedness analysis of infinite delay stochastic differential systems with Lévy noise

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Abstract The problem of boundedness for a class of stochastic differential systems with infinite delay driven by Lévy noise is considered in this article. Several sufficient conditions are derived to ensure that the solution is mean square exponential ultimate boundedness by employing the generalized formula and the stochastic analysis, in which there is no need to construct Lyapunov functions.

Keywords boundedness; stochastic differential systems; infinite delay

1. Introduction

In recent years, the theory of stochastic differential equations has become an active area of investigation due to their applications in many fields, see [1]. In systems analysis, stability and boundedness theory are two important aspects to be studied. The stability for stochastic differential equations has been paid more attention by more authors and we here only mention [2-7]. In fact , there is usually no equilibrium point for stochastic differential systems under conditions of random disturbances. Therefore, the discussion of boundeness is far more meaningful than the discussion of stability for stochastic differential systems. Brownian motion is often used to model random factors that occur in a system. Up to known, a number of important results have been obtained for stochastic differential equations driven by Brownian motion, see [8]. However, the study of Lévy noise-driven stochastic differential equation theory is still limited compared to the study of Brown motion-driven stochastic differential equation theory. The study of bounded delay [9] has attracted a great deal of attention from academics. Infinite delay are a further extension of bounded delay, making them more general. Therefore, the study of infinite delay is even more meaningful.

Based on the above discussion, by applying the Ito's formula, we discuss the globally exponentially ultimately bounded in mean square of stochastic differential systems with infinite delay with Lévy white noise, the sufficient condition of the globally exponentially ultimately bounded in mean square of the system are derived. Finally, an example is given to verify the effectiveness of the results.

2. Preliminaries

In the section, we start with some useful notation in this paper. Let $N=\{1,2,\dots\}$, and $\mathbb{R}_+=[0,\infty)$. Let $\omega(t)=(\omega_1(t),\dots,\omega_m(t))^T$ be an m-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. For $t\geq 0$, denote by $C^2_{\mathcal{F}_t}((-\infty,0],\mathbb{R}^d)$ the family of $\mathcal{F}_t$-measurable $C((-\infty,0],\mathbb{R}^d)$-valued random variables ζ satisfying $E\|\zeta\|^2_{-\infty}<\infty$, where $C((-\infty,0],\mathbb{R}^d)$ denotes the family of continuous functions $\zeta:(-\infty,0]\to\mathbb{R}^d$ with the norm $\|\zeta\|_{-\infty}=sup_{-\infty\leq s\leq 0}\|\zeta(s)\|$, and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$. Let $C^2_1(\mathbb{R}^d\times\mathbb{R}_+;\mathbb{R}_+)$ be the family of all $\mathbb{R}_+$-valued functions $V(x,t)$ defined on $\mathbb{R}^d\times\mathbb{R}_+$ which are once differentiable in $t\in\mathbb{R}_+$ and continuously twice differentiable in $x\in\mathbb{R}^d$.

In this paper, we consider the following infinite delay stochastic differential equation driven by Lévy white noise:

$$\begin{cases} dx(t) = H(t, x(t), \int_{-\infty}^{t} Q(t,s,x(s))\,ds)dt + G(t, x(t), \int_{-\infty}^{t} Q(t,s,x(s))\,ds)d\omega(t) \\ \quad + \int_{\|\varsigma\|<c} H_1(t^-, \varsigma, x(t))\,\tilde{\Xi}(dt, d\varsigma) + \int_{\|\varsigma\|\geq c} H_2(t^-, \varsigma, x(t))\,\Xi(dt, d\varsigma) \;, t \geq t_0 \\ x(t_0+\theta) = \zeta(\theta) \in C^b_{\mathcal{F}_{t_0}}((-\infty, 0], \mathbb{R}^d) \end{cases} \quad (2\text{-}1)$$

where the mappings $H(.,.,.)$and $G(.,.,.)$are Borel-measurable functions, which are defined on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ and taken values in $\mathbb{R}^d$and $\mathbb{R}^{d\times m}$, respectively. $H(.,.,.)$and $G(.,.,.)$is (locally) Lipschitz continuous with respect to the second and the third arguments on each compact subset of $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$. $Q(.,.,.)\colon \Delta \times \mathbb{R}^d \to \mathbb{R}^d (\Delta := \{(t,s) \in \mathbb{R} \times \mathbb{R} : t \geq s\})$ are matrix-valued continuous functions. $Q(.,.,.)$ is (locally) Lipschitz continuous with respect to the third argument on each compact subset of $\Delta \times \mathbb{R}^d$. $H_i\colon \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$with $i = 1,2$, and the constant $c \in (0, +\infty)$ is the maximum allowable jump size. Let Ξ be the Poisson random measure defined on $\mathbb{R}_+ \times (\mathbb{R} - \{0\})$ with the compensator $\tilde{\Xi}$ and the intensity measure v. In this paper, we always assume that $\Xi$is independent of $\omega$and v is a Lévy measure which satisfies $\tilde{\Xi}(dt, d\varsigma) := \Xi(dt, d\varsigma) - v(d\varsigma)dt$ and $\int_{\mathbb{R}-\{0\}} (\|\varsigma\|^2 \wedge 1)\, v(d\varsigma) < \infty$. Usually, the symbol (ω, Ξ) is called a Lévy noise, $\int_{\|\varsigma\|<c} H_1(t^-, \varsigma, x(t))\,\tilde{\Xi}(dt, d\varsigma)$ is called "small jump" and $\int_{\|\varsigma\|\geq c} H_2(t^-, \varsigma, x(t))\,\Xi(dt, d\varsigma)$ is called "large jump" . Throughout this paper we assume that for any $\zeta \in C^b_{\mathcal{F}_{t_0}}((-\infty, 0], \mathbb{R}^d)$, there exists at least one solution of system (2-1). and $E\|x(t; t_0, \zeta)\|^2$ is continuous.

The definition of the globally exponentially stability in mean square and the globally exponentially ultimately bounded in mean square are given below.

Definition 2.1 System (2-1) is said to be globally exponentially ultimately bounded in mean square if there are positive constants λ、K and ηsuch that for any initial value $\zeta \in C^b_{\mathcal{F}_{t_0}}((-\infty, 0], \mathbb{R}^d)$,

$$E\|x(t; t_0, \zeta)\|^2 \leq K e^{-\lambda(t-t_0)} E\|\zeta\|^2_{-\infty} + \eta, \quad t \geq t_0.$$

In order to better apply Ito's formula, for a function $U \in C_1^2(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}_+)$, define the operator $\mathcal{G}U(t, x(t))\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$,

$$\mathcal{G}U(t,x) = U_t(t,x) + U_x(t,x)H + \frac{1}{2} trace[G^T U_{xx}(t,x) G]$$
$$+ \int_{\|\varsigma\|<c} [U(t, x + H_1(t, \varsigma, x(t))) - U(t,x) -$$
$$H_1(t, \varsigma, x(t)) U_x(t,x)] v(d\varsigma)$$
$$+ \int_{\|\varsigma\|\geq c} [U(t, x + H_2(t, \varsigma, x(t))) - U(t,x)] v(d\varsigma).$$

Assumption 2.1. $E\|x(t; t_0, \zeta)\|^2$ is continuous.

1. Main result

To prove the globally exponentially ultimately bounded in mean square pf the system (2-1), suppose there exist continuous functions $h(t)$, $g(t)$, $\hat{h}(t)$, $\hat{g}(t)\colon \mathbb{R} \to \mathbb{R}_+$, $\gamma(.,.)\colon \Delta \to \mathbb{R}_+$, and give the following assumptions:

(T_1) $x^T H(t,x,y) \leq -h(t)\|x(t)\|^2 + g(t)\|x(t)\|\|y(t)\| + I(t)$, $x, y \in \mathbb{R}^d$; (3-1)

(T_2) $G^T(t,x,y) G(t,x,y) \leq \hat{h}(t)\|x(t)\|^2 + \hat{g}(t)\|y(t)\|^2 + J(t)$, $x, y \in \mathbb{R}^d$; (3-2)

(T_3) $\|Q(t,s,z)\|^2 \leq \gamma(t,s)\|z\|^2, (t,s) \in \Delta, z \in R^d$; $sup_{t\in R} \int_{-\infty}^{t} \gamma(t,s)\,ds < \infty$; (3-3)

(T_4) $\int_{\|\varsigma\|<c} x^T H_1(t, \varsigma, x(t)) v(d\varsigma) \geq l\|x(t)\|^2$; (3-4)

(T_5) $\int_{\|\varsigma\|<c} H_1^T[(t, \varsigma, x(t)) + x] H_1[(t, \varsigma, x(t)) + x] v(d\varsigma) \leq q\|x(t)\|^2$; (3-

5)

(T_6) $\int_{\|\varsigma\|\geq c} H_2^T[(t,\varsigma,x(t))+x]H_2[(t,\varsigma,x(t))+x]v(d\varsigma) \leq r\|x(t)\|^2$; (3-6)

(T_7) Suppose there exist $\delta > 0$, such that

$$-2h(t)+\hat{h}(t)+g(t)+q+r-2l-[\int_{\|\varsigma\|<c} v(d\varsigma)+\int_{\|\varsigma\|\geq c} v(d\varsigma)]$$
$$+(g(t)+\hat{g}(t))\int_{-\infty}^{t}\gamma(t,\theta)e^{\delta(t-\theta)}\,d\theta \leq -\delta, \quad t\in\mathbb{R}. \tag{3-7}$$

Theorem 3.1. Assume$(T_1)-(T_7)$hold，then system (2-1) is globally exponentially ultimately bounded in mean square.

Proof. Consider the functions

$$L(t):=E\|x(t;t_0,\zeta)\|^2,\quad P(t):=Ke^{-\delta(t-t_0)}E\|\zeta\|_{-\infty}^2+\eta\ ,\quad t\geq t_0\ , \tag{3-8}$$

where $\eta=\frac{2I+J}{\delta}$， $I=sup_{t\geq t_0}|I(t)|$， $J=sup_{t\geq t_0}|J(t)|$.

Next we claim that

$$L(t)\leq P(t),\ t\geq t_0. \tag{3-9}$$

If (3-9) is not true, then there must be a $\tilde{t}>t_0$ such that $L(\tilde{t})>P(\tilde{t})$.

Let $t_*:=inf\{t>t_0: L(t)>P(t)\}$.

Using the continuity, we have

$$L(t)\leq P(t),\quad t\in[t_0,t_*),L(t_*)=P(t_*),$$

and for some$t_k\in(t_*,t_*+\frac{1}{k})$， $k\in\mathbb{N}$，such that $L(t_k)>P(t_k)$. (3-10)

Let the function $V(x,t):=e^{\lambda t}\|x(t)\|^2$， $(x,t)\in\mathbb{R}^d\times\mathbb{R}_+$.

Fix $\lambda=sup_{t\in[t_0,\tilde{t}]}\left|2h(t)-\hat{h}(t)-g(t)\right|+1$，let $x(t):=x(t;t_0,\zeta)$ ， $t\in\mathbb{R}$.

For any $m>\|x_0\|$，the stopping time is defined by $\mu_m=inf\{t\geq t_0: m\leq\|x(t)\|\}$.

Apply the Ito's formula to $e^{\lambda t}\|x(t)\|^2$，we obtain

$$e^{\lambda(t\wedge\mu_m)}\|x(t\wedge\mu_m)\|^2\leq e^{\lambda t_0}\|x(t_0)\|^2+\int_{t_0}^{t\wedge\mu_m}\lambda e^{\lambda s}\|x(s)\|^2\,ds$$
$$+2\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}x^T(s)H(s,x(s),\int_{-\infty}^{s}Q(s,\theta,x(\theta))\,d\theta)\,ds$$
$$+\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}G^T(s,x(s),\int_{-\infty}^{s}Q(s,\theta,x(\theta))\,d\theta)$$
$$G(s,x(s),\int_{-\infty}^{s}Q(s,\theta,x(\theta))\,d\theta)ds$$
$$+\int_{t_0}^{t\wedge\mu_m}V_x(x(s),x)G(s,x(s),\int_{-\infty}^{s}Q(s,\theta,x(\theta))\,d\theta)d\omega(s)$$
$$+\int_{t_0}^{t\wedge\mu_m}\int_{\|\varsigma\|<c}[V(x(s^-)+H_1(s^-,\varsigma,x(s)),s^-)-$$
$$V(x(s^-),s^-)-H_1(s^-,\varsigma,x(s))V_x(x(s^-),s^-)]v(d\varsigma)ds$$
$$+\int_{t_0}^{t\wedge\mu_m}\int_{\|\varsigma\|\geq c}[V(x(s^-)+H_2(s^-,\varsigma,x(s)),s^-)-$$
$$V(x(s^-),s^-)]v(d\varsigma)ds$$
$$+\int_{t_0}^{t\wedge\mu_m}\int_{\|\varsigma\|<c}[V(x(s^-)+H_1(s^-,\varsigma,x(s)),s^-)-$$
$$V(x(s^-),s^-)]\tilde{\Xi}(dt,d\varsigma)$$
$$+\int_{t_0}^{t\wedge\mu_m}\int_{\|\varsigma\|\geq c}[V(x(s^-)+H_2(s^-,\varsigma,x(s)),s^-)-$$
$$V(x(s^-),s^-)]\Xi(dt,d\varsigma). \tag{3-11}$$

where，

$\int_{t_0}^{t\wedge\mu_m}V_x(x(s),s)G(s,x(s),\int_{-\infty}^{s}Q(s,\theta,x(\theta))d\theta)\,d\omega(s)$,

$\int_{t_0}^{t\wedge\mu_m}\int_{\|\varsigma\|<c}[V(x(s^-)+H_1(s^-,\varsigma,x(s)),s^-)-V(x(s^-),s^-)]\tilde{\Xi}(dt,d\varsigma)$，

$\int_{t_0}^{t\wedge\mu_m}\int_{\|\varsigma\|\geq c}[V(x(s^-)+H_2(s^-,\varsigma,x(s)),s^-)-V(x(s^-),s^-)]\Xi(dt,d\varsigma)$ are three martingales.

Taking the expectation on both sides of (3-11)，we have

$$Ee^{\lambda(t\wedge\mu_m)}\|x(t\wedge\mu_m)\|^2\leq Ee^{\lambda t_0}\|x(t_0)\|^2+E\int_{t_0}^{t\wedge\mu_m}\lambda e^{\lambda s}\|x(s)\|^2ds$$
$$+2E\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}x^T(s)H(s,x(s),\int_{-\infty}^{s}Q(s,\theta,x(\theta))\,d\theta)ds$$

$$+E\int_{t_0}^{t\wedge\mu_m} e^{\lambda s}G^T(s,x(s),\int_{-\infty}^{s}Q(s,\theta,x(\theta))d\theta)$$
$$G(s,x(s),\int_{-\infty}^{s}Q(s,\theta,x(\theta))d\theta)\,ds$$
$$+E\int_{t_0}^{t\wedge\mu_m}\int_{\|\varsigma\|<c}[V(x(s^-)+H_1(s^-,\varsigma,x(s)),s^-)$$
$$-V(x(s^-),s^-)-H_1(s^-,\varsigma,x(s))V_x(x(s^-),s^-)]v(d\varsigma)ds$$
$$+E\int_{t_0}^{t\wedge\mu_m}\int_{\|\varsigma\|\ge c}[V(x(s^-)+H_2(s^-,\varsigma,x(s)),s^-)$$
$$-V(x(s^-),s^-)]v(d\varsigma)ds. \quad (3\text{-}12)$$

Substituting the conditions given by (3-1)-(3-6) into (3-12)，we obtain

$$Ee^{\lambda(t\wedge\mu_m)}\|x(t\wedge\mu_m)\|^2\le Ee^{\lambda t_0}\|x(t_0)\|^2+E\int_{t_0}^{t\wedge\mu_m}\lambda e^{\lambda s}\|x(s)\|^2ds$$
$$-2E\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}h(s)\|x(s)\|^2\,ds+E\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}g(s)\|x(s)\|^2\,ds$$
$$+E\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}g(s)(\int_{-\infty}^{s}\gamma(s,\theta)\|x(\theta)\|^2\,d\theta)\,ds$$
$$+2E\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}I(s)\,ds\ +E\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}\hat{h}(s)\|x(s)\|^2\,ds$$
$$+E\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}\,\hat{g}(s)(\int_{-\infty}^{s}\gamma(s,\theta)\|x(\theta)\|^2\,d\theta)ds$$
$$+E\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}J(s)\,ds$$
$$+E\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}q\|x(s)\|^2\,ds+E\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}r\|x(s)\|^2\,ds$$
$$-2E\int_{t_0}^{t\wedge\mu_m}e^{\lambda s}l\|x(s)\|^2\,ds$$
$$-E\int_{t_0}^{t\wedge\mu_m}\int_{\|\varsigma\|<c}e^{\lambda s}\|x(s)\|^2\,v(d\varsigma)\,ds$$
$$-E\int_{t_0}^{t\wedge\mu_m}\int_{\|\varsigma\|\ge c}e^{\lambda s}\|x(s)\|^2\,v(d\varsigma)\,ds. \quad (3\text{-}13)$$

Letting $m\to\infty$and using the Fubini theorem，we have

$$e^{\lambda t}E\|x(t)\|^2\le e^{\lambda t_0}E\|x(t_0)\|^2+\int_{t_0}^{t}\lambda e^{\lambda s}E\|x(s)\|^2\,ds$$
$$+\int_{t_0}^{t}e^{\lambda s}\,[-2h(s)+\hat{h}(s)+g(s)+q+r-2l$$
$$-(\int_{\|\varsigma\|<c}v(d\varsigma)+\int_{\|\varsigma\|\ge c}v(d\varsigma))]E\|x(s)\|^2ds$$
$$+\int_{t_0}^{t}e^{\lambda s}[g(s)+\hat{g}(s)]\,(\int_{-\infty}^{s}\gamma(s,\theta)\,E\|x(\theta)\|^2d\theta)ds$$
$$+\int_{t_0}^{t}e^{\lambda s}(2I(s)+J(s))\,ds.$$

Let $K_1:=KE\|\zeta\|_{-\infty}^2$. Combining (3-10)，we have

$$e^{\lambda t_*}E\|x(t_*)\|^2\le e^{\lambda t_0}E\|x(t_0)\|^2+\int_{t_0}^{t_*}\lambda e^{\lambda s}\,(K_1e^{-\delta(s-t_0)}+\eta)ds$$
$$+\int_{t_0}^{t_*}e^{\lambda s}\,[-2h(s)+\hat{h}(s)+g(s)+q+r-2l$$
$$-(\int_{\|\varsigma\|<c}v(d\varsigma)+\int_{\|\varsigma\|\ge c}v(d\varsigma))](K_1e^{-\delta(s-t_0)}+\eta)ds$$
$$+\int_{t_0}^{t_*}e^{\lambda s}[g(s)+\hat{g}(s)]\,[\int_{-\infty}^{s}\gamma(s,\theta)\,(K_1e^{-\delta(\theta-t_0)}+\eta)d\theta]ds$$
$$+\int_{t_0}^{t_*}e^{\lambda s}(2I+J)\,ds$$
$$=e^{\lambda t_0}E\|x(t_0)\|^2+\int_{t_0}^{t_*}\lambda e^{\lambda s}\,(K_1e^{-\delta(s-t_0)}+\eta)ds$$
$$+\int_{t_0}^{t_*}e^{\lambda s}\,K_1e^{-\delta(s-t_0)}[-2h(s)+\hat{h}(s)+g(s)+q+r-2l$$
$$-(\int_{\|\varsigma\|<c}v\,(d\varsigma)+\int_{\|\varsigma\|\ge c}v\,(d\varsigma))+(g(s)+\hat{g}(s))$$
$$\int_{-\infty}^{s}\gamma(s,\theta)\,e^{\delta(s-\theta)}d\theta]ds$$
$$+\int_{t_0}^{t_*}e^{\lambda s}\eta\,[-2h(s)+\hat{h}(s)+g(s)+q+r-2l$$
$$-(\int_{\|\varsigma\|<c}v\,(d\varsigma)+\int_{\|\varsigma\|\ge c}v\,(d\varsigma))+(g(s)$$
$$+\hat{g}(s))\int_{-\infty}^{s}\gamma(s,\theta)\,d\theta]ds$$

$$+\frac{1}{\lambda}(2I+J)(e^{\lambda t_*}-e^{\lambda t_0}).$$

Combining (3-7), we obtain

$$\begin{aligned}
e^{\lambda t_*}E\|x(t_*)\|^2 &\le e^{\lambda t_0}E\|x(t_0)\|^2+\int_{t_0}^{t_*}\lambda e^{\lambda s}\,(K_1e^{-\delta(s-t_0)}+\eta)ds\\
&\quad+\int_{t_0}^{t_*}(-\delta)e^{\lambda s}\,K_1e^{-\delta(s-t_0)}ds\\
&\quad+\int_{t_0}^{t_*}e^{\lambda s}\eta\,[-2h(s)+\hat{h}(s)+g(s)]ds\\
&\quad+\int_{t_0}^{t_*}e^{\lambda s}\,[g(s)+\hat{g}(s)][\int_{-\infty}^{s}\eta\,\gamma(s,\theta)d\theta]ds\\
&\quad+\int_{t_0}^{t_*}e^{\lambda s}\,\eta[q+r-2l-\int_{\|\varsigma\|<c}\upsilon\,(d\varsigma)-\int_{\|\varsigma\|\ge c}\upsilon\,(d\varsigma)]ds\\
&\quad+\frac{1}{\lambda}(2I+J)(e^{\lambda t_*}-e^{\lambda t_0})\\
&=e^{\lambda t_0}E\|x(t_0)\|^2+\int_{t_0}^{t_*}K_1e^{\delta t_0}\,e^{(\lambda-\delta)s}(\lambda-\delta)ds\\
&\quad+\int_{t_0}^{t_*}e^{\lambda s}\,[-2h(s)+\hat{h}(s)+g(s)]\eta ds\\
&\quad+\int_{t_0}^{t_*}e^{\lambda s}\,[g(s)+\hat{g}(s)][\int_{-\infty}^{s}\eta\,\gamma(s,\theta)d\theta]ds\\
&\quad+\frac{2I+J+\lambda\eta+[q+r-2l-\int_{\|\varsigma\|<c}\upsilon(d\varsigma)-\int_{\|\varsigma\|\ge c}\upsilon(d\varsigma)]\eta}{\lambda}(e^{\lambda t_*}-e^{\lambda t_0})\\
&<e^{\lambda t_0}E\|x(t_0)\|^2+\int_{t_0}^{t_*}K_1e^{\delta t_0}\,e^{(\lambda-\delta)s}(\lambda-\delta)ds+\int_{t_0}^{t_*}\eta\lambda e^{\lambda s}\,ds\\
&=e^{\lambda t_0}E\|x(t_0)\|^2+K_1e^{\delta t_0}[e^{(\lambda-\delta)t_*}-e^{(\lambda-\delta)t_0}]+\eta[e^{\lambda t_*}-e^{\lambda t_0}]\\
&=e^{\lambda t_0}(E\|\zeta\|_{-\infty}^2-KE\|\zeta\|_{-\infty}^2)+KE\|\zeta\|_{-\infty}^2e^{\lambda t_*}e^{-\delta(t_*-t_0)}\\
&\quad+\eta[e^{\lambda t_*}-e^{\lambda t_0}]\\
&<KE\|\zeta\|_{-\infty}^2e^{\lambda t_*}e^{-\delta(t_*-t_0)}+\eta e^{\lambda t_*}.
\end{aligned}$$

Since, $E\|x(t_*)\|^2<K_1e^{-\delta(t_*-t_0)}+\eta$, which conflicts with (3-10).

So $L(t)=E\|x(t;t_0,\zeta)\|^2\le P(t)=Ke^{-\delta(t-t_0)}E\|\zeta\|_{-\infty}^2+\eta$, $t\ge t_0$.

Therefore, system (2-1) is globally exponentially ultimately bounded in mean square.

Corollary 3.1 If $I(s)=J(s)=0$, and $(T_1)-(T_7)$ hold, then system (2-1) is the globally exponentially stability in mean square.

Remark 3.1. Although some similar methods for discussing the boundedness of stochastic systems driven by Lévy noise have been provided in [10], the results in [10] are invalid for system (2-1) since it is a system with infinite time delay. Even for the case $Q(.,.,.)\equiv 0$, our conditions are looser than those in [10] since our conditions are non-autonomous.

2. Illustrative example

The present section gives an illustrative example for our results.

Example 4.1. Consider the following infinite delay stochastic differential equation with Lévy white noise, where $c=1$, $H(t,x,y)=-7x+y+\frac{x}{\|x\|^2}$, $G(t,x,y)=x+y$, $Q(t,s,z)=e^{\frac{1}{2}(s-t)}z$, $H_1(t,\varsigma,x)=x\varsigma-x$, $H_2(t,\varsigma,x)=2x\varsigma-x$, and the Lévy measure obeys $\upsilon(d\varsigma)=\frac{d\varsigma}{1+\|\varsigma\|^2}$.

Clearly,

$$x^TH(t,x,y)=x^T(-7x+y+\frac{x}{\|x\|^2})=-7\|x\|^2+\|x\|\|y\|+1;$$

$$G^T(t,x,y)G(t,x,y)=(x+y)^T(x+y)\le 2\|x(t)\|^2+2\|y(t)\|^2;$$

$$\|Q(t,s,z)\|^2=e^{(s-t)}\|z\|^2;\quad sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\gamma(t,s)\,ds=1;$$

$$\int_{\|\varsigma\|<1}x^TH_1(t,\varsigma,x(t))\,\upsilon(d\varsigma)=\int_{\|\varsigma\|<1}x^T(x\varsigma-x)\frac{d\varsigma}{1+\|\varsigma\|^2}=-\frac{\pi}{2}\|x(t)\|^2;$$

$$\begin{aligned}
&\int_{\|\varsigma\|<1}H_1^T\,[(t,\varsigma,x(t))+x]H_1[(t,\varsigma,x(t))+x]\upsilon(d\varsigma)\\
&=\int_{\|\varsigma\|<1}(x\varsigma)^T(x\varsigma)\frac{d\varsigma}{1+\|\varsigma\|^2}=\frac{4-\pi}{2}\|x\|^2;
\end{aligned}$$

$$\int_{\|\varsigma\|\geq 1} H_2^T\,[(t,\varsigma,x(t))+x]H_2[(t,\varsigma,x(t))+x]v(d\varsigma)$$
$$=\int_{\|\varsigma\|\geq 1}(2x\varsigma)^T(2x\varsigma)\frac{d\varsigma}{1+\|\varsigma\|^2}=(8-2\pi)\|x\|^2.$$

Then we obtain

$h(t)=7$, $g(t)=1$, $I(t)=1$, $\hat{h}(t)=2$, $\hat{g}(t)=2$, $J(t)=0$, $l=-\frac{\pi}{2}$, $q=\frac{4-\pi}{2}$, $r=8-2\pi$.

Fix $\delta=\frac{1}{2}$, we can verify

$$-2h(t)+\hat{h}(t)+g(t)+q+r-2l-[\int_{\|\varsigma\|<1} v(d\varsigma)+\int_{\|\varsigma\|\geq 1} v(d\varsigma)]$$
$$+(g(t)+\hat{g}(t))\int_{-\infty}^{t}\gamma(t,\theta)e^{\delta(t-\theta)}\,d\theta=-14+2+1+\frac{4-\pi}{2}+8-2\pi+\pi-\pi$$
$$+3\int_{-\infty}^{t} e^{(\theta-t)}e^{\frac{1}{2}(t-\theta)}\,d\theta=-1-\frac{5}{2}\pi+6=5-\frac{5}{2}\pi=-2.85\leq-\delta=-\frac{1}{2}.$$

Hence, (T_7) holds.

Therefore, according to Theorem 3.1, this stochastic differential equation is globally exponentially ultimately bounded in mean square.

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